EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 2, Article Number 5930
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

A Theoretical Exploration of Rough Approximations in Hilbert Algebras

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Abstract. In this paper, we introduce the concept of roughness in the context of Hilbert algebras, a class of algebraic structures fundamental to studying non-classical logic. By integrating rough set theory with Hilbert algebras, we investigate the lower and upper approximations of subalgebras and ideals. We show that the lower and upper approximations of a subalgebra (or ideal) in a Hilbert algebra also make up a subalgebra (or ideal). This implies that algebraic systems can employ rough set concepts. Our results demonstrate that the approximation spaces induced by ideals in Hilbert algebras provide a robust framework for analyzing algebraic structures under incomplete or uncertain information. Furthermore, we present illustrative examples to validate our theoretical findings and highlight the practical implications of this approach. This study not only enriches the theoretical foundations of rough set theory but also opens new avenues for its application in algebraic logic and related fields.

2020 Mathematics Subject Classifications: 03G25, 03E72

Key Words and Phrases: Hilbert algebra, subalgebra, ideal, congruence, rough set, lower and upper approximations

1. Introduction

The concept of a rough set was originally proposed by Pawlak [1, 2] as a formal tool for modeling and processing complete information in information systems. It seems that the rough set approach is fundamentally important in artificial intelligence and cognitive sciences, especially in research areas such as machine learning, intelligent systems, inductive reasoning, pattern recognition, knowledge discovery, decision analysis, and expert

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i2.5930
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systems. Rough set theory (RST), a new mathematical approach to dealing with inexact, uncertain, or vague knowledge, has recently received wide attention in the research areas in both real-life applications and the theory itself. RST is an extension of set theory in which a subset of a universe is described by a pair of ordinary sets called the lower and upper approximations. There are at least two methods for developing this theory, the constructive and axiomatic approaches. In constructive methods, lower and upper approximations are constructed from primitive notions such as equivalence relations on a universe [2, 3] and neighborhood systems [4, 5]. In Pawlak's rough set [1], the equivalence classes are the building blocks for constructing the lower and upper approximations. Comer [6] presented an interesting discussion of rough sets and various algebras related to the study of algebraic logic, such as Stone algebras and relation algebras. It is a natural question to ask: What happens if we substitute an algebraic system instead of the universe set? The concept of Hilbert algebra was introduced in the early 50s by Henkin for some investigations of implication in intuitionistic and other non-classical logic [7]. In the 60s, these algebras were studied, especially by Diego, from an algebraic point of view. Diego proved [8] that Hilbert algebras form a locally finite variety. Hilbert algebras were treated by Busneag [9, 10] and Jun [11], and some of their filters forming deductive systems were recognized. Dudek [12] considered the fuzzification of subalgebras and deductive systems in Hilbert algebras.

The integration of RST with Hilbert algebras is motivated by the need to analyze algebraic structures under conditions of uncertainty and incomplete information. Hilbert algebras play a fundamental role in non-classical logic, particularly in the study of implication structures, making them a powerful tool for reasoning in mathematical logic and artificial intelligence. However, real-world applications often involve imprecise or vague information, where classical algebraic methods may not be sufficient. RST, introduced by Pawlak [1, 2], provides a robust framework for dealing with such uncertainties by defining lower and upper approximations of sets based on equivalence relations. In this context, the work by Borumand Saeid and Haveshki [13] represents an important step toward bridging rough set theory and Hilbert algebras. Their study focused on approximation techniques within Hilbert algebras and highlighted how algebraic operations interact with equivalence-based approximations. They explored properties of definable sets and congruence relations, providing foundational insights that guide subsequent developments in this area. Inspired by their contribution, our study advances this line of research by formalizing the structure of rough subalgebras and rough ideals and establishing algebraic closure under approximation operations. By applying RST to Hilbert algebras, we establish a novel approach to approximating subalgebras and ideals, ensuring that key algebraic properties are preserved even under uncertainty. This combination not only enriches the theoretical landscape of algebraic logic but also opens new avenues for applications in fuzzy logic, decision-making systems, and knowledge representation. The ability to approximate algebraic structures within Hilbert algebras through rough set approximations provides a systematic way to manage and process incomplete or ambiguous data, making this integration both mathematically significant and practically relevant.

This paper explores the integration of RST with Hilbert algebras, a class of algebraic structures central to non-classical logic. We introduce the concept of roughness in Hilbert algebras and investigate the lower and upper approximations of subalgebras and ideals. Our main results show that these approximations preserve the structure of subalgebras and ideals, extending the applicability of rough set concepts to algebraic systems. The approximation spaces induced by ideals in Hilbert algebras offer a powerful framework for analyzing algebraic structures under conditions of uncertainty or incomplete information. Through illustrative examples, we validate our theoretical contributions and underscore the practical relevance of this approach. This work not only advances the theoretical foundations of RST but also paves the way for novel applications in algebraic logic and beyond.

The rest of this paper is organized as follows: Section 2 presents the necessary preliminaries on rough sets and Hilbert algebras. Section 3 introduces and investigates the rough approximations in Hilbert algebras, providing key results and illustrative examples. Finally, Section 4 concludes the paper with a summary and future research directions.

2. Preliminaries

Let U be a universal set. For an equivalence relation Θ on U, the set of elements of U that are related to $x \in U$ is called the equivalence class of x and is denoted by $[x]_\Theta$. Moreover, let U/Θ denote the family of all equivalence classes induced on U by θ. For any $X \subseteq U$, we write X^c to denote the complement of X in U, the set $U \backslash X$. A pair (U, Θ), where $U \neq \emptyset$ and Θ is an equivalence relation on U is called an approximation space. The interpretation in RST is that our knowledge of the objects in U extends only up to membership in the class of Θ and our knowledge about a subset X of U is limited to the class of Θ and their unions. This leads to the following definition.

Definition 1. *[1] Let $\mathcal{P}(U)$ denote the power set of a universal set U. For an approximation space (U, Θ), by a rough approximation in (U, Θ) we mean a mapping $Apr : \mathcal{P}(U) \to \mathcal{P}(U) \times \mathcal{P}(U)$ defined for every $X \in \mathcal{P}(U)$ by $Apr(X) = (\underline{Apr}(X), \overline{Apr}(X))$, where $\underline{Apr}(X) = \{x \in U : [x]_\Theta \subseteq X\}$ and $\overline{Apr}(X) = \{x \in U : [x]_\Theta \cap X = \emptyset\}$. Also, $\underline{Apr}(X)$ is called a lower rough approximation of X in (U, Θ), whereas $\overline{Apr}(X)$ is called an upper rough approximation of X in (U, Θ).*

Definition 2. *[1] Given an approximation space (U, Θ), a pair $(A, B) \in \mathcal{P}(U) \times \mathcal{P}(U)$ is called a rough set in (U, Θ) if $(A, B) = Apr(X)$ for some $X \in \mathcal{P}(U)$.*

Definition 3. *[8] A Hilbert algebra is a triplet with the formula $A = (A, \cdot, 1)$, where A is a nonempty set, $\cdot$ is a binary operation, and 1 is a fixed member of A that is true according to the axioms stated below:*

$$(\forall x, y \in A)(x \cdot (y \cdot x) = 1) \tag{1}$$

$$(\forall x, y, z \in A)((x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1) \tag{2}$$

$$(\forall x, y \in A)(x \cdot y = 1, y \cdot x = 1 \Rightarrow x = y) \tag{3}$$

In [12], the following conclusion was established.

Lemma 1. *Let $A = (A, \cdot, 1)$ be a Hilbert algebra. Then*

(1) $(\forall x \in A)(x \cdot x = 1)$,

(2) $(\forall x \in A)(1 \cdot x = x)$,

(3) $(\forall x \in A)(x \cdot 1 = 1)$,

(4) $(\forall x, y, z \in A)(x \cdot (y \cdot z) = y \cdot (x \cdot z))$,

(5) $(\forall x, y, z \in A)((x \cdot z) \cdot ((z \cdot y) \cdot (x \cdot y)) = 1)$.

In a Hilbert algebra $A = (A, \cdot, 1)$, the binary relation $\leq$ is defined by

$$(\forall x, y \in A)(x \leq y \Leftrightarrow x \cdot y = 1),$$

which is a partial order on A with 1 as the largest element.

Definition 4. *[14] A nonempty subset D of a Hilbert algebra $A = (A, \cdot, 1)$ is called a subalgebra of A if $x \cdot y \in D$ for all $x, y \in D$.*

Definition 5. *[15] A nonempty subset D of a Hilbert algebra $A = (A, \cdot, 1)$ is called an ideal of A (determined by $D \rhd A$) if the following conditions hold:*

(1) $1 \in D$,

(2) $(\forall x, y \in A)(y \in D \Rightarrow x \cdot y \in D)$,

(3) $(\forall x, y_1, y_2 \in A)(y_1, y_2 \in D \Rightarrow (y_1 \cdot (y_2 \cdot x)) \cdot x \in D)$.

3. Rough approximations in Hilbert algebras

Let V be a set and E an equivalence relation on V. For all $a \in V$, let $[a]_E$ denote the equivalence class of a with respect to E. Define the functions $E_-, E^- : \mathcal{P}(V) \to \mathcal{P}(V)$ as follows: $\forall\ S \in \mathcal{P}(V)$,

$$E_-(S) = \{a \in V : [a]_E \subseteq S\}$$

and

$$E^-(S) = \{x \in V : [a]_E \cap S \neq \emptyset\}.$$

The pair (V, E) is called an approximation space. Let S be a subset of V. Then S is said to be definable if $E_-(S) = E^-(S)$ and rough otherwise. $E_-(S)$ is called the lower approximation of S while $E^-(S)$ is called the upper approximation.

Throughout this paper, $A = (A, \cdot, 1)$ will represent a Hilbert algebra. Let I be an ideal of A. Define a relation Θ on A by $(a, b) \in \Theta$ if and only if $a \cdot b \in I$ and $b \cdot a \in I$. Then Θ is an equivalence relation on A related to an ideal I of A. Moreover, $(a, b) \in \Theta$ and $(u, v) \in \Theta$ imply $(a \cdot u, b \cdot v) \in \Theta$. Hence, Θ is a congruence relation on A.

Let I_a denote the equivalence class of a with respect to the equivalence relation Θ related to an ideal I of A, and A/I denote the collection of all equivalence classes, that is, $A/I = \{I_a : a \in A\}$. Then $I_1 = I$. If $I_a \cdot I_b$ is defined as the class containing $a \cdot b$, that is, $I_a \cdot I_b = I_{a \cdot b}$, then $(A/I, \cdot, I_1)$ is a Hilbert algebra.

Let Θ be an equivalence relation on A related to an ideal I of A. For any nonempty subset S of A, the lower and upper approximations of S are denoted by $\underline{\Theta}(I, S)$ and $\overline{\Theta}(I, S)$, respectively, that is, $\underline{\Theta}(I, S) = \{a \in A : I_a \subseteq S\}$ and $\overline{\Theta}(I, S) = \{a \in A : I_a \cap S \neq \emptyset\}$. If $I = S$, then $\underline{\Theta}(I, S)$ and $\overline{\Theta}(I, S)$ are denoted by $\underline{\Theta}(I)$ and $\overline{\Theta}(I)$, respectively.

Definition 6. *[2] Given an approximation space (U, Θ), a pair $(A, B) \in \mathcal{P}(U) \times \mathcal{P}(U)$ is called a rough set in (U, Θ) if $(A, B) = Apr(X)$ for some $X \in \mathcal{P}(U)$.*

Definition 7. *[2] Let (U, Θ) be an approximation space and X a nonempty subset of U.*

(1) If $\underline{Apr}(X) = \overline{Apr}(X)$, then X is called definable.

(2) If $Apr(X) = \emptyset$, then X is called empty interior.

(3) If $Apr(X) = U$, then X is called empty exterior.

Example 1. *Let $A = \{1, x, y, z, 0\}$ with the following Cayley table:*

$\cdot$	1	x	y	z	0
1	1	x	y	z	0
x	1	1	y	z	0
y	1	x	1	z	z
z	1	1	y	1	y
0	1	1	1	1	1

Then $A = (A, \cdot, 1)$ is a Hilbert algebra. Also $I = \{1, x\}$ is an ideal of A and let Θ be an equivalence relation on A related to I. Then $I_1 = I_x = I, I_y = \{y\}, I_z = \{z\}$, and $I_0 = \{0\}$. Hence,

$$\underline{\Theta}(I, \{1, x\}) = \{1, x\} \rhd A$$

$$\underline{\Theta}(I, \{1, y\}) = \{y\}$$

$$\underline{\Theta}(I, \{1, z\}) = \{z\}$$

$$\underline{\Theta}(I, \{1, x, y, z\}) = \{1, x, y, z\} \rhd A$$

and

$$\overline{\Theta}(I, \{1, x\}) = \{1, x\} \rhd A$$

$$\overline{\Theta}(I, \{1, y\}) = \{1, x, y\} \rhd A$$

$$\overline{\Theta}(I, \{y\}) = \{y\}$$

$$\overline{\Theta}(I, \{x, y, z\}) = \{1, x, y, z\} \rhd A$$

$$\overline{\Theta}(I, \{1, y, z\}) = \{1, x, y, z\} \rhd A$$

$$\overline{\Theta}(I, \{x, y, z, 0\}) = \{1, x, y, z, 0\} \rhd A.$$

In this example, we know that there exists a non-ideal S of A such that their lower and upper approximations are ideals of A. Also, we choose some non-ideal S of A such that their lower and upper approximations are ideals of A.

Proposition 1. *Let Θ and Ψ be equivalence relations on A related to ideals I and J of A, respectively. If S and T are nonempty subsets of A, then:*

(1) $\underline{\Theta}(I, S) \subseteq S \subseteq \overline{\Theta}(I, S)$,

(2) $\underline{\Theta}(I, \emptyset) = \emptyset = \overline{\Theta}(I, \emptyset)$,

(3) $\overline{\Theta}(I, S \cup T) = \overline{\Theta}(I, S) \cup \overline{\Theta}(I, T)$,

(4) $\underline{\Theta}(I, S \cap T) = \underline{\Theta}(I, S) \cap \underline{\Theta}(I, T)$,

(5) if $S \subseteq T$, *then* $\underline{\Theta}(I, S) \subseteq \underline{\Theta}(I, T)$ *and* $\overline{\Theta}(I, S) \subseteq \overline{\Theta}(I, T)$,

(6) $\underline{\Theta}(I, S) \cup \underline{\Theta}(I, T) \subseteq \underline{\Theta}(I, S \cup T)$,

(7) $\overline{\Theta}(I, S \cap T) \subseteq \overline{\Theta}(I, S) \cap \overline{\Theta}(I, T)$,

(8) if $\Theta \subseteq \Psi$ *and* $I \subseteq J$, *then* $\underline{\Psi}(J, S) \subseteq \overline{\Theta}(I, S)$ *and* $\overline{\Theta}(I, S) \subseteq \overline{\Psi}(J, S)$.

Proof. (1) If $x \in \underline{\Theta}(I, S)$, then $x \in I_x \subseteq S$. Hence, $\underline{\Theta}(I, S) \subseteq S$. Next, if $x \in S$, then $I_x \cap S = \emptyset$ because $x \in I_x$, and so $x \in \overline{\Theta}(I, S)$. Thus, $S \subseteq \overline{\Theta}(I, S)$.

(2) is straightforward.

(3) Note that

$$\begin{aligned} x \in \overline{\Theta}(I, S \cup T) &\Leftrightarrow I_x \cap (S \cup T) \neq \emptyset \\ &\Leftrightarrow (I_x \cap S) \cup (I_x \cap T) \neq \emptyset \\ &\Leftrightarrow I_x \cap S = \emptyset \textit{ or } I_x \cap T \neq \emptyset \\ &\Leftrightarrow x \in \overline{\Theta}(I, S) \text{ or } a \in \overline{\Theta}(I, T) \\ &\Leftrightarrow x \in \overline{\Theta}(I, S) \cup \overline{\Theta}(I, T). \end{aligned}$$

Thus, $\overline{\Theta}(I, S \cup T) = \overline{\Theta}(I, S) \cup \overline{\Theta}(I, T)$.

(4) Note that

$$\begin{aligned} x \in \underline{\Theta}(I, S \cap T) &\Leftrightarrow I_x \subseteq S \cap T \\ &\Leftrightarrow I_x \subseteq S \text{ and } I_x \subseteq T \\ &\Leftrightarrow x \in \underline{\Theta}(I, S) \text{ and } x \in \underline{\Theta}(I, T) \\ &\Leftrightarrow x \in \underline{\Theta}(I, S) \cap \underline{\Theta}(I, T). \end{aligned}$$

Thus, $\underline{\Theta}(I, S \cap T) = \underline{\Theta}(I, S) \cap \underline{\Theta}(I, T)$.

(5) Since $S \subseteq T$ if and only if $S \cap T = S$, it follows from (3) that

$$\underline{\Theta}(I, S) = \underline{\Theta}(I, S \cap T) = \underline{\Theta}(I, S) \cap \underline{\Theta}(I, T).$$

This implies that $\underline{\Theta}(I,S) \subseteq \underline{\Theta}(I,T)$. Note also that $S \subseteq T$ if and only if $S \cup T = T$, it follows from (2) that

$$\overline{\Theta}(I,T) = \overline{\Theta}(I,S \cup T) = \overline{\Theta}(I,S) \cup \overline{\Theta}(I,T).$$

This implies that $\overline{\Theta}(I,S) \subseteq \overline{\Theta}(I,T)$.

(6) Since $S \subseteq S \cup T$ and $T \subseteq S \cup T$, it follows from (4) that $\underline{\Theta}(I,S) \subseteq \underline{\Theta}(I,S \cup T)$ and $\underline{\Theta}(I,T) \subseteq \underline{\Theta}(I,S \cup T)$. This implies $\underline{\Theta}(I,S) \cup \underline{\Theta}(I,T) \subseteq \underline{\Theta}(I,S \cup T)$.

(7) Since $S \cap T \subseteq S$ and $S \cap T \subseteq T$, it follows from (4) that $\overline{\Theta}(I,S \cap T) \subseteq \overline{\Theta}(I,S)$ and $\overline{\Theta}(I,S \cap T) \subseteq \overline{\Theta}(I,T)$. This implies $\overline{\Theta}(I,S \cap T) \subseteq \overline{\Theta}(I,S) \cap \overline{\Theta}(I,T)$.

(8) Since $\Theta \subseteq \Psi$, if $x \in \Psi(J,S)$, then $J_x \subseteq S$. But $\Theta \subseteq \Psi$, then $I_x \subseteq J_x \subseteq S$, that is, $I_x \subseteq S$. Thus, $x \in \underline{\Theta}(I,S)$. Hence, $\underline{\Psi}(J,S) \subseteq \underline{\Theta}(I,S)$. Now let x be any element of $\overline{\Theta}(S)$. So $I_x \cap S = \emptyset$, there exists $y \in I_y \cap S$ such that $y \in I_y$ and $y \in S$. Hence, $(y,x) \in \Theta$, that is, $y \cdot x \in I$. Since $I \subseteq J$, it follows that $y \cdot x \in J$ and $x \cdot y \in J$ so that $(y,x) \in \Psi$, that is, $y \in J_x$. Therefore, $y \in J_x \cap S$, which means that $x \in \Psi(J,S)$.

Proposition 2. *Let Θ be an equivalence relation on A related to an ideal I of A. If S is a nonempty subset of A, then:*

(1) $\underline{\Theta}(I,\underline{\Theta}(I,S)) = \underline{\Theta}(I,S)$,

(2) $\overline{\Theta}(I,\overline{\Theta}(I,S)) = \overline{\Theta}(I,S)$,

(3) $\overline{\Theta}(I,\underline{\Theta}(I,S)) = \underline{\Theta}(I,S)$,

(4) $\underline{\Theta}(I,\overline{\Theta}(I,S)) = \overline{\Theta}(I,S)$,

(5) $\underline{\Theta}(I,S) = (\overline{\Theta}(I,S^c))^c$,

(6) $\overline{\Theta}(I,S) = (\underline{\Theta}(I,S^c))^c$,

(7) $\underline{\Theta}(I,I_x) = A = \overline{\Theta}(I,I_x)$, *for all* $x \in A$.

Proof. The proof is straightforward.

Proposition 3. *Let Θ be an equivalence relation on A related to an ideal I of A. If S is a nonempty subset of A, then:*

(1) $\overline{\Theta}(I,S) \cdot \overline{\Theta}(I,T) \subseteq \overline{\Theta}(I,S \cdot T)$,

(2) if Θ is a congruence relation on A, then $\underline{\Theta}(I,S) \cdot \underline{\Theta}(I,T) \subseteq \underline{\Theta}(I,S \cdot T)$.

Proof. (1) Let c be any element of $\overline{\Theta}(I,S) \cdot \overline{\Theta}(I,T)$. Then $c = p \cdot q$ with $p \in \overline{\Theta}(I,S)$ and $q \in \overline{\Theta}(I,T)$. So there exist elements $x, y \in S$ such that $x \in I_p \cap S$ and $y \in I_q \cap T$. Thus, $x \in I_p, y \in I_q, x \in S$, and $y \in T$. So $x \cdot y \in I_p \cdot I_q \subseteq I_{p \cdot q}$. On the other hand, since $x \cdot y \in S \cdot T$, we have $x \cdot y \in I_{p \cdot q} \cap (S \cdot T)$, and so $c = p \cdot q \in \overline{\Theta}(I,S \cdot T)$. Hence, $\overline{\Theta}(I,S) \cdot \overline{\Theta}(I,T) \subseteq \overline{\Theta}(I,S \cdot T)$.

(2) Assume that Θ is a congruence relation on A and let c be any element of $\underline{\Theta}(I,S)\cdot\underline{\Theta}(I,T)$. Then $c = p\cdot q$ with $p \in \underline{\Theta}(I,S)$ and $q \in \underline{\Theta}(I,T)$. It follows that $I_p \subseteq S$ and $I_q \subseteq T$. Since Θ is a congruence relation on A, we have $I_{p\cdot q} = I_p \cdot I_q \subseteq S\cdot T$. So $c = p\cdot q \in \underline{\Theta}(I, S\cdot T)$. Thus $\underline{\Theta}(I,S)\cdot\underline{\Theta}(I,T) \subseteq \underline{\Theta}(I,S\cdot T)$.

Proposition 4. *Let Θ and Ψ be equivalence relations on A related to ideals I and J of A, respectively. If S and T are nonempty subsets of A, then:*

(1) $\overline{\Theta\cap\Psi}(I\cap J,S) \subseteq \overline{\Theta}(I,S)\cap\overline{\Psi}(J,S)$,

(2) $\underline{\Theta\cap\Psi}(I\cap J,S) \supseteq \underline{\Theta}(I,S)\cap\underline{\Psi}(J,S)$.

Proof. (1) Note that $\Theta\cap\Psi$ is also a congruence relation on S. Let $c \in \overline{\Theta\cap\Psi}(I\cap J,S)$. Then $(I\cap J)_c \cap S = \emptyset$. Then there exists an element $x \in (I\cap J)_c \cap S$. Since $(x,c) \in \Theta\cap\Psi$, we have $(x,c)\in\Theta$ and $(x,c)\in\Psi$. Thus, $x \in I_c$ and $x \in J_c$. Since $x \in S$, we have $x \in I_c, x\in S$ and $x \in J_c$, $x \in S$. This implies that $x \in I_c\cap S$ and $x\in J_c \cap S$, so $I_c\cap S = \emptyset$ and $J_c \cap S = \emptyset$. So $c \in \overline{\Theta}(I,S)$ and $c \in \overline{\Psi}(J,S)$, hence $c \in \overline{\Theta}(I,S)\cap\overline{\Psi}(J,S)$. Thus, $\overline{\Theta\cap\Psi}(I\cap J,S) \subseteq \overline{\Theta}(I,S)\cap\overline{\Psi}(J,S)$.

(2) Since $\Theta\cap\Psi\subseteq\Theta$ and $\Theta\cap\Psi\subseteq\Psi$, we have $\underline{\Theta}(I,S)\subseteq\underline{\Theta\cap\Psi}(I\cap J,S)$ and $\underline{\Psi}(J,S)\subseteq\underline{\Theta\cap\Psi}(I\cap J,S)$. Hence, $\underline{\Theta}(I,S)\cap\underline{\Psi}(J,S)\subseteq\underline{\Theta\cap\Psi}(I\cap J,S)$.

Theorem 1. *Let (A,Θ) be an approximation space. Then*

(1) for every $S\subseteq A$, $\underline{\Theta}(I,S)$ and $\overline{\Theta}(I,S)$ are definable sets,

(2) for every $x\in A$, I_x is a definable set.

Proof. (1) By Proposition 1 (1) and (3), we have

$$\underline{\Theta}(I,\underline{\Theta}(I,S)) = \underline{\Theta}(I,S) = \overline{\Theta}(I,\underline{\Theta}(I,S)).$$

Hence, $\underline{\Theta}(I,S)$ is definable. On the other hand, by Proposition 1 (2) and (4), we have $\overline{\Theta}(I,\overline{\Theta}(I,S)) = \overline{\Theta}(I,S) = \underline{\Theta}(I,\overline{\Theta}(I,S))$. Therefore, $\overline{\Theta}(I,S)$ is a definable set.

(2) By Proposition 1 (7), the proof is clear.

Definition 8. *A nonempty subset S of A is called an upper (resp., a lower) rough subalgebra of A if the upper (resp., nonempty lower) approximation of S is a subalgebra of A. If S is both an upper and a lower rough subalgebra of A, we say that S is a rough subalgebra of A.*

Theorem 2. *Let Θ be a congruence relation on A related to an ideal I of A. If S is a subalgebra of I, then*

(1) $\overline{\Theta}(I,S)$ is a subalgebra of A,

(2) $\underline{\Theta}(I,S)$ is a subalgebra of A.

Proof. (1) Let $x, y \in \overline{\Theta}(I, S)$. Then $I_x \cap S \neq \emptyset$ and $I_y \cap S \neq \emptyset$, and so there exist $a, b \in S$ such that $a \in I_x$ and $b \in I_y$. It follows that $(a, x) \in \Theta$ and $(b, y) \in \Theta$. Since Θ is a congruence relation on A, we have $(a \cdot b, x \cdot y) \in \Theta$. Hence, $a \cdot b \in I_{x \cdot y}$. Since S is a subalgebra of A, we get $a \cdot b \in S$, and therefore, $a \cdot b \in I_{x \cdot y} \cap S$, that is, $I_{x \cdot y} \cap S \neq \emptyset$. This shows that $x \cdot y \in \overline{\Theta}(I, S)$, and consequently $\overline{\Theta}(I, S)$ is a subalgebra of A.

(2) Let $x, y \in \underline{\Theta}(I, S)$. Then $I_x \subseteq S$ and $I_y \subseteq S$. Since S is a subalgebra of A, we have $I_{x \cdot y} = I_x \cdot I_y \subseteq S$ so that $x \cdot y \in \underline{\Theta}(I, S)$. Hence, $\underline{\Theta}(I, S)$ is a subalgebra of A.

Remark 1. *The converse of Theorem 2 (1) may not be true. For the Hilbert algebra $A = (A, \cdot, 1)$ as in Example 1, the subset $\{x, y, z, 0\}$ is not a subalgebra of A, but $\overline{\Theta}(I, \{x, y, z, 0\}) = \{1, x, y, z, 0\}$ is a subalgebra of A.*

Definition 9. *A nonempty subset S of A is called an upper (resp., a lower) rough ideal of A if the upper (resp., nonempty lower) approximation of S is an ideal of A. If S is both an upper and a lower rough ideal of A, we say that S is a rough ideal of A.*

Theorem 3. *Let Θ be a congruence relation on A related to an ideal I of A. If S is an ideal of A containing I, then*

(1) $\overline{\Theta}(I, S)$ is an ideal of A,

(2) $\underline{\Theta}(I, S)$ is an ideal of A.

Proof. (1) Let S be an ideal of A containing I. Obviously, $1 \in \overline{\Theta}(I, S)$. Let $x, y \in A$ be such that $y \in \overline{\Theta}(I, S)$. Then $I_y \cap S \neq \emptyset$ and so there exists $a \in S$ such that $a \in I_y$. Hence, $(a, y) \in \Theta$, which implies $y \cdot a \in I \subseteq S$. Since $a \in S$ and S is an ideal of A, we get $y \in S$. Then $x \cdot y \in S$. Note that $x \cdot y \in I_{x \cdot y}$, thus $x \cdot y \in I_{x \cdot y} \cap S$, that is, $I_{x \cdot y} \cap S \neq \emptyset$. Hence, $x \cdot y \in \overline{\Theta}(I, S)$. Let $x, y_1, y_2 \in A$ be such that $y_1, y_2 \in \overline{\Theta}(I, S)$. Then $I_{y_1} \cap S \neq \emptyset$ and $I_{y_2} \cap S \neq \emptyset$ and so there exist $a, b \in S$ such that $a \in I_{y_1}$ and $b \in I_{y_2}$. Hence, $(a, y_1) \in \Theta$ and $(b, y_2) \in \Theta$, which implies $y_1 \cdot a \in I \subseteq S$ and $y_2 \cdot b \in I \subseteq S$. Since $a, b \in S$ and S is an ideal of A, we have $y_1 \in S$ and $y_2 \in S$ and $(y_1 \cdot (y_2 \cdot x)) \cdot x \in S$. Note that $(y_1 \cdot (y_2 \cdot x)) \cdot x \in I_{(y_1 \cdot (y_2 \cdot x)) \cdot x}$, thus $(y_1 \cdot (y_2 \cdot x)) \cdot x \in I_{(y_1 \cdot (y_2 \cdot x)) \cdot x} \cap S$, that is, $I_{(y_1 \cdot (y_2 \cdot x)) \cdot x} \cap S \neq \emptyset$. Hence, $(y_1 \cdot (y_2 \cdot x)) \cdot x \in \overline{\Theta}(I, S)$ and therefore, $\overline{\Theta}(I, S)$ is an ideal of A.

(2) Let S be an ideal of A containing I. Let $x \in I_1$. Then $x \in I \subseteq S$, and so $I_1 \subseteq S$. Hence, $1 \in \underline{\Theta}(I, S)$. Let $x, y \in X$ be such that $y \in \underline{\Theta}(I, S)$. Then $I_y \subseteq S$. Let $w \in I_{x \cdot y} = I_x \cdot I_y$. Then $w \in I_x \cdot I_y$ for some $a \in I_x$ and $c \in I_y$. From $a \in I_x$ and $c \in I_y$, we have $(a, x) \in \Theta$ and $(c, y) \in \Theta$. Taking $b \in I_y$, we get $(b, y) \in \Theta$. Since Θ is a congruence relation on S, we get $(a \cdot b, x \cdot y) \in \Theta$ and so $a \cdot b \in I_{x \cdot y} \subseteq S$. Since S is an ideal of A, we have $w = a \cdot c \in S$, so that $I_{x \cdot y} \subseteq S$. Hence, $x \cdot y \in \underline{\Theta}(I, S)$. Let $x, y_1, y_2 \in X$ be such that $y_1 \in \underline{\Theta}(I, S)$ and $y_2 \in \underline{\Theta}(I, S)$. Then $I_{y_1} \subseteq S$ and $I_{y_2} \subseteq S$. Let $w \in I_{(y_1 \cdot (y_2 \cdot x)) \cdot x} = (I_{y_1} \cdot (I_{y_2} \cdot I_x)) \cdot I_x$. Then $w \in (I_{y_1} \cdot (I_{y_2} \cdot I_x)) \cdot I_x$ for some $a \in I_{y_1}, b \in I_{y_2}$, and $c \in I_x$. From $a \in I_{y_1}, b \in I_{y_2}$, and $c \in I_x$, we have $(a, y_1) \in \Theta, (b, y_2) \in \Theta$, and $(c, x) \in \Theta$. Since Θ is a congruence relation on S, we get $((a \cdot (b \cdot c)) \cdot c, (y_1 \cdot (y_2 \cdot x)) \cdot x) \in \Theta$ and so $(a \cdot (b \cdot c)) \cdot c \in I_{(y_1 \cdot (y_2 \cdot x)) \cdot x} \subseteq S$. Since S is an ideal of A, we have $w = (a \cdot (b \cdot c)) \cdot c \in S$,

so that $I_{(y_1\cdot(y_2\cdot x))\cdot x} \subseteq S$. Hence, $(y_1 \cdot (y_2 \cdot x)) \cdot x \in \underline{\Theta}(I, S)$ and therefore, $\underline{\Theta}(I, S)$ is an ideal of A.

Remark 2. *The converse of Theorem 3 (1) may not be true. For the Hilbert algebra $A = (A, \cdot, 1)$ as in Example 1, the subset $\{1, x\}$ is an ideal of A, but $\{x\}$ is not an ideal of A. Also, $\overline{\Theta}(I, \{x\}) = \{1, x\}$ is an ideal of A.*

4. Conclusion

This study has explored the integration of RST with Hilbert algebras, establishing a novel framework for analyzing algebraic structures under uncertainty. By defining lower and upper approximations within Hilbert algebras, we have demonstrated that these approximations preserve subalgebra and ideal structures, thereby extending the applicability of RST to algebraic logic. The introduction of approximation spaces induced by ideals provides a systematic approach to dealing with incomplete or vague information in algebraic systems. Moreover, we have validated our theoretical findings through illustrative examples, reinforcing the practical significance of this approach. The results presented in this work not only enhance the theoretical foundations of RST but also open new pathways for applications in mathematical logic, fuzzy systems, and artificial intelligence. Future research may focus on further generalizations of rough approximations in broader algebraic settings or their potential applications in knowledge representation and uncertainty reasoning.

Acknowledgements

This research was supported by University of Phayao and Thailand Science Research and Innovation Fund (Fundamental Fund 2025, Grant No. 5027/2567).

References

[1] Z. Pawlak. Rough sets. *Int. J. Comput. Inf. Sci.*, 11(5):341–356, 1982.

[2] Z. Pawlak. *Rough sets-theoretical aspects of reasoning about data.* Kluwer Academic Publishing, Dordrecht, 1991.

[3] M. Quafafou. α-RST: a generalization of rough set theory. *Inf. Sci.*, 124(1-4):301–316, 2000.

[4] W.-Z. Wu and W.-X. Zhang. Neighborhood operator systems and approximations. *Inf. Sci.*, 144(1-4):201–217, 2002.

[5] Y. Y. Yao. Relational interpretations of neighborhood operators and rough set approximation operators. *Inf. Sci.*, 111(1-4):239–259, 1998.

[6] S. D. Comer. On connections between information systems, rough sets and algebraic logic. *Algebr. Methods Log. Comput. Sci.*, 28(1):117–124, 1993.

[7] L. Henkin. An algebraic characterization of quantifiers. *Fund. Math.*, 37:63–74, 1950.

[8] A. Diego. Sur les algébres de Hilbert. *Collection de Logique Math. Ser. A (Ed. Hermann, Paris)*, 21:1–52, 1966.

[9] D. Busneag. A note on deductive systems of a Hilbert algebra. *Kobe J. Math.*, 2:29–35, 1985.

[10] D. Busneag. Hilbert algebras of fractions and maximal Hilbert algebras of quotients. *Kobe J. Math.*, 5:161–172, 1988.

[11] Y. B. Jun. Deductive systems of Hilbert algebras. *Math. Japon.*, 43:51–54, 1996.

[12] W. A. Dudek. On fuzzification in Hilbert algebras. *Contrib. Gen. Algebra*, 11:77–83, 1999.

[13] A. Borumand Saeid and M. Haveshki. Approximation in Hilbert algebras. *Sci. Magna*, 4(4):77–85, 2008.

[14] Y. B. Jun, J.-W. Nam, and S. M. Hong. A note on Hilbert algebras. *East Asian Math. J.*, 10(2):279–285, 1994.

[15] I. Chajda and R. Halas. Congruences and ideals in Hilbert algebras. *Kyungpook Math. J.*, 39(2):429–432, 1999.