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Novel Exact Solutions for a Biological Population Model Using the Power Index Method

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Abstract. In this paper, we study a nonlinear biological population model that describes the spatiotemporal evolution of population density, incorporating nonlinear diffusion and reaction effects. Using the Power Index Method, we derive exact solutions for this model. In this approach, we select appropriate indexes for the independent variable in the similarity transformation, allowing the unknown functions to take polynomial, rational, or other elementary forms. These transformations reduce the nonlinear partial differential equation (NLPDE) to nonlinear ordinary differential equations (NLODEs). We then solve the nonlinear ordinary differential equations (NLODEs) exactly using Maple. Finally, by applying the similarity transformations and the exact solutions of the nonlinear ordinary differential equations (NLODEs), we obtain the exact solutions of the nonlinear biological population model. The behavior of the solutions is illustrated through 3D graphs. The results demonstrate that the proposed method is straightforward, powerful, and capable of yielding exact solutions, revealing biological phenomena such as nonlinear diffusion, wave propagation, and decay effects.

2020 Mathematics Subject Classifications: 35Qxx, 26A33

Key Words and Phrases: Partial Differential Equation, Ordinary Differential Equation, Exact Solution, Power Index Method, Biological Population Model

1. Introduction

Nonlinear partial differential equations (NLPDEs) play a fundamental role in modeling complex phenomena across various scientific and engineering disciplines, including hydrodynamics [1], [2], fluid dynamics, plasma physics [3], [4], nonlinear optics [5], [6], [7], [8], and other fields [9], [10], [11]. The study of exact solutions to nonlinear partial differential equations (NLPDEs) remains highly relevant, as it provides deeper insights into nonlinear wave phenomena and their inherent characteristics.

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This article examines nonlinear degenerate parabolic equations arising in the spatial diffusion of biological populations. We consider the following first-order time-dependent nonlinear biological population model:

$$u_t - 2u_x^2 - 2uu_{xx} - 2u_y^2 - 2uu_{yy} - u = 0. \tag{1}$$

In Equation (1), $u(x, y, t)$ represents the population density, where x, y, and t are the independent variables. The term u_t denotes the population growth rate, while the terms $-2u_x^2$ and $-2u_y^2$ describe population movement influenced by local density gradients, potentially modeling phenomena like density-dependent dispersal. The nonlinear diffusion terms $-2uu_{xx}$ and $-2uu_{yy}$ indicate that the rate of population spread depends on the local population density. Finally, the term $-u$ represents a linear decay in population density.

Many researchers have analytically and numerically explored biological population models, extensively studying exact solutions to these systems. Bekir and Gunner [12] investigated a nonlinear fractional biological population model using the (G'/G)-expansion method, obtaining hyperbolic and trigonometric function solutions. Hassan and Mohyud-Din [13] employed the exp-function method to derive generalized exact solutions for fractional biological population models. Shakeri and Dehghan [14] numerically solved the biological population model using He's variational iteration method and Adomian's decomposition method. Bushnaq et al. [15] applied the optimal homotopy asymptotic method to derive exact solutions for the nonlinear fractional-order biological population model. Muhammad Shakeel et al. [16] obtained closed-form solutions via the modified exp-function method, while Mohammad Ali Zirak and Abbas Poya [17] developed analytical solutions for fractional biological population differential equations using reconstructed variational iteration. Khater et al. [18] conducted a comprehensive computational and numerical study of nonlinear biological population models, demonstrating the efficacy of advanced numerical techniques in yielding accurate solutions. Wang et al. [19] investigated a generalized nonlinear evolution equation relevant to shallow water waves, employing the Hirota method with a new ansatz function. Their approach yielded novel exact solutions, including singular and non-singular complexitons, expanding the understanding of this equation's solution space. Wang [20] explored the perturbed Chen-Lee-Liu equation, a key model in nonlinear optics, obtaining diverse optical soliton and other wave solutions, contributing significant insights into the equation's dynamics. Wang et al. [21] studied the (2+1)-dimensional Sawada–Kotera–Kadomtsev Petviashvili (SKP) equation, obtaining diverse wave solutions including lump and breather waves, enhancing understanding of nonlinear fluid dynamics.

Solving nonlinear population models exactly is often a formidable task due to their complexity. Traditional methods like separation of variables, traveling wave reductions, and perturbation techniques have limitations in handling strongly nonlinear or high-dimensional systems. This motivates the exploration of alternative analytical approaches, such as the Power Index Method, which can systematically transform and simplify such equations into solvable forms. No previous studies have applied the Power Index Method to solve this nonlinear biological population model. In this work, we present the first successful imple-

mentation of this method to obtain exact solutions for the nonlinaer biological population model. The Power Index Method represents an efficient analytical technique for solving nonlinear partial differential equations (NLPDEs) that eliminates the need for series expansions. In contrast to conventional approaches like perturbation methods or homotopy analysis that require series-form solutions, this direct method utilizes targeted algebraic manipulations of nonlinear terms to derive exact closed-form solutions.
Novel exact solutions of nonlinear partial differential equations (NLPDEs) provide both analytical and graphical insights into complex nonlinear phenomena. These solutions significantly enhance our understanding of the qualitative structures governing natural processes across scientific and engineering disciplines. Consequently, researchers have extensively investigated exact solutions of nonlinear partial differential equations (NLPDEs), which arise in diverse fields ranging from theoretical physics to applied engineering. In recent years, substantial progress has been made in developing exact solution methods for nonlinear partial differential equations (NLPDEs), with numerous powerful techniques emerging including, Lie symmetry analysis [22], first integral method [23], modified exp-function method [24], $(G^{'}/G)$-expansion method [25], Power Index Method [26], generalized exp-function Method [27], $\exp(-\phi(\xi))$-expansion method [28], simple equations method [29], modified extended direct algebraic method [30] to solve nonlinear partial differential equations. The effectiveness of the Power Index Method (PIM) for deriving exact solutions to nonlinear partial differential equations (NLPDEs) was evaluated through a systematic comparison with two established analytical approaches: Lie Symmetry Analysis and the Sine-Cosine Method. As summarized in Table 1, these methods demonstrate distinct characteristics, applicability ranges, and solution strategies, highlighting their complementary strengths and limitations.

The similar function transformation is a mathematical technique that employs specific

Table 1:- Comparison of Analytical Methods for solving NLPDEs

Method	Applicability	Solution Type	Strengths	Limitations
Power Index Method(PIM)	Suited for polynomial-type nonlinearities	Exact closed-form solutions, Self-similar solutions	Exact solutions for power-law systems	Restricted to scaling-invariant PDEs
Lie Symmetry Analysis	Very broad (applicable to wide classes of NLPDEs)	Similarity solutions, Invariant solutions	Systematic reduction	Computationally intensive
Sine-Cosine Method	Mainly for wave-type PDEs with periodic behavior	Periodic wave solutions, Soliton solutions	Simple for periodic waves	Only works for specific wave forms

similarity transformations to convert nonlinear biological population models into more

tractable forms, enabling the derivation of novel exact solutions through methods like the Power Index Method (PIM). The Power Index Method (PIM) yields closed-form solutions that are crucial for understanding population dynamics, stability analysis, and pattern formation. Unlike numerical methods, this analytical approach provides exact solutions without approximations, facilitating the interpretation of biologically meaningful parameters. While the Power Index Method (PIM) generates exact solutions, which is a rare achievement for nonlinear models, it is only applicable to equations admitting power-law or self-similar solutions. The method proves most effective for autonomous equations, self-similar equations, and equations with power-law nonlinearities. However, Power Index Method (PIM) is limited to autonomous equations with specific nonlinearities, excluding non-autonomous and fully nonlinear polynomial cases.
The structure of this paper is organized as follows. Section 2 provides a brief introduction to the Power Index Method. In Section 3, we employ this method to construct exact solutions and visualize their behaviors through 3D graphical representations. Section 4 presents physical interpretations of the obtained solutions. Finally, Section 5 contains concluding remarks.

2. Power Index Method

We define the new wave variable ξ and the corresponding transformation for u as follows:

$$\xi = x^m Z(y). \tag{2}$$

$$u = x^n A(y) e^t f(\xi). \tag{3}$$

Here, $A(y)$ and $Z(y)$ are unknown functions that can be determined after selecting the indexes of independent variable x. These functions may be polynomial, rational, or other elementary functions. Now, differentiate Equations (2) and (3) using Equation (1). Substituting all derivative values into Equation (1), we can express the given PDE (1) in terms of the differentials of the new variables as follows:

$$\begin{aligned} &x^{2n+2m}(x^{-2}m^2Z^2(y) + Z'^2(y))f'^2(\xi) + x^{2n+2m}(x^{-2}m^2Z^2(y) + Z'^2(y)) \\ &f(\xi)f''(\xi) + x^{2n+m}(x^{-2}m^2Z(y) - x^{-2}mZ(y) + \frac{4A'(y)Z'(y)}{A(y)} + Z''(y) + \\ &4x^{-2}nmZ(y))f(\xi)f'(\xi) + x^{2n}(2x^{-2}n^2 - nx^{-2} + \frac{A'^2(y)}{A^2(y)} + \frac{A''(y)}{A(y)})f^2(\xi) = 0. \end{aligned} \tag{4}$$

We have selected those terms whose coefficients cannot be coverted into the new variable ξ.

$$F_1(x, y, t) = x^{2m+2}Z'^2(y)(f'^2(\xi) + f(\xi)f''(\xi)). \tag{5}$$

$$F_2(x, y, t) = (\frac{4A'(y)Z'(y)}{A(y)} + Z''(y))x^{m+2}f(\xi)f'(\xi). \tag{6}$$

$$F_3(x, y, t) = (2n^2 + \frac{x^2A'^2(y)}{A^2(y)} + \frac{x^2A''(y)}{A(y)})f^2(\xi). \tag{7}$$

In Equations (5), (6), and (7), we must select the elementary functions $A(y)$ and $Z(y)$ and the indexes m and n of independent variable x such that equation (4) transforms into a nonlinear ODE.
The optimal choices for the indexes m and n of the independent variable x are as follows:
Table:- 2 List of Optimal Values for the Indexes of the Independent Variable x

$2m+2n$	$n=-m$	$m=-1,\ n=1$
$2m+2n$	$n=-m$	$m=1,\ n=-1$
$2n+m$	$n=-\frac{m}{2}$	$m=-2,\ n=1$
$2n+m-2$	$n=\frac{2-m}{2}$	$m=1,\ n=\frac{1}{2}$

3. New Exact Solutions of Biological Population Model

Case:- 1 ($n=1$, $m=-1$)
For $n=1$ and $m=-1$, the wave variable $\xi = x^m Z(y)$ becomes $\xi = \frac{Z(y)}{x}$ and the transformation $u = x^n A(y) e^t f(\xi)$ becomes $u = xA(y)e^t f(\xi)$. Substituting the wave variable $\xi = \frac{Z(y)}{x}$ and the transformation $u = xA(y)e^t f(\xi)$ into Equation (4), we obtain:

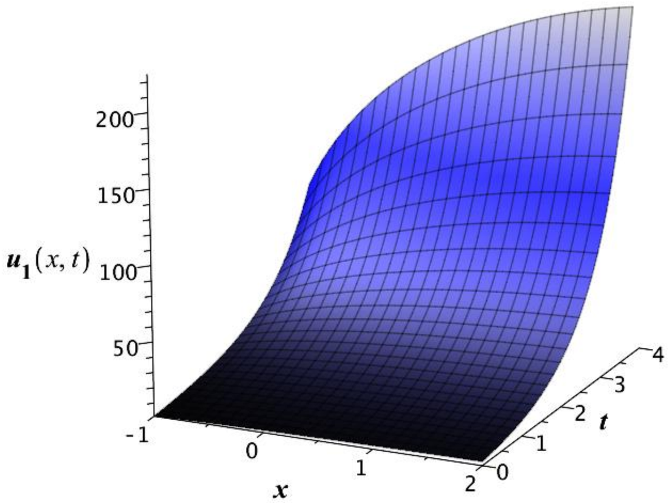

Figure 1: 3D plot of solution u_1 given in (11) of PDE (1) for $C_1 = 1$, $C_2 = -1$ and $y = -3$.

$$
\begin{aligned}
&(\xi^2 + Z'^2(y) + \frac{x^2 A'^2(y)}{A^2(y)} + \frac{x^2 A''(y)}{A(y)}) f'^2(\xi) + (\xi^2 + Z'^2(y)) f(\xi) f''(\xi) + ... \\
&(xz''(y) - 2\xi + \frac{4xA'(y)Z'(y)}{A(y)}) f(\xi) f'(\xi) + f^2(\xi) = 0.
\end{aligned}
\tag{8}
$$

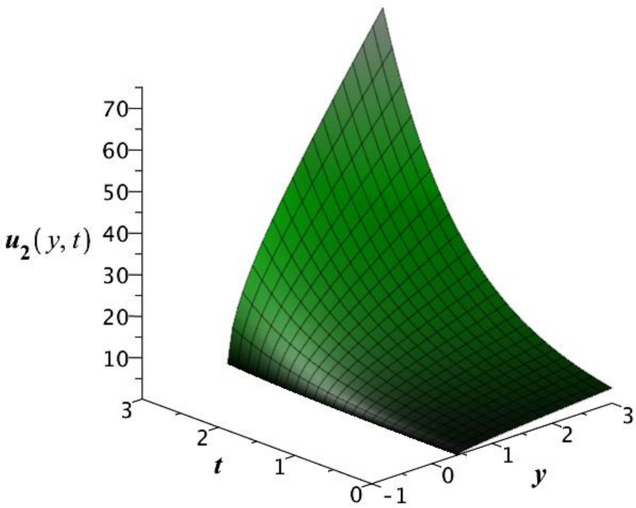

Figure 2: 3D plot of solution u_2 given in (11) of PDE (1) for $C_1 = 1$, $C_2 = -1$ and $x = -1$.

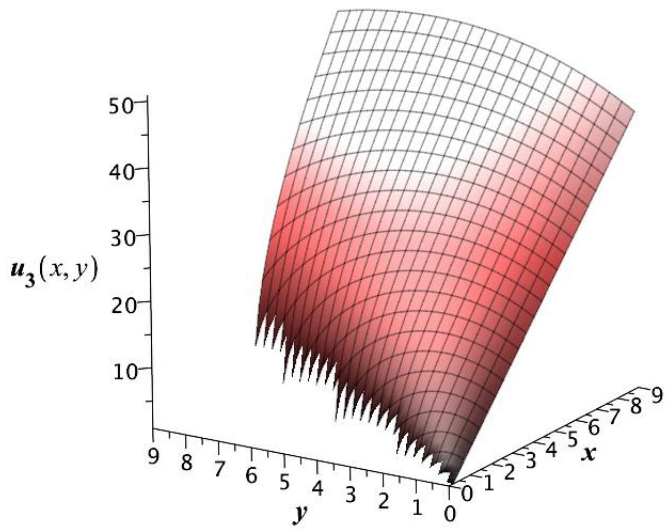

Figure 3: 3D plot of solution u_3 given in (11) of PDE (1) for $C_1 = -2$, $C_2 = 3$ and $t = 1$.

To express the terms in Equation (8) in terms of ξ, we set $Z'(y) = 1$ and $\frac{A'(y)}{A(y)} = \frac{Z''(y)}{Z'(y)}$. This yields $Z(y) = y$ and $A(y) = 1$. Consequently, the wave variable becomes $\xi = \frac{y}{x}$ and the transformation takes the form $u = xe^t f(\xi)$. Substituting $\xi = \frac{y}{x}$ and $u = xe^t f(\xi)$ into Equation (8), we obtain the following ODE:

$$(\xi^2 + 1)f'^2(\xi) + (\xi^2 + 1)f(\xi)f''(\xi) - 2\xi f(\xi)f'(\xi) + f^2(\xi) = 0. \tag{9}$$

The exact solution of ODE (9) is

$$f(\xi) = \sqrt{-C_1 \sin(2\tan^{-1}\xi)(\xi^2+1) + C_2 \cos(2\tan^{-1}\xi)(\xi^2+1))}. \tag{10}$$

The exact solution of PDE (1) is

$$u(x,y,t) = e^t \sqrt{-C_1 \sin(2\tan^{-1}\frac{y}{x})(y^2+x^2) + C_2 \cos(2\tan^{-1}\frac{y}{x})(y^2+x^2)}. \tag{11}$$

Case:- 2 ($n = 1$, $m = -1$)

To express the terms in Equation (8) in terms of ξ, we set $Z'(y) = 1$ and $\frac{A'^2(y)}{A^2(y)} = -\frac{A''(y)}{A(y)}$. Solving these equations yields $Z(y) = y$ and $A(y) = \sqrt{y}$. Consequently, the wave variable becomes $\xi = \frac{y}{x}$ and the transformation takes the form $u = x\sqrt{y}e^t f(\xi)$. Substituting $\xi = \frac{y}{x}$ and $u = x\sqrt{y}e^t f(\xi)$ into Equation (8), we derive the following ODE:

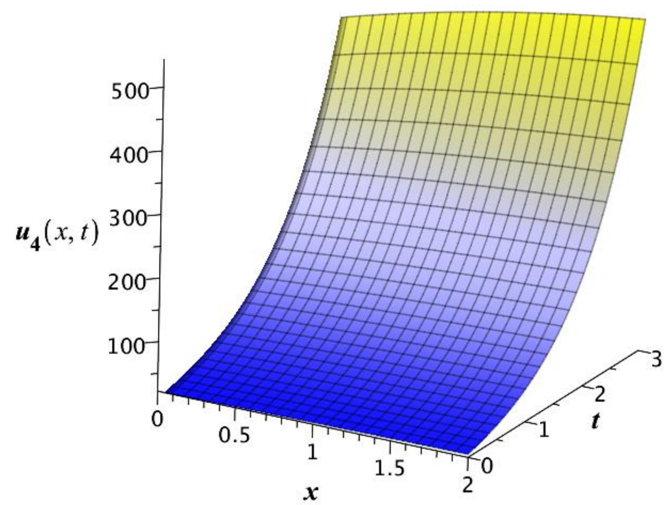

Figure 4: 3D plot of solution u_4 given in (14) of PDE (1) for $C_1 = 1$, $C_2 = -1$ and $y = 9$.

$$(\xi^2+1)f'^2(\xi) + (\xi^2+1)f(\xi)f''(\xi) + (\frac{2}{\xi} - 2\xi)f(\xi)f'(\xi) + f^2(\xi) = 0. \tag{12}$$

The exact solution of ODE (12) is

$$f(\xi) = \frac{\sqrt{6\xi(-C_2\xi^3 + 3C_1\xi^2 + 3C_2\xi - C_1)}}{3\xi}. \tag{13}$$

The exact solution of PDE (1) is

$$u(x,y,t) = \frac{\sqrt{6}x^2e^t}{3\sqrt{y}} \sqrt{-C_2\frac{y^4}{x^4} + 3C_1\frac{y^3}{x^3} + 3C_2\frac{y^2}{x^2} - C_1\frac{y}{x}}. \tag{14}$$

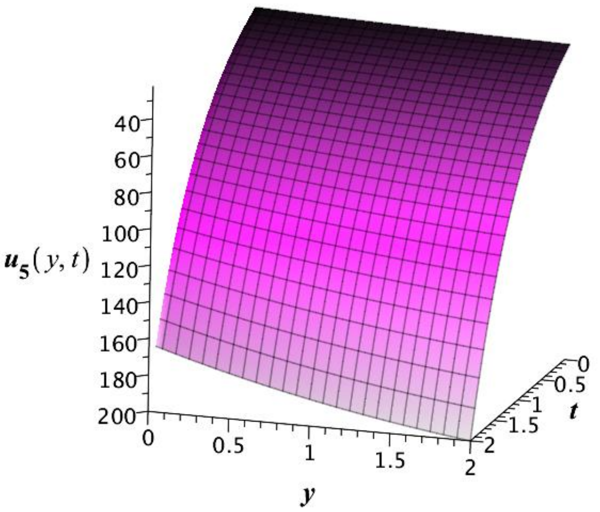

Figure 5: 3D plot of solution u_5 given in (14) of PDE (1) for $C_1 = 1$, $C_2 = 1$ and $x = -9$.

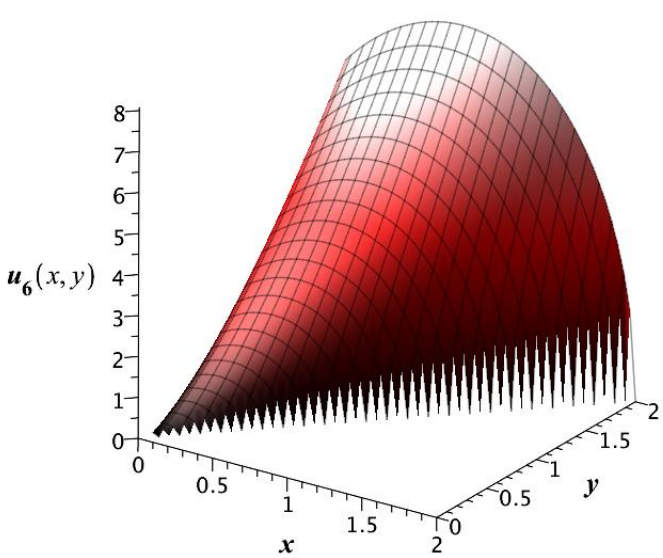

Figure 6: 3D plot of solution u_6 given in (14) of PDE (1) for $C_1 = 1$, $C_2 = -1$ and $t = 1$.

Case:- 3 ($n = 1$, $m = -2$)

For $n = 1$ and $m = -2$, the wave variable $\xi = x^m Z(y)$ becomes $\xi = \frac{Z(y)}{x^2}$, and the transformation $u = x^n A(y)e^t f(\xi)$ reduces to $u = xA(y)e^t f(\xi)$. Substituting $\xi = \frac{Z(y)}{x^2}$ and $u = xA(y)e^t f(\xi)$ into Equation (4), we obtain:

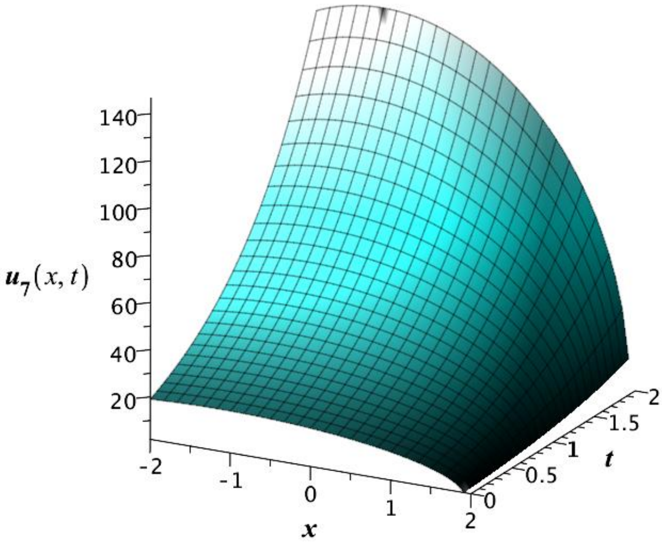

Figure 7: 3D plot of solution u_7 given in (18) od PDE (1) for $C_1 = -1$, $C_2 = -2$ and $y = 5$.

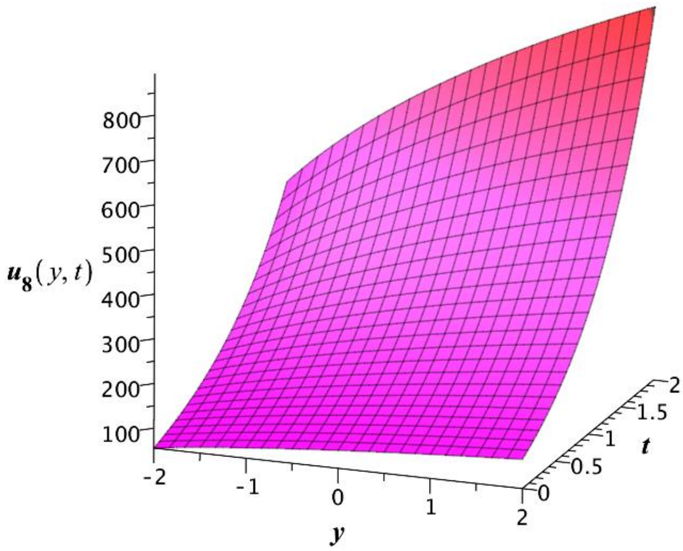

Figure 8: 3D plot of solution u_8 given in (18) of PDE (1) for $C_1 = -1$, $C_2 = -2$ and $x = -19$.

$$(4\xi^2 + \frac{Z'^2(y)}{x^2})f'^2(\xi) + (4\xi^2 + \frac{Z'^2(y)}{x^2})f(\xi)f''(\xi) + (\frac{4Z'(y)A'(y)}{A(y)} - 2\xi + Z''(y))f(\xi)f'(\xi) + (\frac{x^2A'^2(y)}{A(y)} + \frac{x^2A''(y)}{A(y)} + 1)f^2(\xi) = 0. \tag{15}$$

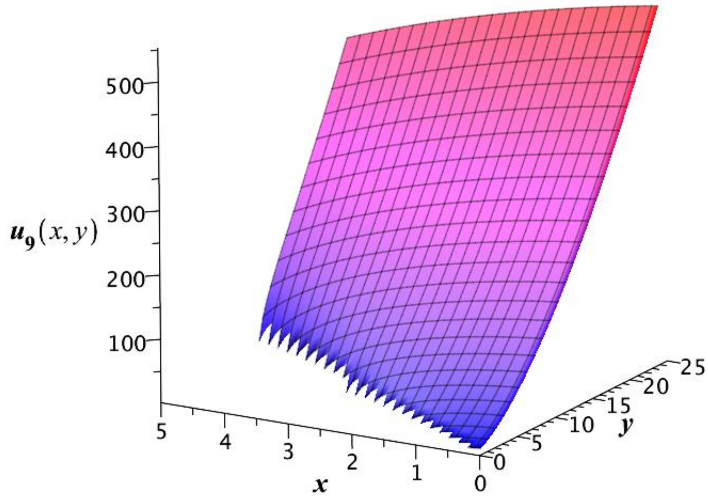

Figure 9: 3D plot of solution u_9 given in (18) of PDE (1) for $C_1 = -1$, $C_2 = -2$ and $t = 1$.

To express the terms in Equation (15) in terms of ξ, we set $\frac{Z'^2(y)}{x^2} = \frac{Z(y)}{x^2}$ and $\frac{A'^2(y)}{A^2(y)} = -\frac{A''(y)}{A(y)}$. Solving these equations yields $Z(y) = y^2$ and $A(y) = \sqrt{y}$. Consequently, the wave variable becomes $\xi = \frac{y^2}{x^2}$, and the transformation takes the form $u = x\sqrt{y}e^t f(\xi)$. Substituting $\xi = \frac{y^2}{x^2}$ and $u = x\sqrt{y}e^t f(\xi)$ into Equation (15), we derive the following ODE:

$$4(\xi^2 + \xi)f'^2(\xi) + 4(\xi^2 + \xi)f(\xi)f''(\xi) + (6 - 2\xi)f(\xi)f'(\xi) + f^2(\xi) = 0. \tag{16}$$

The analytic solution of ODE (16) is

$$f(\xi) = \frac{2}{\sqrt{3\xi}}\sqrt{-C_2\xi^2 + 3C_2\xi + 3C_1\xi^{\frac{3}{2}} - C_1\xi^{\frac{1}{2}}}. \tag{17}$$

The exact solution of PDE (1) is

$$u(x, y, t) = \frac{2x^2e^t}{\sqrt{3}y}\sqrt{-C_2\frac{y^4}{x^4} + 3C_2\frac{y^2}{x^2} + 3C_1\frac{y^3}{x^3} - C_1\frac{y}{x}}. \tag{18}$$

Case:- 4 ($n = -1$, $m = 1$)

For $n = -1$ and $m = 1$, the wave variable $\xi = x^m Z(y)$ becomes $\xi = xZ(y)$, and the transformation $u = x^n A(y)e^t f(\xi)$ reduces to $u = \frac{A(y)e^t f(\xi)}{x}$. Substituting $\xi = xZ(y)$ and $u = \frac{A(y)e^t f(\xi)}{x}$ into Equation (4), we obtain:

$$(\xi^2 + x^4Z'^2(y))f(\xi)f''(\xi) + (\xi^2 + x^4Z'^2(y))f'^2(\xi) + (x^3Z''(y) + \frac{4x^3Z'(y)A'(y)}{A(y)} - 4\xi)f(\xi)f'(\xi) + (\frac{x^2A''(y)}{A(y)} + 3 + \frac{x^2A'^2(y)}{A^2(y)})f^2(\xi) = 0. \tag{19}$$

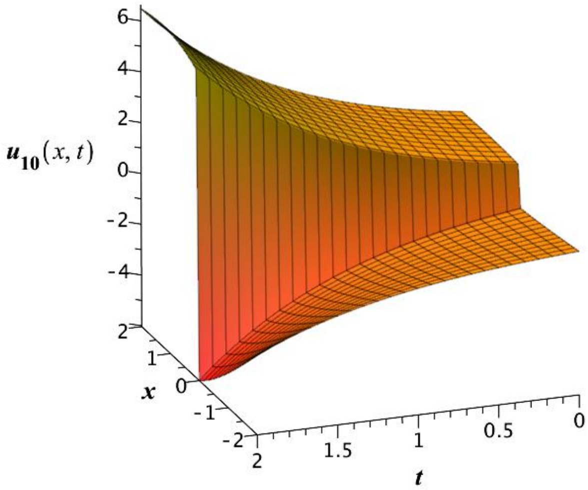

Figure 10: 3D plot of solution u_{10} given in (22) of PDE (1) for $C_1 = 1$, $C_2 = -1$ and $y = 3$.

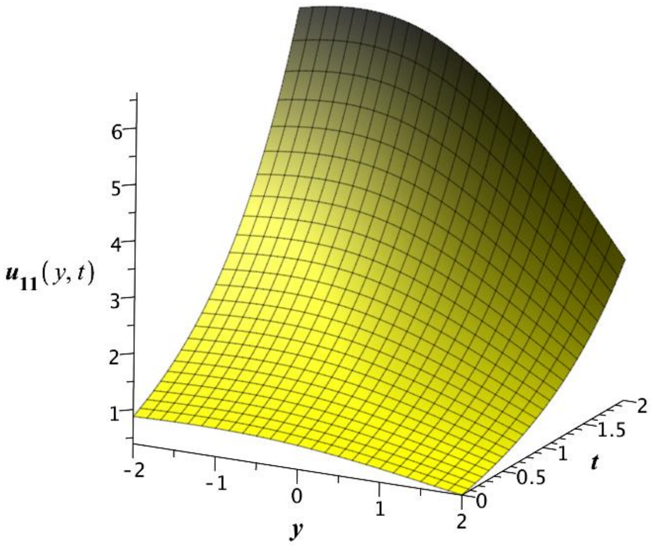

Figure 11: 3D plot of solution u_{11} given in (22) of PDE (1) for $C_1 = 1$, $C_2 = 1$ and $x = 3$.

To express the terms in Equation (19) in terms of ξ, we set $x^4Z'^2(y) = x^4Z^4(y)$ and $\frac{A''(y)}{A'(y)} = -\frac{A'^2(y)}{A(y)}$. Solving these Equations yields $Z(y) = \frac{1}{y}$ and $A(y) = \sqrt{y}$. Consequently, the wave variable becomes $\xi = \frac{x}{y}$ and, the transformation takes the form $u = \frac{\sqrt{y}e^t f(\xi)}{x}$.

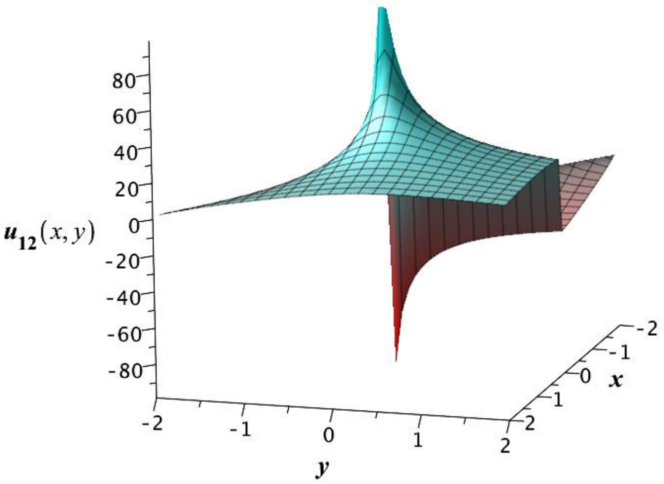

Figure 12: 3D plot of solution u_{12} given in (22) of PDE (1) for $C_1 = 1$, $C_2 = -1$ and $t = 3$.

Substituting these expressions into Equation (19), we obtain the following ODE:

$$(\xi^4 + \xi^2)f(\xi)f''(\xi) + (\xi^4 + \xi^2)f'^2(\xi) - 4\xi f(\xi)f'(\xi) + 3f^2(\xi) = 0. \tag{20}$$

The analytic solution of ODE (20) is

$$f(\xi) = \frac{\sqrt{2}\sqrt{-C_2\xi^5 + C_1\xi^4 - C_2\xi^3 + C_1\xi^2}}{\xi^2 + 1}. \tag{21}$$

The exact solution of PDE (1) is

$$u(x, y, t) = \frac{\sqrt{2}e^t\sqrt{-C_2x^5 + C_1x^4y - C_2x^3y^2 + C_1x^2y^3}}{x^3 + xy^2}. \tag{22}$$

Case:- 5 ($n = \frac{1}{2}$, $m = 1$)
For $n = \frac{1}{2}$ and $m = 1$, the wave variable $\xi = x^m Z(y)$ becomes $\xi = xZ(y)$ and the transformation $u = x^n A(t)e^t f(\xi)$ reduces to $u = \sqrt{x}A(y)e^t f(\xi)$. Substituting $\xi = xZ(y)$ and $u = \sqrt{x}A(y)e^t f(\xi)$ into Equation (4) yields:

$$\begin{aligned}&(\xi^2 + x^4Z'^2(y))f(\xi)f''(\xi) + (\xi^2 + x^4Z'^2(y))f'^2(\xi) + (2\xi + x^3Z''(y) + \\ &\frac{4x^3Z'(y)A'(y)}{A(y)})f(\xi)f'(\xi) + (\frac{x^2A''(y)}{A(y)} + \frac{x^2A'^2(y)}{A^2(y)})f^2(\xi) = 0.\end{aligned} \tag{23}$$

To express the terms in equation (23) in terms of ξ, we set $x^4Z'^2(y) = x^4Z^4(y)$ and $\frac{A''(y)}{A'(y)} = -\frac{A'(y)}{A(y)}$. Solving these equations yields $Z(y) = \frac{1}{y}$ and $A(y) = \sqrt{y}$. Consequently,

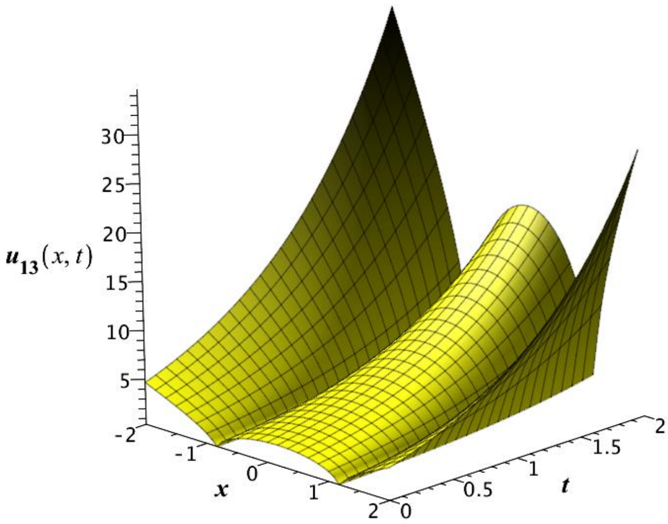

Figure 13: 3D plot of solution u_{13} given in (26) of PDE (1) for $C_1 = -3$, $C_2 = -1$ and $y = -1$.

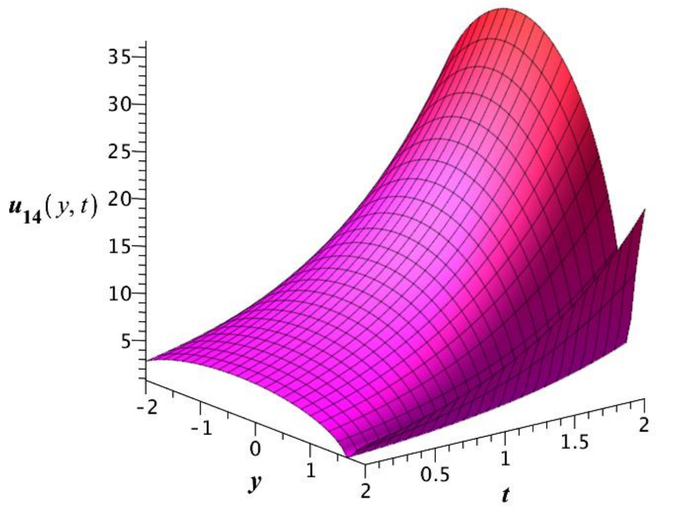

Figure 14: 3D plot of solution u_{14} given in (26) of PDE (1) for $C_1 = -3$, $C_2 = 1$ and $x = 2$.

the wave variable becomes $\xi = \frac{x}{y}$ and the transformation takes the form $u = \sqrt{xy}e^t f(\xi)$. Substituting these expressions into Equation (23), we derive the following ODE:

$$(\xi^4 + \xi^2)f(\xi)f''(\xi) + 2\xi f(\xi)f'(\xi) + (\xi^4 + \xi^2)f'^2(\xi) = 0. \tag{24}$$

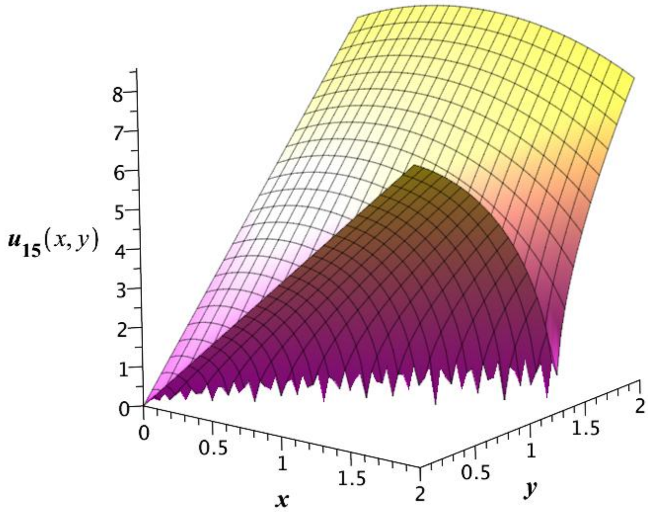

Figure 15: 3D plot of solution u_{15} given in (26) of PDE (1) for $C_1 = -1$, $C_2 = 1$ and $t = 1$.

The analytic solution of ODE (24) is

$$f(\xi) = \sqrt{2C_1\xi + 2C_2 - \frac{2C_1}{\xi}}. \tag{25}$$

The exact solution of PDE (1) is

$$u(x, y, t) = e^t\sqrt{2C_1x^2 + 2C_2xy - 2C_1y^2}. \tag{26}$$

4. Results and Discussion

By developing an analytical approach to solving the nonlinear biological population model, this paper makes a significant contribution to the field of nonlinear science. The successful applications of the Power Index Method reveal a variety of solutions with unique wave structures, demonstrating its effectiveness in solving complex nonlinear problems. The paper serves as a valuable resource for scientists and researchers due to its comprehensive analysis and graphical representations, which provide deeper insights into the equation's dynamics and related physical phenomena. Moreover, this study opens new research avenues across multiple scientific disciplines by showcasing the Power Index Method capability to solve nonlinear differential equations. Overall, the paper's unique methodology and findings make it a significant and influential contribution to the field.
A variety of graphs from Fig. 1 to Fig. 15 presented illustrates a range of behaviors typical of solutions to nonlinear PDEs, including growth, decay, diffusion, wave propagation, and singularities. These behaviors are commonly observed in models across biological, physical, and engineering systems. The graphical examination presented showcases a variety of

how the solutions evolve in both space and time, contingent on the specific equations and parameter values employed. The plots for $u_1(x,t)$, $u_2(y,t)$, $u_4(x,t)$, $u_5(y,t)$, $u_7(x,t)$, and $u_8(y,t)$ consistently highlight a significant increase in the solution's magnitude over time across their respective spatial extents, without displaying typical wave-like movement. In these instances, the spatial patterns remain largely stationary, with their amplitudes scaling uniformly as time progresses. Conversely, the spatial distributions captured at single moments in time for $u_3(x,y)$, $u_6(x,y)$, $u_9(x,y)$, and $u_{15}(x,y)$ reveal intricate and spatially varying amplitudes, suggesting intrinsic spatial frameworks that could impact the temporal development of the complete solutions. Notably, the snapshot of $u_{12}(x,y)$ at a particular time point shows a highly concentrated and potentially problematic behavior near the center. Lastly, the behaviors depicted in $u_{10}(x,t)$, $u_{13}(x,t)$, and $u_{14}(y,t)$ demonstrate a more complex interaction between temporal growth and specific spatial curvatures, indicating an amplification that differs depending on location rather than straightforward wave propagation. In summary, the analysis underscores a spectrum of solution behaviors, ranging from predominant temporal growth to intricate static spatial configurations and dynamic evolutions in both space and time, emphasizing the crucial role of parameter selection and variable fixation. Further exploration of the fundamental mathematical forms is necessary to comprehensively categorize and elucidate the potential for wave-like characteristics within these solution sets.

5. Conclusions

This study successfully derived novel exact solutions for a nonlinear biological population model using the Power Index Method (PIM), demonstrating its effectiveness for degenerate parabolic equations with power-law nonlinearities. The closed-form solutions provide explicit descriptions of population density dynamics, capturing essential features like finite-speed propagation and density-dependent dispersal. Graphical analysis of these solutions (Figures 1-15) revealed distinct spatial-temporal patterns, offering valuable insights into how nonlinear diffusion interacts with population growth processes. Compared to numerical or perturbation approaches, the PIM's strength lies in delivering exact analytical solutions while preserving the system's inherent scaling properties.

While the PIM proves powerful for autonomous equations with specific nonlinearities, its limitations motivate important future research directions. The method cannot currently handle non-autonomous systems or fully nonlinear polynomial equations, suggesting the need for hybrid analytical-numerical approaches. Further work should focus on empirical validation against ecological data, extensions to multi-species interactions, and adaptations for stochastic environments. These advancements would strengthen the method's practical utility while maintaining its mathematical rigor, potentially expanding its applications to other biological and physical systems governed by nonlinear diffusion processes.

The Power Index Method has proven to be an effective analytical tool for deriving novel exact solutions to biological population models, offering deeper insights into the dynamics of species interactions, diffusion processes, and growth patterns. By transforming nonlinear partial differential equations (PDEs) into solvable forms, this method provides closed-form solutions that are crucial for understanding complex ecological behaviors, such as wave propagation, spatial patterning, and stability regimes. However, the true potential of this approach extends beyond the current findings. Future research should focus on bridging the gap between theoretical solutions and real-world biological systems by incorporating data-driven parameter estimation, stochastic influences, and multi-scale interactions. Additionally, integrating machine learning and hybrid numerical-analytical techniques could further enhance the method's efficiency in handling more intricate models, such as those involving cross-diffusion, fractional dynamics, or time-dependent environmental factors. Moreover, the mathematical rigor of the Power Index Method should be complemented by dynamical systems analysis, including stability studies, bifurcation theory, and phase-space exploration, to ensure that the derived solutions are not only exact but also biologically meaningful. Collaboration between mathematicians, ecologists, and computational scientists will be key in validating these solutions against experimental and field data, ensuring their applicability.

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