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Proximity Prestige of a Vertex in Some Graph Families

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Abstract. Let $G = (V, E)$ be an undirected graph where V, E are the set of vertices and edges respectively. The proximity prestige (PP) of a vertex v_i is the sum of the shortest path distance between vertex v_i and v_j all over the number of vertices in the graph. Proximity prestige (PP) emphasizes the importance of both reachability and distance. Here, general properties of proximity prestige in some classes of graph, including path, cycle, complete, friendship, complete bipartite, star, fan and wheel were determined.

2020 Mathematics Subject Classifications: 05C12, 91D30
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1. Introduction

Graph theory has become an essential tool for analyzing complex systems and networks, providing insights into the structure and dynamics of various real-world systems, from social networks to biological systems. One of the key aspects of graph analysis is centrality, which aims to identify the most important or influential nodes in a network. Traditional centrality measures, such as degree centrality, betweenness centrality, and closeness centrality, have been widely used to capture different aspects of node importance based on direct and indirect connections within a network. However, as networks grow increasingly complex, these conventional measures may fail to fully capture the nuanced roles that certain nodes play in facilitating information flow and influencing others.

A critical aspect of network analysis is centrality, which represents the importance of a node by its position in the network. Using this type of information about social network, Linton Freeman in 1976 [1] first proposed a measure of prestige called proximity prestige. This measure, introduced by Freeman in 1970's, considers not only the number of connections a node has but also how accessible it is to others in the network. Proximity prestige emphasizes strategically positioned nodes, by providing a perspective on influence and importance that goes beyond connectivity. This focuses on reach offers a more

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complex conception of influence than is commonly used in disciplines like sociology and organizational studies.

In 2015, these foundational studies by Freeman [1] and Zhao [2] motivated the current investigation into proximity prestige. In this paper, the researcher's employed the concept of proximity prestige to study the prestige of vertices in nontrivial, connected, and undirected graphs utilizing the results established by Zhao et al. [2]. By extending their work, this study contributes to a deeper understanding of how proximity prestige can be used to evaluate node importance in various types of networks, particularly in settings where indirect influence is a critical factor.

This paper explores the formal definition of proximity prestige as it applies to vertices in a graph, providing a mathematical framework for calculating node importance based on their indirect connections. Other studies that deal with the concept of centralities are located in [3] and [4]. For graph-theoretic terminologies not specifically defined nor described in this study,please refer to either [5] or [6]. Therefore, all graphs considered in this study are nontrivial, connected and undirected.

2. Terminology and Notation

2.1. Preliminary Concepts

A ***graph*** G is a finite nonempty set V of objects called ***vertices*** together with a possibly empty set E of 2-element sets of V called ***edges***. To indicate that a graph G has vertex set V and edge set E, we write $G = (V, E)$. To emphasize that V and E are the vertex set and edge set of a graph G, we often write V as $V(G)$ and E as $E(G)$. Each edge $\{u, v\}$ of G is usually denoted by uv or vu. The number of vertices in a graph G is the ***order*** of G and the number of edges is the ***size***of G. The ***degree of a vertex*** v in a graph G is the number of edges incident with v and is denoted by $\deg v$ or simply by $\deg v$. The degree of a vertex v is denoted by $deg(v)$ and the ***minimum degree*** of G is denoted by $\delta(G)$ and the ***maximum degree*** of G is denoted by $\Delta(G)$ [7].

If uv is an edge of G, then u and v are adjacent vertices. Two ***adjacent vertices*** are referred to as ***neighbors*** of each other. The set of neighbors of a vertex v is called the ***open neighborhood*** of v (or simply the neighborhood of v) and is denoted by $N_G(v)$ or $N(v)$ if the graph is understood. The set $N[v] = N(v) \cup \{v\}$ is called the ***closed neighborhood*** of v. If uv and vw are distinct edges in G, then uv and vw are ***adjacent edges***. The vertex u and the edge uv are said to be ***incident*** with each other. Similarly, v and uv are incident [7].

A graph of order 1 is called a ***trivial graph***. A ***nontrivial graph*** therefore has two or more vertices. A graph of size 0 is called an ***empty graph***. A ***nonempty graph*** then has one or more edges. In any empty graph, no two vertices are adjacent [7].

A $u - v$ ***walk*** W in G is a sequence of vertices in G, beginning with u and ending at v such that consecutive vertices in the sequence are adjacent. A $u - v$ walk in a graph in which no vertices are repeated is a $u - v$ ***path***. The ***distance*** $d_G(u, v)$ from a vertex u to a vertex v in a connected graph G is the length of a shortest $u - v$ path in G. If the graph

G being considered is understood, then this distance is written more simply as $d(u, v)$. A $u - v$ path of length $d(u, v)$ is called a $u - v$ ***geodesic*** [7].

For an integer $n \geq 1$, the ***path*** P_n is a graph of order n and size $n - 1$ whose vertices can be labeled by $v_1, v_2, ..., v_n$ and whose edges are $v_i v_{i+1}$ for $i = 1, 2, ..., n - 1$ [7].

For an integer $n \geq 3$, the ***cycle*** C_n is a graph of order n and size n whose vertices can be labeled by $v_1, v_2, ..., v_n$ and whose edges are $v_1 v_n$ and $v_i v_{i+1}$ for $i = 1, 2, ..., n - 1$. The cycle C_n is also referred to as an $n-$***cycle***[7] .

A ***complete graph*** of order $n \geq 2$, denoted by K_n, is a graph with n vertices where in every pair of distinct vertices are adjacent [7].

The ***friendship graph*** denoted by Fr_n is a set of n triangles having a common central vertex [8].

A graph G is a ***complete bipartite graph*** denoted by $K_{m,n}$ if its vertices can be partitioned into two disjoint nonempty sets V_1 and V_2 such that two vertices u and v are adjacent if and only if $u \in V_1$ and $v \in V_2$. If $|V_1| = m$ and $|V_2| = n$ [9].

A ***star graph*** denoted by $K_{1,n}$ is a graph of order $n + 1$ whose one vertex has degree n which is called the apex u and the remaining n vertices have a degree equal to 1 [6].

For $n \geq 2$, the ***fan graph*** F_n of order $n + 1$ is a graph obtained by connecting a new vertex v to each vertex of the path P_n [6].

A ***wheel graph*** W_n is a graph of order $n + 1$, where $n \geq 3$, which is obtained by joining a new vertex called the root vertex of W_n to each of the vertices of the cycle C_n produced from the complete product of an isolated vertex and a cycle C_n [6].

3. Results

This paper employs the term proximity prestige in social network analysis to represent specific concepts in graph. Furthermore, for a graph G, the vertex set is denoted as$V(G)$ and the edge set as $E(G)$, abbreviated to V and E, respectively.

Definition 1. *Let $G = (V, E)$ be a graph where V represents the set of vertices and E represents the set of edges. The **proximity prestige** of a vertex $v_i \in V$ is defined as:*

$$PP_G(v_i) = \frac{\sum d_G(v_i, v_j)}{|V(G)|}$$

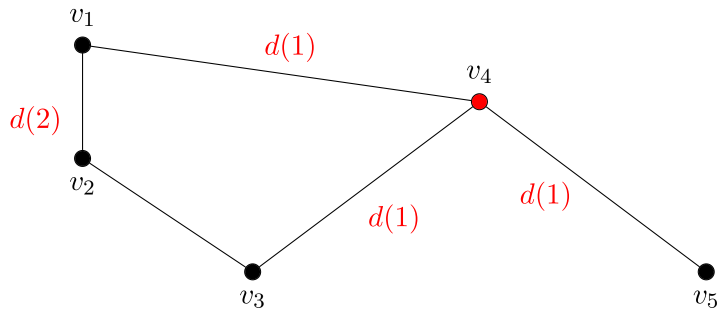
where,

- $PP_G(v_i)$ *: proximity prestige of vertex v_i;*
- $d_G(v_i, v_j)$*: length of the shortest path from v_i to any v_j; and*
- $|V(G)|$*: number of vertices in the graph.*

Example 1. *Consider the example below. By definition, if we choose v_4, we have,*

$$PP_G(v_4) = \frac{\sum d_G(v_4, v_j)}{|V(G)|} = \frac{1+1+1+2}{5} = \frac{5}{5} = 1.$$

Figure 3.1: *The Proximity Prestige of v_4*



Special graph families considered in this paper are path P_n, cycle C_n, complete K_n, friendship Fr_n, complete bipartite $K_{m,n}$, star $K_{1,n}$, fan F_n, and wheel W_n.

Theorem 1. *Let $G = (V, E)$ be a path graph $P_n = [v_1, v_2, \ldots v_n]$ of order $n \geq 2$, then the proximity prestige of any vertex v_i where $1 \leq i \leq n$ is given by,*

$$PP_{P_n}(v_i) = \begin{cases} \frac{\binom{n}{2}}{n}, & \text{if } i = 1 \text{ or } i = n; \\ \frac{(i^2 - i)}{2n} + \frac{(n-i)(n-i+1)}{2n}, & \text{if } 2 \leq i \leq n - i. \end{cases}$$

Proof. Considering the structure of path graph $P_n = [v_1, v_2, \ldots, v_n]$ of order $n \geq 2$, $PP_{P_n}(v_1) = PP_{P_n}(v_n) = \frac{1+2+3+\ldots+n-1}{n} = \frac{\binom{n}{2}}{n}$. But for $2 \leq i \leq n-1$,

$$\begin{aligned} \sum_{i=2}^{n-1} d_{P_n}(v_i, v_j) &= \sum_{i=2}^{i-1} d_{P_n}(v_i, v_j) + \sum_{i=2}^{n-1} d_{P_n}(v_i, v_j) \\ &= [1 + 2 + \ldots + (i-1)] + [1 + 2 + \ldots + (n-i)] \\ &= \frac{(i^2 - 1)}{2} + \frac{(n-i)(n-i+1)}{2} \end{aligned}$$

Thus,

$$PP_{P_n}(v_i) = \frac{(i^2 - 1)}{2n} + \frac{(n-i)(n-i+1)}{2n}.$$

Therefore, we have

$$PP_{P_n}(v_i) = \begin{cases} \dfrac{\binom{n}{2}}{n}, & \text{if } i = 1 \text{ or } i = n; \\ \\ \dfrac{(i^2 - i)}{2n} + \dfrac{(n-i)(n-i+1)}{2n}, & \text{if } 2 \le i \le n - i. \end{cases}$$

■

Theorem 2. *Let $G = (V, E)$ be a cycle graph $C_n = [v_1, v_2, \ldots, v_n, v_1]$ of order $n \ge 3$, then the proximity prestige of any vertex v_i where $1 \le i \le n$ is given by,*

$$PP_{C_n}(v_i) = \begin{cases} \dfrac{n}{4} & \textit{if } n \textit{ is even}; \\ \\ \dfrac{n^2 - 1}{4n} & \textit{if } n \textit{ is odd}. \end{cases}$$

Proof. Suppose first that n is even. By the structure of cycle $C_n = [v_1, v_2, \ldots, v_n, v_1]$, the sum of the distance of v_i and v_j where $i \neq j$ can be derived as follows. For each i, we have distance $d(v_i, v_j) = 1 + 1 + 2 + 2 + \ldots + 2\left(\frac{n}{2} - 1\right) + \frac{n}{2}$. Thus,

$$\sum_{i \neq j} d_{C_n}(v_i, v_j) = 2\left(1 + 2 + \ldots + \left(\frac{n}{2} - 1\right)\right) + \frac{n}{2} = \frac{n^2}{4}.$$

Hence,

$$PP_{C_n}(v_i) = \frac{\frac{n^2}{4}}{n} = \frac{n}{4}.$$

On the other hand, if n is odd, where $i \neq j$. For each i, we have distance $d(v_i, v_j) = 1 + 1 + 2 + 2 + \ldots + 2\left(\frac{n-1}{2}\right)$. Thus,

$$\sum_{i \neq j} d_{C_n}(v_i, v_j) = 2\left[1 + 2 + \ldots + \frac{n-1}{2}\right] = \frac{n^2 - 1}{4}.$$

Hence,

$$PP_{C_n}(v_i) = \frac{\frac{n^2 - 1}{4}}{n} = \frac{n^2 - 1}{4n}.$$

Therefore, we have

$$PP_{Cn}(v_i) = \begin{cases} \dfrac{n}{4} & \text{if } n \text{ is even;} \\ \\ \dfrac{n^2-1}{4n} & \text{if } n \text{ is odd.} \end{cases}$$

■

Theorem 3. *Let G be a complete graph K_n of order $n \geq 3$, then the proximity prestige of any vertex v_i where $1 \leq i \leq n$ is given by,*

$$PP_{Kn}(v_i) = \frac{n-1}{n}.$$

Proof. For each i,

$$\sum_{i \neq j} d_{Kn}(v_i, v_j) = \underbrace{1 + 1 + \ldots + 1}_{\text{n-1 addends}} = n - 1.$$

Therefore,

$$PP_{Kn}(v_i) = \frac{n-1}{n}.$$

■

Theorem 4. *Let G be a friendship graph $Fr_n = [v_1, v_2, \ldots, v_{2n,}, v_{2n+1}]$ where $\deg(v_{2n+1}) = 2n$, then the proximity prestige of any vertex v_i where $1 \leq i \leq 2n+1$ is given by,*

$$PP_{Frn}(v_i) = \begin{cases} \dfrac{2n}{2n+1} & \textit{if } \deg(v_i) = 2n; \\ \\ \dfrac{4n-2}{2n+1} & \textit{if } \deg(v_i) = 2. \end{cases}$$

Proof. Consider the structure of a friendship graph which consists of n triangles sharing a common vertex often called the center vertex v_{2n+1}. Thus the total number of vertices in Fr_n is $2n+1$, with one center vertex of degree $2n$ and outer vertices each of degree 2.

Case 1: $\deg(v_i) = 2n$.
There is only one vertex with a degree of $2n$, which is v_{2n+1} and for any $i = 2n+1 \neq j$ the distance $d(v_i, v_j) = 1$. Thus,

$$\sum d_{Frn}(v_i, v_j) = \underbrace{1 + 1 + \ldots + 1}_{\text{2n addends}} = 2n.$$

Hence,

$$PP_{Frn}(v_i) = \frac{2n}{2n+1}.$$

Case 2: deg $(v_i) = 2$.
Choose $v_1 \in V(Fr_n)$, observe that the distance

$$d(v_1, v_j) = d(v_1, v_{2n+1}) + d(v_1, v_2) + \sum_{j \notin \{2,2n+1\}} d(v_1, v_j).$$

Now, the distance $d(v_1, v_{2n+1}) = 1 = d(v_1, v_2)$ and $d(v_1, v_j) = 2$ for $j \notin \{2, 2n+1\}$. Thus,

$$\sum_{j \notin \{2,2n+1\}} d(v_1, v_j) = \underbrace{2 + 2 + \ldots + 2}_{\text{2n-2 addends}} = 2(2n-2) = 4n-4$$

Thus,

$$\sum d_{Frn}(v_1, v_j) = 1 + 1 + \underbrace{2 + 2 + \ldots + 2}_{\text{2n-2 addends}} = 2 + (4n-4) = 4n-2$$

Hence,

$$PP_{Frn}(v_i) = \frac{4n-2}{2n+1}.$$

Therefore, we have

$$PP_{Frn}(v_i) = \begin{cases} \dfrac{2n}{2n+1} & \text{if } \deg(v_i) = 2n; \\ \\ \dfrac{4n-2}{2n+1} & \text{if} \deg(v_i) = 2. \end{cases}$$

■

Theorem 5. *Let $G = (V, E)$ be a fan graph $F_n = [v_1, \ldots, v_n, v_{n+1}]$ where $\deg(v_{n+1}) = n$, then the proximity prestige of any vertex v_i where $1 \leq i \leq n+1$ is given by,*

$$PP_{F_n}(v_i) = \begin{cases} \dfrac{n}{n+1}, & \textit{if } \deg(v_i) = n; \\ \\ \dfrac{2n-2}{n+1}, & \textit{if } \deg(v_i) = 2; \\ \\ \dfrac{2n-3}{n+1}, & \textit{if } \deg(v_i) = 3. \end{cases}$$

Proof. Using the structure of fan graph F_n of order $n \geq 3$, obtained by connecting a single vertex v_{n+1} to each vertex of the path. Here, we need to consider three cases separately.
Case 1: $\deg(v_i) = n$.

There is only one vertex with a degree n, which is v_{n+1} and for $i = n+1 \neq j$ the distance $d(v_i, v_j) = 1$. Thus,

$$\sum_{i \neq j} d_{Fn}(v_i, v_j) = \underbrace{1 + 1 + \ldots + 1}_{\text{n addends}} = n.$$

Hence,

$$PP_{Fn}(v_i) = \frac{n}{n+1}.$$

Case 2: $\deg(v_i) = 2$.
If $\deg(v_i) = 2$, then there are only two vertices with a degree of 2 in F_n, that is v_1 and v_n *i.e.*, $i \in \{1, n\}$. Choose v_1, thus the distance

$$d(v_1, v_j) = d(v_1, v_2) + d(v_1, v_{n+1}) + \sum_{j \notin \{2, n+1\}} d(v_1, v_j).$$

Now, $d(v_1, v_2) = 1 = d(v_1, v_{n+1})$ and $d(v_1, v_j) = 2$ for $j \notin \{2, n+1\}$. Thus,

$$\sum_{j \notin \{2, n+1\}} d(v_1, v_j) = \underbrace{2 + 2 + \ldots + 2}_{\text{n-2 addends}} = 2(n-2) = 2n - 4.$$

Thus,

$$\sum d_{F_n}(v_1, v_j) = 1 + 1 + (2n - 4) = 2 + (2n - 4) = 2n - 2.$$

Hence,

$$PP_{Fn}(v_i) = \frac{2n-2}{n+1}.$$

Case 3: $\deg(v_i) = 3$.
Choose $v_2 \in V(F_n)$. Then the distance

$$d(v_2, v_j) = d(v_2, v_1) + d(v_2, v_3) + d(v_2, v_{n+1}) + \sum_{j \notin \{1,3,n+1\}} d(v_2, v_j).$$

Now, $d(v_2, v_1) = d(v_2, v_3) = d(v_2, v_{n+1}) = 1$ and $d(v_2, v_j) = 2$, $j \notin \{1, 3, n+1\}$. Thus,

$$\sum_{j \notin \{1,3,n+1\}} d(v_2, v_j) = \underbrace{2 + 2 + \ldots + 2}_{\text{n-3 addends}} = 2(n-3) = 2n - 6$$

Thus,

$$\sum d_{Fn}(v_2, v_j) = 1 + 1 + 1 + \underbrace{2 + 2 + \ldots + 2}_{\text{n-3 addends}} = 3 + (2n - 6) = 2n - 3.$$

Hence,

$$PP_{Fn}(v_i) = \frac{2n-3}{n+1}.$$

Therefore, we have

$$PP_{Fn}(v_i) = \begin{cases} \dfrac{n}{n+1}, & \text{if } \deg(v_i) = n; \\ \\ \dfrac{2n-2}{n+1}, & \text{if } \deg(v_i) = 2; \\ \\ \dfrac{2n-3}{n+1}, & \text{if } \deg(v_i) = 3. \end{cases}$$

■

Theorem 6. *Let $G = (V, E)$ be a wheel graph $W_n = [v_1, v_2, \ldots, v_n, v_{n+1}]$ where $\deg(v_{n+1}) = n$, then the proximity prestige of any vertex v_i where $1 \leq i \leq n+1$ is given by,*

$$PP_{Wn}(v_i) = \begin{cases} \dfrac{n}{n+1}, & \textit{if } \deg(v_{v_i}) = n; \\ \\ \dfrac{2n-3}{n+1}, & \textit{if } \deg(v_i) = 3. \end{cases}$$

Proof. Using the structure of a wheel graph, formed by adjoining central vertex (v_{n+1}) to each vertex of the cycle $C_n = [v_1, v_2, \ldots, v_n, v_1]$ the following cases are need to be considered.

Case 1: $\deg(v_i) = n$.

In this case, $v_i = v_{n+1}$, that is the central vertex of a wheel graph. There is only one vertex with a degree n, that is v_{n+1} and distance $d(v_{n+1}, v_j) = 1$ for $j \neq n+1$ since each vertex $\{v_1, v_2, \ldots, v_n\}$ is directly connected to the central vertex. Thus,

$$\sum d_{Wn}(v_i, v_j) = \underbrace{1 + 1 + \ldots + 1}_{\text{n addends}} = n.$$

Hence,

$$PP_{Wn}(v_i) = \frac{n}{2n+1}.$$

Case 2: $\deg(v_i) = 3$.

In this case, v_i is any of the vertices $\{v_1, v_2, \ldots, v_n\}$. The distance from any vertex v_i where $1 \leq i \leq n$ to the central vertex v_{n+1} is 1 since each v_i is adjacent to v_{n+1} and also the distance from v_i to its two adjacent vertices v_{i-1} and v_{i+1} in the cycle C_n is 1. Then the distance from v_i to any other vertex v_j in the cycle where $j \notin \{i, i-1, i+1\}$ is 2. Thus the distance

$$d(v_i, v_j) = d(v_i, v_{i-1}) + d(v_i, v_{i+1}) + d(v_i, v_{n+1}) + \sum_{j \notin \{i-1, i+1, n+1\}} d(v_i, v_j).$$

Now, from v_i there are 3 vertices that have distance 1, that is $d(v_i, v_{i-1}), d(v_i, v_{i+1})$and $d(v_i, v_{n+1})$. Also, the remaining $n-3$ vertices has distance 2 from v_i. Thus,

$$\sum d_{F_n}(v_i, v_j) = 1 + 1 + 1 + \underbrace{2 + 2 + \ldots + 2}_{\text{n-3 addends}} = 3 + 2n - 6 = 2n - 3$$

Hence,

$$PP_{Wn}(v_i) = \frac{2n-3}{n+1}.$$

Therefore, we have

$$PP_{Wn}(v_i) = \begin{cases} \dfrac{n}{n+1}, & \text{if } \deg(v_i) = n; \\ \\ \dfrac{2n-3}{n+1}, & \text{if } \deg(v_i) = 3. \end{cases}$$

■

Theorem 7. *Let $G = (V, E)$ be a complete bipartite $K_{m,n}$, then the proximity prestige of any vertex v_i where $1 \leq i \leq m + n$ is given by,*

$$PP_{K_{m,n}}(v_i) = \begin{cases} \dfrac{n + (2m-2)}{m+n}, & \textit{if } \deg(v_i) = n; \\ \\ \dfrac{m + (2n-2)}{m+n}, & \textit{if } \deg(v_i) = m. \end{cases}$$

Proof. Using the structure of complete bipartite graph, formed if its vertices can be partitioned into two disjoint nonempty sets V_1 and V_2 such that two vertices u and v are adjacent if and only if $u \in V_1$ and $v \in V_2$ then the following cases are needed to be consider.
Case 1: $\deg(v_i) = n$.
Then,

$$\sum_{i \neq j} d_{K_{m,n}}(v_i, v_j) = \sum_{1 \leq q \leq n} d(v_i, v_q) + \sum_{k \neq i, 1 \leq k \leq m} d(v_i, v_k).$$

Now,

$$\sum_{1\leq q\leq n} d(v_i, v_q) = \underbrace{1 + 1 + \ldots + 1}_{\text{n addends}} = n$$

and

$$\sum_{k\neq i, 1\leq k\leq m} d(v_i, v_k) = \underbrace{2 + 2 + \ldots + 2}_{\text{m-1 addends}}$$
$$= 2(m-1) = 2m - 2.$$

Thus,

$$\sum_{i\neq j} d_{K m,n}(v_i, v_j) = n + 2m - 2.$$

Hence,

$$PP_{K m,n}(v_m) = \frac{n + 2m - 2}{m + n}.$$

Case 2: $\deg(v_i) = m$.
Then,

$$\sum_{i\neq j} d_{K m,n}(v_i, v_j) = \sum_{1\leq k\leq m} d(v_i, v_k) + \sum_{q\neq i, 1\leq q\leq n} d(v_i, v_q).$$

Now,

$$\sum_{1\leq k\leq m} d(v_i, v_k) = \underbrace{1 + 1 + \ldots + 1}_{\text{m addends}} = m$$

and

$$\sum_{l\neq i, 1\leq q\leq n} d(v_i, v_q) = \underbrace{2 + 2 + \ldots + 2}_{\text{n-1 addends}} = 2(n-1) = 2n - 2.$$

Thus,

$$\sum_{i\neq j} d_{K m,n}(v_i, v_j) = m + (2n - 2).$$

Hence,

$$PP_{Km,n}(v_n) = \frac{m + (2n-2)}{m+n}.$$

Therefore, we have

$$PP_{Km,n}(v_i) = \begin{cases} \dfrac{n + (2m-2)}{m+n}, & \text{if } \deg(v_i) = n; \\ \\ \dfrac{m + (2n-2)}{m+n}, & \textit{if } \deg(v_i) = m. \end{cases}$$

■

Theorem 8. *Let $G = (V, E)$ be a star $K_{1,n} = [v_1, v_2, \ldots, v_n, v_{n+1}]$ where $\deg(v_{n+1}) = n$, then the proximity prestige of any vertex v_i where $1 \leq i \leq n+1$ is given by,*

$$P_P(v_i) = \begin{cases} \dfrac{n}{n+1}, & \textit{if } \deg(v_i) = n; \\ \\ \dfrac{2n-1}{n+1}, & \textit{if } \deg(v_i) = 1. \end{cases}$$

Proof. Suppose first that $\deg(v_i) = n$. There is only one vertex with a degree n in $K_{1,n}$, that is v_{n+1}, and the distance $d(v_i, v_j) = 1$, for $j \neq n+1$. Thus,

$$\sum_{j \neq n+1} d_{K1,n}(v_i, v_j) = \underbrace{1 + 1 + \ldots + 1}_{\text{n addends}} = n.$$

Hence,

$$PP_{K1,n}(v_i) = \frac{n}{n+1}.$$

On the other hand, if $\deg(v_i) = 1$. Then,

$$\sum_{i \neq j} d_{K1,n}(v_i, v_j) = 1 + \underbrace{2 + 2 + \ldots + 2}_{\text{n-1 addends}} = 1 + 2(n-1) = 2n - 1$$

Hence,

$$PP_{K1,n}(v_i) = \frac{2n-1}{n+1}.$$

Therefore, we have

$$PP_{K1,n}(v_i) = \begin{cases} \dfrac{n}{n+1}, & \text{if } \deg(v_i) = n; \\ \\ \dfrac{2n-1}{n+1}, & \textit{if } \deg(v_i) = 1. \end{cases}$$

■

4. Conclusion

This paper introduced proximity prestige (PP) as a centrality measure in fixed graphs, defined by the average shortest path distance from a vertex to all other vertices, focusing on indirect connections. Proximity prestige (PP) offers a valuable approach for quantifying vertex importance based on its reach within the network.

Future research could explore the application of proximity prestige (PP) to random and dynamic graphs, as well as its integration with other centrality measures, to enhance understanding of vertex influence in evolving network structures and real-world, complex networks.

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