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Existence of Solutions for Fractional Order Differential Equations: Extended Results

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Abstract. This study investigates the existence of solutions for nonlinear fractional differential equations of order $q \in (1, 2]$. We establish new existence results for the boundary conditions $\xi(\kappa) = \alpha \neq 0$ and $\xi(\omega) = \beta \neq 0$ by incorporating an intermediate point, extending existing methodologies. Our results rely on fixed point theorems and the contraction principle, which provide a robust framework for analyzing these equations. We also provide several illustrative examples to demonstrate our results, showcasing their relevance in theoretical and applied contexts.

2020 Mathematics Subject Classifications: 26A33, 34A08, 33E30, 34A35, 34A34, 34K37

Key Words and Phrases: Fractional derivatives, differential equations, fractional differential equations, antiperiodic, nonlocal boundary conditions, existence

1. Introduction

In this research article, we study the existence of solution for the following:

$$\begin{cases} {}^cD^q\xi(\rho) = \Xi(\rho, \xi(\rho)), \quad 1 < q \leq 2, \rho \in [0, \omega] \\ \xi(\kappa) = \alpha, \quad \xi(\omega) = \beta, \quad 0 \leq \kappa < \omega, \quad 0 < \alpha < \beta \end{cases} \tag{1}$$

where, $\xi \in C([0, \omega], \mathbb{R})$ and $\Xi : [0, \omega] \times \mathbb{R} \longrightarrow \mathbb{R}$, and ${}^cD^q$ represents the Caputo fractional derivative of order $q \in (1, 2]$, by applying contraction principal and fixed point theorem of

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Krasnoselskii's .

Fractional differential equations (FDEs) extend the concepts of traditional calculus, offering mathematical frameworks to analyze systems where memory effects, ongoing influences, or irregular spatial characteristics present challenges for standard modeling techniques [1, 2]. The roots of fractional calculus are linked to the 17th century, as mathematicians such as Leibniz explored the concept of derivatives of varying orders. Nevertheless, in the 19th century, foundational formulations by Riemann, Liouville, and their successors established the basis for contemporary fractional calculus [3]. Nowadays, FDEs are a crucial link between theoretical constructs and practical applications [3]. For instance, engineers utilize these equations to forecast stress relaxation in viscoelastic materials, which "remember" previous deformations. Geophysicists model unusual diffusion patterns in fractured geological formations, where particles exhibit unpredictable movement [4]. In biology, researchers apply them to study cellular processes affected by delayed responses, while economists investigate their potential to predict market fluctuations driven by long-term patterns [5, 6]. Unlike traditional integer-order models, fractional approaches inherently consider historical context through operators such as the Caputo derivative, compatible with physical initial conditions, or the Riemann-Liouville integral, which emphasizes past states in varied ways [7].

The increasing integration of FDEs into research highlights a transformation in scientific understanding: Many natural and engineering systems are complex and influenced by memory. From climate dynamics shaped by years of greenhouse gas emissions to health treatments based on cumulative drug effects, fractional calculus offers valuable insights into navigating this complexity [1].

In the literature, the stability, existence, and uniqueness of the solution for fractional differential equations have been discussed widely with different methods due to the importance of the equations in practical applications [8–12]. In [13], Zhang found the expression of the solution, under the boundary conditions $\xi(0) = \alpha \neq 0$ and $\xi(1) = \beta \neq 0$, with the aid of Laplace transformation, by highlighting the role of Schauder's fixed-point theorem and Mittag-Leffler functions in solving these boundary value problems. Bashir and Nieto [13] gave some interesting results for fractional differential equations with anti-periodic boundary conditions, $\xi(0) = -\xi(0)$, and $\xi^{'}(\omega) = -\xi^{'}(\omega)$, by using Leray-Schauder degree theory. R. Agarwal, B. Ahmad, and J. Nieto in [14] introduce and solve fractional and sequential FDEs with parametric type conditions where they consider intermediate points using standard fixed point theorem. In [15], extended the existence and uniqueness results of [16] with nonlocal boundary conditions under essential conditions using the fixed-point theorem of Krasnoselskii and the contraction principle, the research broadens the scope of these equations, demonstrating applications to classical fractional differential equations.

Fractional derivatives can be defined in various ways, with notable formulations including the Grünwald–Letnikov, Liouville, Hadamard, Riesz, and Caputo derivatives. These definitions have been widely utilized to explore solutions, analyze system stability, and define and characterize various functional spaces. This study specifically focuses on the

Caputo fractional derivative due to its close relationship with classical differential equations and its proven effectiveness in addressing antiperiodic boundary value problems. The Caputo fractional derivative has gained significant attention because of its ability to model real-world phenomena involving memory and hereditary properties since it offers a solid framework for addressing boundary value problems.

This paper extends these results by introducing significantly broadening the existing solution frameworks. Introducing this novel intermediate condition facilitates a more diverse array of boundary behaviors and significantly enhances the applicability of various solution methodologies. Unlike prior works that focus primarily on boundary conditions at the endpoints or involve periodicity or anti-periodicity, the inclusion of an intermediate condition provides a more flexible and general approach to solving fractional differential equations. Moreover, our results offer new insight into the existence and uniqueness of solutions under more complex boundary scenarios, thereby expanding the scope of previous research. For more interesting results, see [17–20].

The paper is organized as follows: Section 2 presents the material and methods used in the research article, detailing the theoretical framework. This is followed by the results derived from the analysis, highlighting key findings and their implications. Section 4 provides an example to illustrate and validate the results. Finally, the last section presents the conclusion.

2. Materials and methods

Definition 1. *[1] We define the Caputo fractional derivative of order $q > 0$, denoted ${}^cD^q$, for a given function $\Psi \in C^k([0,\omega])$, is defined by:*

$$ {}^cD^q\Psi(\rho) = \frac{1}{\Gamma(k-q)} \int_0^\rho (\rho - \varpi)^{k-q-1}\Psi^{(k)}(\varpi)d\varpi, $$

where, $k = [q] + 1$.

Definition 2. *[1] The Riemann-Liouville fractional integral of order $q > 0$, denoted I^q, for a defined function $\Psi \in C([0,\omega])$, is defined by:*

$$ I^q\Psi(\rho) = \frac{1}{\Gamma(q)} \int_0^\rho (\rho - \varpi)^{q-1}\Psi(\varpi)d\varpi. $$

Lemma 1. *[1] For $q > 0$, the general solution of ${}^cD^q\xi(\rho) = 0$ is given by,*

$$ \xi(\rho) = a_0 + a_1\rho + a_2\rho^2 + ... + a_{k-1}\rho^{k-1}, $$

where, $a_i \in \mathbb{R}$, for $i = 1, 2$, and $k = [q] + 1$.

Lemma 2. *The unique solution of the following problem*

$$ \begin{cases} {}^cD^q\xi(\rho) = \delta(\rho), & 1 < q \le 2, \rho \in [0,\omega] \\ \xi(\kappa) = \alpha, \quad \xi(\omega) = \beta, & 0 \le \kappa < \omega, \quad 0 < \alpha < \beta, \end{cases} \tag{2} $$

is given by:

$$\xi(\rho) = \frac{\alpha\omega - \beta\kappa}{\omega - \kappa} + \frac{\beta - \alpha}{\omega - \kappa}\,\rho + \frac{1}{\Gamma(q)}\int_0^\rho (\rho - \varpi)^{q-1}\delta(\varpi)\,d\varpi \\ - \frac{\rho - \kappa}{\Gamma(q)(\omega - \kappa)}\left[\int_0^\omega (\omega - \varpi)^{q-1}\delta(\varpi)\,d\varpi\right] + \frac{\rho - \omega}{\Gamma(q)(\omega - \kappa)}\left[\int_0^\kappa (\kappa - \varpi)^{q-1}\delta(\varpi)\,d\varpi\right]. \tag{3}$$

Proof. By Lemma 1, the solution of (2) is given by $\xi(\rho) = I^q\delta(\rho) - a_0 - a_1\rho$ where, $a_i \in \mathbb{R}$, for $i = 1, 2$.
By applying the boundary conditions we get,

$$a_0 = \frac{\beta\kappa - \alpha\omega}{\omega - \kappa} - \frac{1}{\Gamma(q)(\omega - \kappa)}\left[\kappa\int_0^\omega (\omega - \varpi)^{q-1}\delta(\varpi)d\varpi - \omega\int_0^\kappa (\kappa - \varpi)^{q-1}\delta(\varpi)\right]$$

$$a_1 = \frac{\alpha - \beta}{\omega - \kappa} + \frac{1}{\Gamma(q)(\omega - \kappa)}\left[\int_0^\omega (\omega - \varpi)^{q-1}\delta(\varpi)d\varpi - \int_0^\kappa (\kappa - \varpi)^{q-1}\delta(\varpi)d\varpi\right]$$

by using the values of a_0, a_1 we get

$$\xi(\rho) = \frac{\alpha\omega - \beta\kappa}{\omega - \kappa} + \frac{\beta - \alpha}{\omega - \kappa}\,\rho + \frac{1}{\Gamma(q)}\int_0^\rho (\rho - \varpi)^{q-1}\delta(\varpi)\,d\varpi \\ - \frac{\rho - \kappa}{\Gamma(q)(\omega - \kappa)}\left[\int_0^\omega (\omega - \varpi)^{q-1}\delta(\varpi)\,d\varpi\right] + \frac{\rho - \omega}{\Gamma(q)(\omega - \kappa)}\left[\int_0^\kappa (\kappa - \varpi)^{q-1}\delta(\varpi)\,d\varpi\right],$$

completing the proof.

Remark 1. *Note that as $\kappa \longrightarrow 0^+$ and $\omega = 1$ in 3 we will get the integral solution in [13]. Meaning that the results in this paper extend and generalize the results in [13].*

Theorem 1. *[21] Let $\mathcal{C}$ be a complete nonempty metric space into itself, Then every contraction mapping on $\mathcal{C}$ has a unique fixed point in $\mathcal{C}$.*

Theorem 2. *[21] Let $\mathcal{B}$ be a Banach space, and ϕ be convex nonempty closed subset of $\mathcal{B}$, suppose that $\vartheta_i : \phi \longrightarrow \mathcal{B}$ for $i = 1, 2$ and $\vartheta_1\xi_1 + \vartheta_2\xi_2 \in \phi$ for all $\xi_1, \xi_2 \in \phi$, ϑ_1 is continuous and compact, ϑ_2 is a contraction mapping. Then there exists a $\xi \in \phi$ such that $\vartheta_1\ \xi + \vartheta_2\ \xi = \xi$.*

3. Results

Let $\mathcal{Z} = C([0, \omega], \mathbb{R})$ denotes the Banach space equipped with the norm $||\xi|| = \sup|\xi(\rho)|, \forall\rho \in [0, \omega]$.
Define the operator $\mathcal{L} : \mathcal{Z} \longrightarrow \mathcal{Z}$ as

$$(\mathcal{L}\xi)(\rho) = \frac{\alpha\omega - \beta\kappa}{\omega - \kappa} + \frac{\beta - \alpha}{\omega - \kappa}\,\rho + \frac{1}{\Gamma(q)}\int_0^\rho (\rho - \varpi)^{q-1}\Xi(\varpi, \xi(\varpi))\,d\varpi$$

$$-\frac{\rho-\kappa}{\Gamma(q)(\omega-\kappa)}\left[\int_0^{\omega}(\omega-\varpi)^{q-1}\Xi(\varpi,\xi(\varpi))\,d\varpi\right]+\frac{\rho-\omega}{\Gamma(q)(\omega-\kappa)}\left[\int_0^{\kappa}(\kappa-\varpi)^{q-1}\Xi(\varpi,\xi(\varpi))\,d\varpi\right]. \tag{4}$$

Lemma 3. *Assume Ξ is continuous function then $\mathcal{L}$ is a completely compact operator.*

Proof. Define the operator $\mathcal{L}$ as in (4):

Since Ξ is continuous and $\mathcal{L}$ is continuous. Assume M$=\max_{\rho\in[0,\omega]}|\Xi(\rho,\xi(\rho))|$. Then, for $\xi\in\mathcal{B}=\{\xi\in C([0,\omega]);||\xi||<m\}$ we have,

$$|(\mathcal{L}\xi)(\rho)|\leq\left|\frac{\alpha\omega-\beta\kappa}{\omega-\kappa}+\frac{\beta-\alpha}{\omega-\kappa}\rho\right|+\left|\frac{M}{\Gamma(q)}\int_0^{\rho}(\rho-\varpi)^{q-1}\,d\varpi\right|+\left|\frac{M(\rho-\kappa)}{\Gamma(q)(\omega-\kappa)}\int_0^{\omega}(\omega-\varpi)^{q-1}\,d\varpi\right|$$
$$+\left|\frac{M(\rho-\omega)}{\Gamma(q)(\omega-\kappa)}\int_0^{\kappa}(\kappa-\varpi)^{q-1}\,d\varpi\right|$$
$$\leq\left|\frac{\alpha\omega-\beta\kappa+(\beta-\alpha)\rho}{\omega-\kappa}\right|+\left|\frac{M\rho^q}{\Gamma(q+1)}\right|+\left|\frac{M(\rho-\kappa)\omega^q}{\Gamma(q+1)(\omega-\kappa)}\right|+\left|\frac{M(\rho-\omega)\kappa^q}{\Gamma(q+1)(\omega-\kappa)}\right|\leq\beta+\frac{2M\omega^q}{\Gamma(q+1)}=M_1$$

Therefore, $\mathcal{L}$ is bounded. Now proving equicontinuity of $\overline{\mathcal{L}(M)}$

For all $\xi\in\mathcal{B}$, $\forall\epsilon>0,\rho_1<\rho_2\in[0,\omega]$, choose $\rho_2-\rho_1<\nu<\left\{\frac{\epsilon(\omega-\kappa)}{3(\beta-\alpha)},\frac{\epsilon\Gamma(q)}{3M(\omega^q+\kappa^q)},(\frac{\epsilon\Gamma(q+1)}{6M})^{\frac{1}{q-1}}\right\}$. Then, we have

$$|\mathcal{L}\xi(\rho_2)-\mathcal{L}\xi(\rho_1)|=\left|\frac{\beta-\alpha}{\omega-\kappa}(\rho_2-\rho_1)+I^{q-1}I^1\Xi(\rho_2,\xi(\rho_2))-I^{q-1}I^1\Xi(\rho_1,\xi(\rho_1))\right.$$
$$\left.-\frac{\rho_2-\rho_1}{\Gamma(q)(\omega-\kappa)}\left[\int_0^{\omega}(\omega-\varpi)^{q-1}\Xi(\varpi,\xi(\varpi))\,d\varpi\right]+\frac{\rho_2-\rho_1}{\Gamma(q)(\omega-\kappa)}\left[\int_0^{\kappa}(\kappa-\varpi)^{q-1}\Xi(\varpi,\xi(\varpi))\,d\varpi\right]\right|$$
$$\leq\left|\frac{\beta-\alpha}{\omega-\kappa}\right|(\rho_2-\rho_1)+\frac{M}{\Gamma(q-1)}\int_0^{\rho_1}((\rho_1-\varpi)^{q-2}-(\rho_2-\varpi)^{q-2})d\varpi$$
$$+\frac{M}{\Gamma(q-1)}\int_{\rho_1}^{\rho_2}(\rho_2-\varpi)^{q-2}d\varpi+\frac{M(\rho_2-\rho_1)}{\Gamma(q)(\omega-\kappa)}\left[\int_0^{\omega}(\omega-\varpi)^{q-1}\,d\varpi\right]+\frac{M(\rho_2-\rho_1)}{\Gamma(q)(\omega-\kappa)}\left[\int_0^{\kappa}(\kappa-\varpi)^{q-1}\,d\varpi\right]$$
$$\leq\frac{\beta-\alpha}{\omega-\kappa}(\rho_2-\rho_1)+\frac{M}{\Gamma(q)}(\rho_2^{q-1}+2(\rho_2-\rho_1)^{q-1}-\rho_2^{q-1})+\frac{M(\omega^q+\kappa^q)}{\Gamma(q+1)}(\rho_2-\rho_1)$$
$$\leq\frac{\beta-\alpha}{\omega-\kappa}\nu+\frac{2M}{\Gamma(q)}\nu^{q-1}+\frac{M(\omega^q+\kappa^q)}{\Gamma(q+1)}\nu<\frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}=\epsilon.$$

Therefore, we proved that $\mathcal{L}$ is equicontinuous. Hence, the operator $\mathcal{L}$ is completely continuous by Arzela-Ascoli theorem [22].

Theorem 3. *Suppose* $\Xi : [0,\omega] \times \mathbb{R} \longrightarrow \mathbb{R}$ *is a continuous function satisfying* $\big|\Xi(\rho,\xi_1) - \Xi(\rho,\xi_2)\big| \le L\big|\xi_1 - \xi_2\big|$*, for all* $\rho \in [0,\omega]$*,* $L > 0$*, and* $\big|\Xi(\rho,\xi(\rho))\big| \le \big|\psi(\rho)\big|$*, for all* $(\rho,\xi) \in [0,\omega] \times \mathbb{R}$*, and* $\psi \in L_1([0,\omega],\mathbb{R}^+)$*. Then (1) has at least one solution on* $[0,\omega]$ *if* $\frac{L\omega^q}{\Gamma(q+1)} < 1$.

Proof. Define $\sup_{(\rho,\xi)\in[0,\omega]\times\mathcal{B}_r} ||\Xi(\rho,\xi)|| = \Xi_{max} < \infty$, and let the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ defined as follows:

$$(\mathcal{L}_1\xi)(\rho) = \frac{1}{\Gamma(q)}\int_0^\rho (\rho-\varpi)^{q-1}\Xi(\varpi,\xi(\varpi))d\varpi + \frac{\alpha\omega-\beta\kappa}{\omega-\kappa} + \frac{\beta-\alpha}{\omega-\kappa}\,\rho,$$

$$(\mathcal{L}_2\xi)(\rho) = -\frac{\rho-\kappa}{\Gamma(q)(\omega-\kappa)}\left[\int_0^\omega (\omega-\varpi)^{q-1}\Xi(\varpi,\xi(\varpi))\,d\varpi\right] + \frac{\rho-\omega}{\Gamma(q)(\omega-\kappa)}\left[\int_0^\kappa (\kappa-\varpi)^{q-1}\Xi(\varpi,\xi(\varpi))\,d\varpi\right].$$

Also, define a ball $\mathcal{B}_r = \{\xi \in C([0,\omega],\mathbb{R}) : ||\xi|| \le r\}$, such that $r > \left\{2\beta, \frac{4||\psi||\omega^q}{\Gamma(q+1)}\right\}$

Then for $\xi_1,\xi_2 \in \mathcal{B}_r$,

$$\begin{aligned}
\big|\big|\mathcal{L}_1\xi_1 + \mathcal{L}_2\xi_2\big|\big| &\le \Bigg|\frac{1}{\Gamma(q)}\int_0^\rho (\rho-\varpi)^{q-1}\Xi(\varpi,\xi_1(\varpi))d\varpi + \frac{\alpha\omega-\beta\kappa}{\omega-\kappa} + \frac{\beta-\alpha}{\omega-\kappa}\,\rho \\
&- \frac{\rho-\kappa}{\Gamma(q)(\omega-\kappa)}\left[\int_0^\omega (\omega-\varpi)^{q-1}\Xi(\varpi,\xi_2(\varpi))\,d\varpi\right] + \frac{\rho-\omega}{\Gamma(q)(\omega-\kappa)}\left[\int_0^\kappa (\kappa-\varpi)^{q-1}\Xi(\varpi,\xi_2(\varpi))\,d\varpi\right]\Bigg| \\
&\le \frac{||\psi||}{\Gamma(q)}\left[\int_0^\rho (\rho-\varpi)^{q-1}d\varpi + \left|\frac{\rho-\kappa}{\omega-\kappa}\Big[\int_0^\omega (\omega-\varpi)^{q-1}d\varpi\Big]\right| + \left|\frac{\rho-\omega}{\omega-\kappa}\Big[\int_0^\kappa (\kappa-\varpi)^{q-1}d\varpi\Big]\right|\right] + \beta \\
&\le \frac{||\psi||}{\Gamma(q+1)}\Big[2\omega^q\Big] + \beta < \frac{r}{2} + \frac{r}{2} = r
\end{aligned}$$

Therefore, $\mathcal{L}_1\xi_1 + \mathcal{L}_2\xi_2 \in \mathcal{B}_r$. Also, since Ξ is continuous, $\mathcal{L}_1$ is also continuous and uniformly bounded as $||\mathcal{L}_1\xi|| \le \frac{\omega^q||\psi||}{\Gamma(q+1)} + \beta$.

Moreover, for $\rho_1,\rho_2 \in [0,\omega]$,we see that

$$\begin{aligned}
\Big|\Big|(\mathcal{L}_1\xi)(\rho_1) - (\mathcal{L}_1\xi)(\rho_2)\Big|\Big| &\le \frac{1}{\Gamma(q)}\Big|\Big|\int_0^{\rho_1}\big[(\rho_1-\varpi)^{q-1} - (\rho_2-\varpi)^{q-1}\big]\Xi(\varpi,\xi(\varpi))d\varpi \\
&+ \int_{\rho_1}^{\rho_2}(\rho_2-\varpi)^{q-1}\Xi(\varpi,\xi(\varpi))d\varpi\Big|\Big| + ||\frac{\beta-\alpha}{\omega-\kappa}(\rho_1-\rho_2)|| \\
&\le \frac{\Xi_{max}}{\Gamma(\xi+1)}[2(\rho_2-\rho_1)^q + \rho_1^q - \rho_2^q] + ||\frac{\beta-\alpha}{\omega-\kappa}(\rho_1-\rho_2)||
\end{aligned}$$

as $\rho_2 \longrightarrow \rho_1$ we see that $\Big|\Big|(\mathcal{L}_1\xi)(\rho_1) - (\mathcal{L}_1\xi)(\rho_2)\Big|\Big| \longrightarrow 0$, proving that $\mathcal{L}_1$ is uniformly bounded and relatively compact on $\mathcal{B}_r$. Therefore, $\mathcal{L}_1$ is compact. Finally, $\mathcal{L}_2$ is a contraction by assumption since $\frac{L\omega^q}{\Gamma(q+1)} < 1$. Therefore, Theorem (2) guarantee that (1) has at least on solution on $[0,\omega]$.

Theorem 4. *For a continuous function* $\Xi : [0,\omega]\times\mathbb{R} \longrightarrow \mathbb{R}$*, assume* $\big|\big|\Xi(\rho,\xi_1)-\Xi(\rho,\xi_2)\big|\big| \leq L\big|\big|\xi_1-\xi_2\big|\big|$ *holds for all* $\rho \in [0,\omega]$*,* $L > 0$*,* ξ_1,ξ_2*, and* $\frac{2L\omega^q}{\Gamma(q+1)} < 1$*. Then (1) has a unique solution on* $[0,\omega]$*.*

Proof. Let $\mathcal{L}$ be an operator defined as in 4

$$(\mathcal{L}\xi)(\rho) = \frac{\alpha\omega-\beta\kappa}{\omega-\kappa} + \frac{\beta-\alpha}{\omega-\kappa}\,\rho + \frac{1}{\Gamma(q)}\int_0^{\rho}(\rho-\varpi)^{q-1}\Xi(\varpi,\xi(\varpi))\,d\varpi$$
$$-\frac{\rho-\kappa}{\Gamma(q)(\omega-\kappa)}\Bigg[\int_0^{\omega}(\omega-\varpi)^{q-1}\Xi(\varpi,\xi(\varpi))\,d\varpi\Bigg] + \frac{\rho-\omega}{\Gamma(q)(\omega-\kappa)}\Bigg[\int_0^{\kappa}(\kappa-\varpi)^{q-1}\Xi(\varpi,\xi(\varpi))\,d\varpi\Bigg],$$

and for each $\rho \in [0,\omega]$ and any $\xi_1,\xi_2 \in C([0,\omega])$ we have

$$\big|\big|\mathcal{L}\xi_1-\mathcal{L}\xi_2\big|\big| = \Bigg|\Bigg|\frac{1}{\Gamma(q)}\int_0^{\rho}(\rho-\varpi)^{q-1}\big[\Xi(\varpi,\xi_1(\varpi))-\Xi(\varpi,\xi_2(\varpi))\big]\,d\varpi$$
$$-\frac{\rho-\kappa}{\Gamma(q)(\omega-\kappa)}\Bigg[\int_0^{\omega}(\omega-\varpi)^{q-1}\big[\Xi(\varpi,\xi_1(\varpi))-\Xi(\varpi,\xi_2(\varpi))\big]\,d\varpi\Bigg]$$
$$+\frac{\rho-\omega}{\Gamma(q)(\omega-\kappa)}\Bigg[\int_0^{\kappa}(\kappa-\varpi)^{q-1}\big[\Xi(\varpi,\xi_1(\varpi))-\Xi(\varpi,\xi_2(\varpi))\,d\varpi\Bigg]\Bigg|\Bigg|$$
$$\leq \frac{L}{\Gamma(q)}\Bigg[\int_0^{\rho}(\rho-\varpi)^{q-1}d\varpi+\frac{|\rho-\kappa|}{\omega-\kappa}\Big[\int_0^{\omega}(\omega-\varpi)^{q-1}d\varpi\Big]+\frac{|\rho-\omega|}{\omega-\kappa}\int_0^{\kappa}(\kappa-\varpi)^{q-1}d\varpi\Bigg]||\xi_1-\xi_2||$$
$$\leq \frac{L}{\Gamma(q+1)}\Bigg[2\omega^q\Bigg]||\xi_1-\xi_2|| = \frac{2L\omega^q}{\Gamma(q+1)}\Bigg[||\xi_1-\xi_2||.$$

Thus, $\frac{2L\omega^q}{\Gamma(q+1)} < 1$, meaning that $\mathcal{L}$ is a contraction depending on L,q,ω,κ.
Now setting $\sup_{\rho\in[0,\omega]}\big|\Xi(\rho,0)\big| = \mathcal{N}$, and choosing $\mathcal{B}_r = \{\xi \in C([0,\omega],\mathbb{R}) : ||\xi|| \leq r\}$ be a ball with a radius $r \geq \frac{\beta+\mathcal{N}d}{1-Ld}$ where $d = \frac{2\omega^q}{\Gamma(q+1)}$ so we have

$$||(\mathcal{L}\xi)(\rho)|| \leq \max_{\rho\in[0,\omega]}\Bigg[\Bigg|\frac{\alpha\omega-\beta\kappa}{\omega-\kappa}+\frac{\beta-\alpha}{\omega-\kappa}\,\rho\Bigg| + \frac{1}{\Gamma(q)}\int_0^{\rho}(\rho-\varpi)^{q-1}\Bigg[\Bigg|\Xi(\varpi,\xi(\varpi))-\Xi(\varpi,0))\Bigg| + \Bigg|\Xi(\varpi,0))\Bigg|\Bigg]\,d\varpi$$
$$+\frac{|\rho-\kappa|}{\Gamma(q)(\omega-\kappa)}\Bigg[\int_0^{\omega}(\omega-\varpi)^{q-1}\Bigg[\Bigg|\Xi(\varpi,\xi(\varpi))-\Xi(\varpi,0))\Bigg| + \Bigg|\Xi(\varpi,0))\Bigg|\Bigg]\,d\varpi\Bigg]$$
$$+\frac{|\rho-\omega|}{\Gamma(q)(\omega-\kappa)}\int_0^{\kappa}(\kappa-\varpi)^{q-1}\Bigg[\Bigg|\Xi(\varpi,\xi(\varpi))-\Xi(\varpi,0))\Bigg| + \Bigg|\Xi(\varpi,0))\Bigg|\Bigg]\,d\varpi\Bigg]$$
$$\leq \beta + (Lr+\mathcal{N})\Big[\frac{2\omega^q}{\Gamma(q+1)}\Big] < r$$

Implying that $\mathcal{L}\mathcal{B}_r \subset \mathcal{B}_r$. Hence we proved the uniqueness for (1).

Remark 2. *By using κ in the boundary condition, we allow the possibility of intermediate boundary conditions that are more applicable in many practical scenarios. Instead of assuming that the system's behavior at the endpoints dictates the solution, this formulation allows the solution to be influenced by conditions at an interior point κ. Moreover, flexibility in dealing with nonlocal or nonlinear systems, where the condition is at some intermediate point (rather than at the boundaries), could be crucial for the system's evolution, providing accuracy when placing boundary conditions where they are most relevant.*

4. Example

Fractional differential equations play a major rule in many models such as viscoelastic material models where the displacement $\xi(\rho)$ is influenced by both local elasticity and nonlocal effects. Moreover, some system exhibits typical viscoelastic behavior, where the elastic term dominates at higher frequencies, and the memory (nonlocal) effects become more significant at lower frequencies or for longer times. The next examples show how fractional derivatives can be used to describe systems with memory or delayed response.

Example 1.

$$\begin{cases} {}^cD^{\frac{1}{2}}\xi(\rho) = \frac{\Gamma(q+1)}{10}\frac{|\xi|}{1+|\xi|} + \rho^{\nu}, \;\; \rho \in [0,1], \;\; \nu > 0 \\ \xi(0) = \alpha \neq 0, \quad \xi(1) = \beta \neq 0, \quad 0 < \alpha < \beta \end{cases} \tag{5}$$

Note that, $\omega = 1$, $q = \frac{1}{2}$, $\Xi(\rho, \xi(\rho)) = \frac{\Gamma(q+1)}{10}\frac{|\xi|}{1+|\xi|} + \rho^{\nu}$, and $\left|\Xi(\rho,\xi_1) - \Xi(\rho,\xi_2)\right| \leq \frac{\sqrt{\pi}}{20}|\xi_1 - \xi_2|$, where $L = \frac{\sqrt{\pi}}{20}$.
Also, $\frac{2L\omega^q}{\Gamma(q+1)} = \frac{1}{5} < 1$. Therefore, Theorem (4) guarantee that (5) has a unique solution in $[0,1]$.

Example 2.

$$\begin{cases} {}^cD^{\frac{1}{2}}\xi(\rho) = \frac{cos(\rho)}{7}\xi(\rho), \;\; \rho \in [0,1], \\ \xi(\frac{1}{2}) = \alpha \neq 0, \quad \xi(\frac{1}{2}) = \beta \neq 0, \quad 0 < \alpha < \beta, \end{cases} \tag{6}$$

Note that, $\omega = 1$, $q = \frac{1}{2}$, $\Xi(\rho, \xi(\rho)) = \frac{cos(\rho)}{7}\xi(\rho)$, $\left|\Xi(\rho,\xi_1) - \Xi(\rho,\xi_2)\right| \leq \frac{1}{7}|\xi_1 - \xi_2|$, where $L = \frac{1}{7}$, and $\left|\Xi(\rho,\xi(\rho))\right| \leq \frac{1}{7}\left|\xi(\rho)\right|$.
Also, $\frac{L\omega^q}{\Gamma(q+1)} = \frac{2}{7\sqrt{\pi}} \approx 0.161228 < 1$. Therefore, Theorem (3) guarantee that (6) has at least one solution in $[0,1]$.

5. Conclusions

In this paper, we investigate a boundary value problem of an intermediate point. It has been shown that including additional terms in the integral solutions significantly affects the behavior of the fractional-order problems under consideration. Furthermore, the results presented here are adaptable, particularly in scenarios where there is a shift in the location of the boundary phenomena near the left endpoint of the interval $[0, \omega]$ with $\kappa < \omega$. Notably, it is demonstrated that the classical boundary conditions in [13] can be derived from our results as κ approaches 0^+. Additionally, the results presented in Section 3, which focus on fractional differential equations in the limit $\kappa \to 0^+$, are novel contributions to the field. Ultimately, the nonlocal characteristics of the classical boundary conditions allow the boundary phenomena to occur at any intermediate position within the specified interval.

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References

[1] I. Podlubny. *Fractional Differential Equations*, volume 198 of *Mathematics in Science and Engineering*. Elsevier, San Diego, 1999.

[2] S. Abbas, M. Benchohra, and G. M. N'Guerekata. *Topics in Fractional Differential Equations*. Springer, New York, 2012.

[3] K. Oldham and J. Spanier. *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*. Elsevier, New York, 1974.

[4] R. Metzler and J. Klafter. The random walk's guide to anomalous diffusion: a fractional dynamics approach. *Physics Reports*, 339(1):1–77, 2000.

[5] M. Richard. Fractional calculus in bioengineering. *Critical Reviews in Biomedical Engineering*, 32(1):1–92, 2004.

[6] H. Ming, J. Wang, and M. Feckan. The application of fractional calculus in Chinese economic growth models. *Mathematics*, 7(8):665, 2019.

[7] S. Samko, A. Kilbas, and O. Marichev. *Fractional Integrals and Derivatives: Theory and Applications*. Gordon and Breach, Yverdon, 1993.

[8] S. Aljurbua. Generalized existence results for solutions of nonlinear fractional differential equations with nonlocal boundary conditions. *Ain Shams Engineering Journal*, 15(11):103035, 2024.

[9] H. Hasanen and S. Aljurbua. Solving fractional random differential equations by using fixed point methodologies under mild boundary conditions. *Fractal and Fractional*, 8(7):384, 2024.

[10] S. Aljurbua. Extended existence results of solutions for FDEs of order $1 < \gamma \leq 2$. *AIMS Mathematics*, 9(6):13077–13086, 2024.

[11] S. Aljurbua, H. Hammad, and N. Almutairi. Existence of solutions to a new class of fractional differential equations with antiperiodic boundary conditions. *European Journal of Pure and Applied Mathematics*, 18(1):5671, 2025.

[12] U. Tshering, E. Thailert, and S. Ntouyas. Existence and stability results for a coupled system of Hilfer-Hadamard sequential fractional differential equations with multi-point fractional integral boundary conditions. *AIMS Mathematics*, 9(10):25849–25878, 2024.

[13] Z. Shuqin. Existence of solution for a boundary value problem of fractional order. *Acta Mathematica Scientia*, 26(2):220–228, 2006.

[14] R. P. Agarwal, B. Ahmad, and J. J. Nieto. Fractional differential equations with nonlocal (parametric type) anti-periodic boundary conditions. *Filomat*, 31(5):1207–1214, 2017.

[15] S. Aljurbua. Extended existence results for FDEs with nonlocal conditions. *AIMS Mathematics*, 9(4):9049–9058, 2024.

[16] B. Ahmad and J. J. Nieto. Extended existence results for FDEs with nonlocal conditions. *Topological Methods in Nonlinear Analysis*, 35(2):295–304, 2010.

[17] M. Alaroud, H. Aljarrah, K. Alomari, A. Ishak, and M. Darus. Explicit and approximate series solutions for nonlinear fractional wave-like differential equations with variable coefficients. *Partial Differential Equations in Applied Mathematics*, 10:100680, 2024.

[18] H. Aljarrah, M. Alaroud, K. Alomari, A. Ishak, M. Darus, and S. Momani. Exact and approximate solutions of heat fractional differential equation using Laplace residual power series method. In *2023 International Conference on Fractional Differentiation and Its Applications (ICFDA)*, pages 1–5, Ajman, United Arab Emirates, 2023. IEEE.

[19] S. Khirsariya and S. Rao. Solution of fractional Sawada–Kotera–Ito equation using Caputo and Atangana–Baleanu derivatives. *Mathematical Methods in the Applied Sciences*, 46(15):16072–16091, 2023.

[20] S. Khirsariya, S. Rao, and J. Chauhan. A novel hybrid technique to obtain the solution of generalized fractional-order differential equations. *Mathematics and Computers in Simulation*, 205:272–290, 2023.

[21] D. R. Smart. *Fixed Point Theorems.* Cambridge University Press, Cambridge, 1980.

[22] Y. Zhou. *Fractional Evolution Equations and Inclusions: Analysis and Control.* Academic Press, London, 2016.