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The Total Choosability with Neighbor Sum Distinguishing Properties of Planar Graphs That Are Devoid of C_5 Adjacent to C_3

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Abstract. Let G be a graph such that a proper total coloring $\psi : V(G) \cup E(G) \longrightarrow \mathbb{N}$. Let $s(x)$ denote the sum of colors assigned to x and those incident edges of x. A coloring ψ is a *neighbor sum distinguishing total coloring* if $s(x) \neq s(y)$, whenever xy is an edge in G. Let L be a k-list assignment of a graph G if L is a function, say $L : V(G) \cup E(G) \to 2^{\mathbb{N}}$ such that $|L(t)| = k$ for all t in the set $V(G) \cup E(G)$. If G has a total coloring such that $\alpha(x)$ is the element of $L(x)$ for all x in the set $V(G) \cup E(G)$, then we call α *total-L-coloring*. Moreover, a total-L-coloring α is called a *neighbor sum distinguishing total-L-coloring* if $s(x) \neq s(y)$ where xy is an edge in G. Let $Ch''_{\sum}(G)$ be the minimum number k such that G has such coloring for every k-list assignment L.
In this work, we present that for a graph G has $Ch''_{\sum}(G) \leq \max\{10, \Delta(G) + 3\}$ if G is a planar graph that are devoid of C_5 adjacent to C_3.

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1. Introduction

Simple, undirected, finite graphs are considered in all cases. When referring to the maximum degree, face set, edge set, and vertex set of a graph G, respectively, we use

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the notation $\Delta(G)$, $F(G)$, $E(G)$, and $V(G)$, respectively. If the boundary of two faces is shared, then two faces are said to be *adjacent*.

Let a proper total coloring $\psi : V(G) \cup E(G) \longrightarrow \mathbb{N}$ and $s(x)$ denote the sum of colors assigned to x and those incident edges of x. A coloring ψ is a *neighbor sum distinguishing total coloring* (abridged as *nsdt-coloring*) if $s(x) \neq s(y)$, whenever xy is an edge in G. The *neighbor sum distinguishing total chromatic number* of G, say $\text{tndi}_{\sum}(G)$ is the minimum number such that G has a nsdt-coloring.

Pilśniak and Woźniak [1] introduced neighbor sum distinguishing total coloring and obtained $\text{tndi}_{\sum}(G)$ for cycles, bipartite graphs, cubic graphs, and complete graphs. Additionally, the authors posed the following conjecture.

[1] If $|G| \geq 2$, then $\text{tndi}_{\sum}(G) \leq \Delta(G) + 3$.

The conjecture is verified for K_4-minor free graphs by Li, Liu, and Wang [2] and planar graphs with maximum degree at least 13 by Zhang and Li [3].

Later, the result of Zhang and Li [3] was improved by IC-planar graphs with a maximum degree of 13 by Song et al. [4] and planar graphs with a maximum degree at least 11 by Qu et al. [5]. For other classes, Wang, Ma, and Han [6] and Ge, Li, and Xu [7] confirmed the conjecture by planar graphs without 3-cycles with maximum degree at least 7 and planar graphs without 5-cycles with maximum degree at least 7, respectively.

The conjecture is also shown to be true for planar graphs with a better bound, $\text{tndi}_{\sum}(G) \leq \Delta(G) + 2$, by [4, 8, 9].

Let L be a k-list assignment of a graph G if L is a mapping, say $L : V(G) \cup E(G) \to 2^{\mathbb{N}}$ where $|L(t)| = k$ for each $t \in V(G) \cup E(G)$. If G has a total coloring such that $\alpha(t) \in L(t)$ for all t in the set $V(G) \cup E(G)$, then we call α *total-L-coloring*. Moreover, a total-L-coloring α is called a *neighbor sum distinguishing total-L-coloring* if $s(x) \neq s(y)$ where xy is an edge in G. Let $Ch''_{\sum}(G)$ be the minimum number k such that G has such coloring for every k-list assignment L, which is called the *neighbor sum distinguishing total choosability* of G.

Qu et al. [10] proved that $Ch''_{\sum}(G) \leq \Delta(G) + 3$ for every planar graph G with $\Delta(G) \geq 13$. Yao et al. [11] studied $Ch''_{\sum}(G)$ of d-degenerate graphs.

Conjecture 1 has been verified for several classes of planar graphs as follows: planar graphs without adjacent 3-cycles with maximum degree at least 8 by [12], planar graphs without adjacent 6-cycles with maximum degree at least 7 [13], and planar graphs without 4-cycles adjacent to 3-cycles with maximum degree at least 7 by [14].

More results about the neighbor sum distinguishing total choosability for planar graphs can be seen in [3, 15–19].

In this paper, we give Theorem 1 to confirm the list version of Conjecture 1 on planar graphs without neighboring 5-cycles and 3-cycles. The theorem also improves the results of Wang, Ma, and Han [6] (on planar graphs without 3-cycles) and Ge, Li, and Xu [7] (on planar graphs without 5-cycles).

2. Notations

All subscripts are modulo l unless stated otherwise for the remainder of the paper.

An *l-vertex (face)*, an *l^+-vertex (face)*, or an *l^--vertex (face)* is a vertex (a face) which has degree l, at least l, or at most l, respectively. Moreover, we call f a *$(d_1, d_2, \ldots, d_l)$-face* if f is an l-face where its incident vertices have degree $d_1, d_2, \ldots, d_l$ in clockwise order.

If f_1 and f_2 are adjacent 3-faces with a common incident vertex v, then we call a vertex v a *rough vertex* of f_1 and f_2.

For an l-face f, we let $f_1, \ldots, f_l$ adjacent faces of f in clockwise order. An adjacent k-face f_i of f is called a *thin adjacent k-face* of f if f_{i-1} or f_{i+1} is a 4^+-face.

3. Helpful tools

Consider a minimal planar graph G with $Ch''_{\sum}(G) > \Delta(G) + 3$.

We denote the plane embedding of the graph obtained by removing all 2^--vertices of G by H'.

Lemma 1. *(Claim 1, Observation 1, and Claim 4 in [20]) The graph H' satisfies the following properties:*

(i) A graph H' has no 2^--vertices.

(ii) Every 3-vertex have to be adjacent to a 5^+-vertex.

(iii) Only a $(3, 5^+, 5^+)$-face or a $(4^+, 4^+, 5^+)$-face is a 3-face of a graph H'.

Lemma 2. *(Lemma 7 in [14]) The graph H' satisfies the following properties: A 5-vertex is not adjacent to two 3-vertices.*

From now on, we impose an additional condition that a planar graph G has no the adjacency of 3-cycles and 5-cycles. The additional condition implies the following lemma.

Lemma 3. *Faces and vertices in H' satisfy the following properties.*

(i) For $l \in \{4, 5\}$, an l-face is not adjacent to any 3-faces.

If f is a 3-face, then f is not adjacent to l-face for $l \in \{4, 5\}$.

(ii) Let f_i, f_{i+1}, and f_{i+2} be three consecutive incident faces of an l-vertex v. If $l \geq 4$, then one of f_i, f_{i+1}, and f_{i+2} is a 4^+-face.

(iii) Any 3-face is not adjacent to three 3-faces.

Proof. Let $b(f)$ be the boundary of a face f.

(i) Let f be a 3-face and g be an l-face where $l \in \{4, 5\}$. Suppose that f is adjacent to g. Give $b(f) = v_1v_2v_3$ and $b(g) = v_1v_2u_1 \ldots u_{l-2}$.

Consider $l = 4$. One can see that $b(f)$ and $b(g)$ share three vertices; otherwise, a 5-cycle $v_2u_1u_2v_1v_3$ is adjacent to a 3-cycle $v_1v_2v_3$, a contradiction. If $v_3 = u_1$ or $v_3 = u_2$, then there is a 2-vertex or parallel edges, a contradiction by Lemma 1 (1) or property of a graph H'.

Consider $l = 5$. One can see that a 5-cycle $b(g)$ is adjacent to a 3-cycle $b(f)$, a contradiction.

(ii) Let v be an 4^+-vertex. Let f_i, f_{i+1}, and f_{i+2} be three consecutive incident faces of v. Suppose that f_i, f_{i+1}, and f_{i+2} are 3-faces. Give $b(f_i) = vv_iv_{i+1}$, $b(f_{i+1}) = vv_{i+1}v_{i+2}$, and $b(f_{i+2}) = vv_{i+2}v_{i+3}$. Since each vertex is not a 4^+-vertex by Lemma 1 (i), five vertices

$v, v_i, v_{i+1}, v_{i+2}, v_{i+3}$ should be distinct; otherwise, parallel edges exist, a contradiction. Then a 5-cycle $vv_iv_{i+1}v_{i+2}v_{i+3}$ is adjacent to a 3-cycle vv_iv_{i+1}, a contradiction.

(iii) Let f be a 3-face adjacent to faces f_1, f_2, and f_3. Suppose that f_1, f_2, and f_3 are 3-face. By Lemma 3 (ii), each incident vertex of f is a 3-vertex. This contradicts Lemma 1 (iii).

4. Main Theorem

Theorem 1. *A graph G has $Ch''_{\sum}(G) \leq \max\{10, \Delta(G)+3\}$ if G is a planar graph without C_5 adjacent to C_3.*

Proof. Suppose Theorem 1 to the contrary. Consider a minimal counterexample G. Recall that $H^{'}$ is defined as in the previous section. Discharging method is used to show that $H^{'}$ does not exist. For each $z \in V(H') \cup F(H')$, we let $\mu(z) = d(z) - 4$. One can observe that

$$\sum_{z \in V(H') \cup F(H')} \mu(z) = -8$$

by Handshaking lemma and Euler's formula.

Next, we desire some discharging rules to transfer charge $w(x \to y)$ from x to y for some $x, y \in V(H') \cup F(H')$. In additional, we have a new charge $\mu^*(z)$ for each $z \in V(H') \cup F(H')$ after the transfer. Moreover, $\sum_{z \in V(H') \cup F(H')} \mu^*(z) = \sum_{z \in V(H') \cup F(H')} \mu(z) = -8$. Our goal is to find the discharging rules that make $\mu^*(z) \geq 0$ for each $z \in V(H') \cup F(H')$.

The following are discharging rules.

(R1) For a 3-face f, let g be an adjacent to a l-face g where u is a rough 5^+-vertex of f and g.

(R1.1) Let $l = 3$, $w(u \to f) = \frac{1}{3}$ when the incident vertex of f not incident to g is a 4^+-vertex.

(R1.2) Let $l \geq 6$, $w(g \to f) = \frac{1}{2}$ if g is a thin adjacent face of f, otherwise $w(g \to f) = \frac{1}{3}$.

(R2) For a 5^+-vertex u, $w(u \to v) = \frac{1}{3}$ for each its adjacent 3-vertex v.

Now, it is necessary to claim that $\mu * (z) \geq 0$ follows discharge for each $z \in V(H') \cup F(H')$.

It is clear that if f is a 4-face or 5-face, then $\mu^*(f) = 0$ or 1 respectively.

CASE 1: Consider a 3-face f.

let f_1, f_2, and f_3 be adjacent faces of f where f_i is incident to v_i and v_{i+1}.

We consider three cases by Lemma 3 (iii).

- A face f is not incident to a 3-face.

It follows that $\mu^*(f) \geq \mu(f) + 3 \times \frac{1}{3} = 0$ by (R1.2).

- A face f is incident to exactly one 3-face, say f_1.

It follows that v_1 and v_2 are rough vertices of f and f_1.

If v_1 and v_2 are 4-vertices or v_3 is a 3-vertex, then f_2 and f_3 are thin adjacent 6^+-faces of f by Lemma 3 (i) or Lemma 3 (ii), respectively. Then $\mu^*(f) \geq \mu(f) + 2 \times \frac{1}{2} = 0$ by (R1.2).

If v_1 or v_2 is a 5^+-vertex or v_3 is a 4^+-vertex, then v_1 or v_2 gives charge $\frac{1}{3}$ to f by (R1.1). Combining with (R1.2), we have $\mu^*(f) \geq \mu(f) + 3 \times \frac{1}{3} = 0$.

- A face f is incident to two 3-faces, say f_1 and f_2.

Note that v_2 is incident to three consecutive 3-faces. Then v_2 is a 3-vertex by Lemma 3 (ii). Moreover, v_1 and v_3 are 5^+-vertices by Lemma 1 (iii). Similarly, a vertex incident to f_1 or f_2 but not incident to f is a 5^+-vertex. Then each of v_1, v_2, and f_3 gives charge $\frac{1}{3}$ to f by (R1). Thus $\mu^*(f) \geq \mu(f) + 3 \times \frac{1}{3} = 0$.

CASE 2: Consider a 6^+-face.

Let f be an l-face where $l \geq 6$. We let $f_1, \ldots, f_l$ adjacent faces of f in clockwise order. For the convenience of calculating $\mu^*(f)$, we redistribute the charge that was transferred from f as follows.

First, we give $w(f \to f_i) = \frac{1}{3}$ for each f_i where f_i transfer its charge $\frac{1}{6}$ revived form f to f_{i-1} and f_{i+1}. One can see that the process is as described in (R1.2).

- If f_i is a 3-face being not a thin adjacent face of f, then $w(f \to f_i) \geq \frac{1}{3}$.
- If f_i is a thin adjacent 3-face of f, then $w(f \to f_i) \geq \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$.
- If f_i is a 4^+-face, then $w(f \to f_i) \geq \frac{1}{3} - 2 \times \frac{1}{3} = 0$.

Hence, $\mu^*(f) \geq \mu(f) - l \times \frac{1}{3} = l - 4 - l \times \frac{1}{3} = l \times \frac{2}{3} - 4 \geq 0$ as desired.

CASE 3: Consider a 3-vertex v.

By Lemma 1(ii), v is not adjacent to any 4^--vertices. Thus $\mu^*(v) \geq \mu(v) + 3 \times \frac{1}{3} = 0$ by (R2)

CASE 4: Consider a 5-vertex v.

By Lemma 3 (ii), a vertex v is a rough vertex of at most two of its incident faces. It follows from Lemma 2 that a vertex v is adjacent to at most one 3-vertex. Hence, $\mu^*(v) \geq \mu(v) - 3 \times \frac{1}{3} = 0$ by (R1.1) and (R2).

CASE 5: consider a 6^+-vertex v.

Let v be a l-vertex where $k \geq 6$.

We let $v_1, \ldots, v_l$ adjacent vertices of v in clockwise order. For the convenience of calculating $\mu^*(v)$, we redistribute the charge that was transferred from v as follows. First, we give $w(v \to v_i) = \frac{1}{3}$ for each v_i. Now, we consider a 3-face f bounded by v_i, v_{i+1}, and v in two situations. If (1) v_i is not a 3-vertex, and (2) v_{i+1}, v_{i+2}, and v bound the same 3-face then reduce $w(v \to v_i)$ to 0 and transfer charge $\frac{1}{3}$ to a 3-face f instead. If (1) v_{i+1} is a 4^+-vertex and (2) or v_{i-1}, v_i, and v bound the same 3-face, then adjust $w(v \to v_{i+1})$ to 0 and carry charge $\frac{1}{3}$ from v to f instead. Note that Lemma 3 (ii) implies two previously mentioned situations cannot happen simultaneously. Consequently, each of a 3-vertex v_i receives $\frac{1}{3}$ from v as by (R2) and each of a 3-face that v is a rough vertex as in (R1.1) receive $\frac{1}{3}$ from v. Additionally, we get $\mu^*(v) \geq \mu(v) - l \times \frac{1}{3} = l - 4 - l \times \frac{1}{3} = l \times \frac{2}{3} - 4 \geq 0$ as desired.

This completes the proof.

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