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Hierarchical Fixed Point Results for a Countable Family of Strict Pseudo-Contractive Mappings in Hadamard Manifolds

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Abstract. The aim of this paper is to present convergence results for a countable family of strict pseudocontractive mappings in Hadamard manifolds. More precisely, we employ the shrinking projection method to approximate common hierarchical fixed points of a countable family of strict pseudocontractive mappings in the setting of Hadamard manifolds. We also present some nontrivial examples to illustrate our result.

2020 Mathematics Subject Classifications: 47H10, 47H09

Key Words and Phrases: Hierarchical Fixed point, Hadamard Manifolds, Pseudocontractive Mappings

1. Introduction

In 1966, Hartman and Stampacchia [1] introduced variational inequality theory as a method for studying partial differential equations with applications, primarily in mechanics. The variational inequality problem has a wide range of applications in some practical problems arising in economics, transportation, network and structural analysis, elasticity, engineering and mechanics, supply chain management, finance and game theory. In recent years, many authors discussed variational inequality problems in the context of Banach and Hilbert spaces [2–7]. To solve environmental projects concerning the transmission of pollution in different kind of media we need to transfer pollution along certain bounded surface areas. These restrictions lead to many boundary value problems on manifolds. To overcome, this situation in 2003, Nemeth [8] introduced the variational inequalities in Hadamard manifolds.

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A constrained optimization problem, where the constrained set is the solution set of another optimization problem, is known as a bilevel programming problem. Over the past thirty years, there has been extensive research on these challenges due to their relevance in domains such as mechanics and network design. If the first level problem is one of variational inequality and the second level problem is a collection of fixed points of a mapping, then the bilevel problem is known as a hierarchical variational inequality problem. Stated otherwise, a variational inequality problem defined over the set of fixed points is a hierarchical variational inequality problem, sometimes referred to as a hierarchical fixed point problem.

In 2006, Moudafi and Mainge [9] introduced the hierarchical fixed point problem in the setting of Hilbert space

$$\text{Find } \zeta \in F(H) \text{ such that } \langle \zeta - G(\zeta), \zeta - \nu \rangle \leq 0, \text{ for all } \nu \in F(G). \tag{1}$$

Here G, H are nonexpansive mappings defined on a Hilbert space $\mathcal{M}$. Later in 2010, Xu [10] extended his work in the context of uniformly smooth Banach spaces. After that the viscosity method was developed by a number of researchers to solve variational inequalities defined on the set of fixed points of nonexpansive mapping in the context of Hilbert or Banach spaces. These researchers replaced contraction mapping with weaker forms of contraction mappings, such as pseudo-contraction mapping and weakly contraction mapping [3, 4, 11–14].
In 2020, Al-Homidan, presented a viscosity approach to solve the hierarchical fixed point problem in the context of Hadamard manifolds defined on the set of fixed points of nonexpansive mapping and involving a nonexpansive mapping and another ϕ-contraction mapping. There are many authors who presented a viscosity approach for hierarchical variational inequality problems in the context of Hadamard manifolds [9, 15], and references therein.

Motivated by the above works, the purpose of this paper is to introduce and analyse a new algorithm for solving hierarchical variational inequality problems in the framework of Hadamard manifolds. In this paper, we present a new algorithm to solve hierarchical fixed point problem and present convergence result for a finite family of β-strict pseudo-contractive mapping in Hadamard manifolds.

The manuscript is presented as follows: Section 2 contains basic definitions and facts. Section 3 has the hierarchical variational inequality problem and the proposed algorithm and its convergence analysis. Section 4 has some numerical examples which illustrates the result presented in the manuscript.

2. Preliminaries

Now, we present some basic facts and definitions which are related to Hadamard manifolds.

Let us consider, Γ is the differentiable and finite dimensional manifold. We write the tangent space of Γ, for all $\zeta \in \Gamma$ as $G_\zeta\Gamma$. This $G_\zeta\Gamma$ is also a vector space and its dimension is same as the dimension of Γ. Also, the tangent bundle of Γ is denoted by $G\Gamma = \bigcup_{\zeta\in\Gamma} G_\zeta\Gamma$. If we define an inner product $\mathcal{R}_\zeta(\cdot,\cdot)$ on the tangent space $G_\zeta\Gamma$ then it is said to be a Riemannian metric defined on $G_\zeta\Gamma$. If Γ can be endowed with a Riemannian metric $\mathcal{R}_\zeta(\cdot,\cdot)$ then we say Γ is a Riemannian manifold. We denote the corresponding norm for the inner product on the tangent space $G_\zeta\Gamma$ by $\|\cdot\|_\zeta$. A Riemannian manifold is a manifold, which is differentiable endowed with a Riemannian metric $\mathcal{R}(\cdot,\cdot)$. The length of piecewise smooth curve $\Upsilon : [0,1] \to \Gamma$ joining ζ to η (i.e. $\Upsilon(0) = \zeta$ and $\Upsilon(1) = \eta$) is given by $L(\Upsilon) = \int_0^1 \|\Upsilon'(\mu)\| d\mu$. The Riemannian distance $\rho(\zeta,\eta)$ is the minimal length over the set of all these curves joining ζ to η, which includes the original topology on Γ.

A Riemannian manifold Γ is said to be complete if for all $\zeta \in \Gamma$, all geodesics starting from ζ are defined for all $\mu \in \mathbb{R}$. A geodesic joining ζ to η is said to be minimal in Γ if the length of the geodesic is equal to $\rho(\zeta,\eta)$. The Riemannian manifold Γ having the Riemannian distance ρ is also a metric space (Γ,ρ).

Definition 1. *Let us consider that Γ is a complete Riemannian manifold. We define the exponential map $\exp_\zeta : G_\zeta\Gamma \to \Gamma$ at point $\zeta \in \Gamma$ by $\exp_\zeta v = \Upsilon_v(1,\zeta)$ for all $v \in G_\zeta\Gamma$, where $\Upsilon_v(\cdot,\zeta)$ is the geodesic with the velocity v and starting from the point ζ i.e. $\Upsilon'_v(0,\zeta) = v$ and $\Upsilon_v(0,\zeta) = \zeta$* [16].

We also know that for all $\mu \in \mathbb{R}$ the exponential map $\exp_\zeta \mu v = \Upsilon_v(\mu,\zeta)$. Here we can also see for all zero tangent vector, exponential map $\exp_\zeta 0 = \Upsilon_v(0,\zeta) = \zeta$. The exponential map $\exp_\zeta$ is differentiable on $T_\zeta\Gamma$ for all $\zeta \in \Gamma$ and $\rho(\zeta,\eta) = \|\exp_\zeta^{-1}\eta\|$ for all $\zeta,\eta \in \Gamma$.

Definition 2. *A Riemannian manifold of non positive sectional curvature is said to be a Hadamard Manifold if it is complete and simply connected.*

Lemma 1. [17].

(1) *For all $w, u, \varsigma, \eta, , z \in \Gamma$, $0 \le \mu \le 1$, following hold:*

$$\rho(\exp_\eta(1-\mu)\exp_\eta^{-1}\varsigma, z) \le \mu\rho(\eta,z) + (1-\mu)\rho(\varsigma,z);$$

$$\rho^2(\exp_\eta(1-\mu)\exp_\eta^{-1}\varsigma, z) \le \mu\rho^2(\eta,z) + (1-\mu)\rho^2(\varsigma,z) - \mu(1-\mu)\rho^2(\eta,\varsigma);$$

$$\rho(\exp_\eta(1-\mu)\exp_\eta^{-1}\varsigma, \exp_u(1-\mu)\exp_u^{-1}\zeta) \le \mu\rho(\eta,u) + (1-\mu)\rho(\varsigma,\zeta).$$

(2) *Let $\Upsilon : [0,1] \to \Gamma$ be a geodesic joining points η to ς. Then*

$$\rho(\Upsilon(\mu_1),\Upsilon(\mu_2)) = |\mu_1 - \mu_2|\rho(\eta,\varsigma) \text{ for all } \mu_1,\mu_2 \in [0,1].$$

Proposition 1. [18]. $\exp_\zeta : G_\zeta\Gamma \to \Gamma$ *is said to be a diffeomorphism for all* $\zeta \in \Gamma$*, any pair of points* $\zeta, \eta \in \Gamma$*, there exists a unique normalized geodesic* $\Upsilon : [0,1] \to \Gamma$ *joining the points* $\zeta = \Upsilon(0)$ *to* $\eta = \Upsilon(1)$*, in fact it is a minimal geodesic defined by*

$$\Upsilon(\mu) = \exp_\zeta \mu \exp_\zeta^{-1} \eta \text{ for all } 0 \leq \mu \leq 1.$$

Definition 3. *The map* $G : \mathcal{B} \to \mathcal{B}$ *is called as*

(i) nonexpansive

$$\rho(G(\varsigma), G(\eta)) \leq \rho(\varsigma, \eta) \quad \text{for all } \varsigma, \eta \in \mathcal{B},$$

(ii) firmly nonexpansive, if for all $\varsigma, \eta \in \mathcal{B}$*, function* $\phi : [0,1] \to [0, +\infty]$ *defined by*

$$\phi(t) = \rho\left(\exp_\eta t \exp_\eta^{-1} G(\eta), \exp_\varsigma t \exp_\varsigma^{-1} G(\varsigma)\right) \text{ for all } 0 \leq t \leq 1$$

is nonincreasing [18].

(iii) β*-strict pseudocontractive if there exists* $\beta \in [0,1)$ *such that*

$$\rho^2(G(\eta), G(\varsigma)) \leq \rho^2(\eta, \varsigma) + \beta \|P_{\eta,\varsigma} \exp_\varsigma^{-1} G(\varsigma) - \exp_\eta^{-1} G(\eta)\|^2, \text{ for all } \eta, \varsigma \in \mathcal{B}.$$

Note that the mapping G *is nonexpansive if and only if it is* 0*-strict pseudocontractive.*

Suppose $\mathcal{B}$ is a geodesic convex and closed subset of the given Hadamard manifold Γ. The projection map onto the geodesic convex and closed sets can also defined in the setting of linear metric spaces. A projection mapping $P_\mathcal{B}(\cdot) : \Gamma \to \mathcal{B}$ is given by for any $\zeta \in \Gamma$

$$P_\mathcal{B}(\zeta) = \{w \in \mathcal{B} : \rho(\zeta, w) \leq \rho(\zeta, \mu), \text{ for all } \mu \in \mathcal{B}\}.$$

Proposition 2. [19] *Suppose* $\mathcal{B} \neq \emptyset$ *be closed and geodesic convex subset of* Γ*. Then we have:*

(1) $P_\mathcal{B}$ *is a single valued and firmly nonexpansive mapping;*

(2) *for all* $\zeta \in \Gamma, \omega = P_\mathcal{B}(\zeta)$ *if and only if* $\mathcal{R}(\exp_\omega^{-1} \zeta, \exp_\omega^{-1} \vartheta) \leq 0, \text{ for all } \vartheta \in \mathcal{B}$*;*

(3) *if* $P_\mathcal{B}$ *is firmly nonexpansive.*

$$\rho^2(\omega, \nu) + \rho^2(\omega, \zeta) \leq \rho^2(\zeta, \nu), \text{ for all } \zeta \in \Gamma, \nu \in \mathcal{B},$$

here $\omega = P_\mathcal{B}(\zeta)$*.*

Lemma 2. [20] *Suppose* $\Delta(\zeta_1, \zeta_2, \zeta_3)$ *is a geodesic triangle in* Γ *then there exists a triangle* $\Delta(\overline{\zeta_1}, \overline{\zeta_2}, \overline{\zeta_3})$ *in* $\mathbb{R}^2$ *for* $\Delta(\zeta_1, \zeta_2, \zeta_3)$ *such that* $\rho(\zeta_i, \zeta_{i+1}) = \|\overline{\zeta_i} - \overline{\zeta_{i+1}}\|$*, indices are taken modulo* 3*; and it is unique upto an isometry of* $\mathbb{R}^2$*.*

Proposition 3. [18] *Suppose* $\Delta(\zeta_1, \zeta_2, \zeta_3)$ *be a geodesic triangle in* Γ. *Then*

$$\rho^2(\zeta_1, \zeta_2) + \rho^2(\zeta_2, \zeta_3) - 2\mathcal{R}\left(\exp_{\zeta_2}^{-1} \zeta_1, \exp_{\zeta_2}^{-1} \zeta_3\right) \leq \rho^2(\zeta_3, \zeta_1), \tag{2}$$

and

$$\rho^2(\zeta_1, \zeta_2) \leq \mathcal{R}\left(\exp_{\zeta_1}^{-1} \zeta_3, \exp_{\zeta_1}^{-1} \zeta_2\right) + \mathcal{R}\left(\exp_{\zeta_2}^{-1} \zeta_3, \exp_{\zeta_2}^{-1} \zeta_1\right). \tag{3}$$

Moreover, if θ *is the angle at* ζ_1, *then we have*

$$\mathcal{R}\left(\exp_{\zeta_1}^{-1} \zeta_2, \exp_{\zeta_1}^{-1} \zeta_3\right) = \rho(\zeta_2, \zeta_1)\rho(\zeta_1, \zeta_3)\cos(\theta).$$

Lemma 3. [20] *Suppose* $\Delta(\zeta_1, \zeta_2, \zeta_3)$ *be geodesic triangle in* Γ, $\Delta(\overline{\zeta_1}, \overline{\zeta_2}, \overline{\zeta_3})$ *its comparison triangle.*

(1) *Suppose* $\alpha_1, \alpha_2, \alpha_3$ *and* $\overline{\alpha_1}, \overline{\alpha_2}, \overline{\alpha_3}$ *be the angles of* $\Delta(\zeta_1, \zeta_2, \zeta_3)$ *and* $\Delta(\overline{\zeta_1}, \overline{\zeta_2}, \overline{\zeta_3})$ *at the vertices* $\zeta_1, \zeta_2, \zeta_3$ *and* $\overline{\zeta_1}, \overline{\zeta_2}, \overline{\zeta_3}$, *respectively. Then*

$$\alpha_1 \leq \overline{\alpha_1}, \alpha_2 \leq \overline{\alpha_2} \text{ and } \alpha_3 \leq \overline{\alpha_3}.$$

(2) *Suppose* μ *be any point on the geodesic connecting* ζ_1, ζ_2 *and* $\overline{\mu}$ *its comparison point in interval* $[\overline{\zeta_1}, \overline{\zeta_2}]$. *If* $\rho(\zeta_1, \mu) = \|\overline{\zeta_1} - \overline{\mu}\|$ *and* $\rho(\zeta_2, \mu) = \|\overline{\zeta_2} - \overline{\mu}\|$ *then* $\rho(\zeta_3, \mu) \leq \|\overline{\zeta_3} - \overline{\mu}\|$.

Proposition 4. [21] *Suppose* $\mathcal{B} \neq \emptyset$ *be a geodesic convex subset of a Hadamard manifold* Γ *and* $G : \mathcal{B} \to \mathcal{B}$ *be a* β*-strictly pseudocontractive mapping with* $\beta \in (0, 1]$. *Then* $G_\lambda(\zeta) = \exp_\zeta \lambda \exp_\zeta^{-1} G(\zeta)$ *is a nonexpansive mapping for every* $\lambda \in (0, 1 - \beta)$.

Remark 1. *If* G *and* G_λ *are same as above proposition then* $F(G) = F(G_\lambda)$. *It follows from the following equivalence*

$$\begin{aligned} &\text{Let } \zeta \in F(G_\lambda) \\ &\text{implies } \zeta = G_\lambda(\zeta) \Leftrightarrow \zeta = \exp_\zeta \lambda \exp_\zeta^{-1} G(\zeta) \\ &\Leftrightarrow 0 = \exp_\zeta^{-1} G(\zeta) \\ &\Leftrightarrow \zeta = G(\zeta). \end{aligned}$$

Throughout the article we write the set of all single valued vector fields $\digamma : \Gamma \to G\Gamma$ as $\Xi(\Gamma)$ such that $\digamma(\zeta) \in G_\zeta\Gamma$ for all $\zeta \in \Gamma$. Let $\Psi(\Gamma)$ denote the set of all multivalued vector fields $V : \Gamma \to 2^{G\Gamma}$ such that $V(\zeta) \subseteq G_\zeta\Gamma$ for all $\zeta \in \Gamma$, and we denote the domain of V by $D(V) = \{\zeta \in \Gamma : V(\zeta) \neq \emptyset\}$.

Definition 4. [22] *Any vector field* $\digamma \in \Xi(\Gamma)$ *is monotone if it satisfies*

$$\mathcal{R}\left(\digamma(\zeta), \exp_\zeta^{-1} \nu\right) + \mathcal{R}(\digamma(\nu), \exp_\nu^{-1} \zeta) \leq 0, \text{ for all } \zeta, \nu \in \Gamma.$$

Definition 5. [23] *A multivalued vector field* $V \in \Psi(\Gamma)$ *is said to be monotone if for all* $\zeta, \nu \in D(V)$

$$\mathcal{R}\left(u, \exp_{\zeta}^{-1}\nu\right) \leq \mathcal{R}(w, -\exp_{\nu}^{-1}\zeta), \text{ for all }, u \in V(\zeta), w \in V(\nu);$$

and maximal monotone if it is already monotone, for all $\zeta \in \Gamma$, $u \in G_{\zeta}\Gamma$,

$$\mathcal{R}\left(u, \exp_{\zeta}^{-1}\nu\right) \leq \mathcal{R}(w, -\exp_{\nu}^{-1}\zeta), \text{ for all }, \nu \in D(V), w \in V(\nu) \text{ implies } u \in V(\zeta).$$

Definition 6. [22] *Suppose* $G : \Gamma \to \Gamma$ *is a mapping and vector field* $\digamma \in \Xi(\Gamma)$ *is defined as*

$$\digamma(\zeta) = -\exp_{\zeta}^{-1} G(\zeta), \text{ for all } \zeta \in \Gamma,$$

is the complementary vector field.

Theorem 1. [22] *The complimentary vector field* $\digamma = -\exp^{-1} G$ *defined for any nonexpansive mapping* $G : \Gamma \to \Gamma$ *is always monotone.*

3. Main Results

Suppose Γ is a Hadamard Manifold, $G_i, H_i : \Gamma \to \Gamma$ be two countable family of β-strict pseudocontractive mappings with $\beta \in (0, 1]$ and the set of common fixed points of mappings H_i and G_i are given by $\bigcap_i F(H_i)$ and $\bigcap_i F(G_i)$. We define the hierarchical fixed point problem as follows

$$\text{Find } \zeta \in \bigcap_i F(H_i) : \mathcal{R}\left(\exp_{\zeta}^{-1} G_{\lambda_i}(\zeta), \exp_{\zeta}^{-1}\nu\right) \leq 0, \quad \text{for all } \nu \in \bigcap_i F(H_i). \tag{4}$$

If $\bigcap_i F(H_i) \neq \emptyset$, then using Proposition 2 (2) the above problem defined as

$$\text{Find } \zeta \in \Gamma \text{ such that } \zeta = P_{\bigcap_i F(H_i)} G_{\lambda_i}(\zeta), \tag{5}$$

here $P_{\bigcap_i F(H_i)}$ is the metric projection of Γ onto $\bigcap_i F(H_i)$. We denote the set of solution of (4) as $\Phi = \left\{\zeta^{\dagger} \in \Gamma : \zeta^{\dagger} = P_{\bigcap_i F(H_i)} G_{\lambda_i}(\zeta^{\dagger})\right\}$.

Hierarchical Variational Inequality Problem: Suppose $\mathcal{B} \neq \emptyset$ is a geodesic convex, closed subset of Γ and $\digamma : \mathcal{B} \to H\Gamma$ be the monotone vector field. The problem is to find $\zeta \in \mathcal{B}$:

$$\mathcal{R}\left(\digamma(\zeta), \exp_{\zeta}^{-1}\nu\right) \geq 0, \text{ for all } \nu \in \mathcal{B}. \tag{6}$$

Suppose $G_i : \Gamma \to \Gamma$ be a countable family of β-strict pseudocontractive mappings. Then the complimentary vector field of the corresponding family of mappings G_{λ_i} is monotone by Theorem 1. Now, we can reduce the problem (4) as:

$$\text{Find } \zeta \in \bigcap_i F(H_i) : \mathcal{R}\left(\digamma(\zeta), \exp_{\zeta}^{-1}\nu\right) \geq 0, \text{ for all } \nu \in \bigcap_i F(H_i). \tag{7}$$

Where $F = -\exp^{-1} G_{\lambda_i}$ is the complimentary vector field of G_{λ_i}. In the context of normal cone the set $\bigcap_i F(H_i)$, we can easily see that problem (7) is equivalent to the following

$$\text{Find } \zeta \in \bigcap_i F(H_i) : 0 \in -\exp_\zeta^{-1} G_{\lambda_i}(\zeta) + N_{\bigcap_i F(H_i)}(\zeta). \tag{8}$$

Here $N_{\bigcap_i F(H_i)}$ is normal cone onto set $\bigcap_i F(H_i)$ at $\zeta \in \bigcap_i F(H_i)$, and defined as

$$N_{\bigcap_i F(H_i)}(\zeta) = \left\{ \mu \in H_{\lambda_{i\zeta}}\Gamma : \mathcal{R}\left(\mu, \exp_\zeta^{-1} \nu\right) \leq 0, \text{ for all } \nu \in \bigcap_i F(H_i) \right\}.$$

Now, we present a new algorithm to approximate the solution of the problem (4). Suppose Γ be a Hadamard Manifold and $G_i, H_i : \Gamma \to \Gamma$ be two countable family of β-strict pseudocontractive mappings with $\beta \in (0, 1]$. Assume that $\zeta_1 \in \Gamma$ and $\mathcal{B}_1 = \Gamma$, we can generate sequences $\{\zeta_n\}$ and $\{\vartheta_n\}$ as follows:

$$\begin{cases} \vartheta_n = \exp_{G_{\lambda_i}(\zeta_n)}(1-\delta_n)\exp^{-1}_{G_{\lambda_i}(\zeta_n)} H_{\lambda_i}(\zeta_n), \\ \mathcal{B}_{n+1} = \{\nu \in \mathcal{B}_n : \rho(\vartheta_n, \nu) \leq \rho(\zeta_n, \nu)\}, \\ \zeta_{n+1} = P_{\mathcal{B}_{n+1}}(\zeta_1), \text{ for all } n \geq 1. \end{cases} \tag{9}$$

Here, $G_{\lambda_i}(\zeta) = \exp_\zeta \lambda \exp_\zeta^{-1} G_i(\zeta)$, $H_{\lambda_i}(\zeta) = \exp_\zeta \lambda \exp_\zeta^{-1} H_i(\zeta)$, $\{\delta_n\} \in (0, 1)$ satisfying $0 < \delta_1 \leq \delta_n \leq \delta_2 < 1$ and $\lim_{n\to+\infty} \delta_n = 0$.

Theorem 2. *Suppose Γ, G_i, H_i are same as defined above and $\Pi = \Phi \bigcap_i F(G_i) \neq \emptyset$. For $\mathcal{B}_1 = \Gamma$ the sequence generated by (9) converges to $P_\Pi(\zeta_1)$.*

Proof. Since $\Phi = F\left(P_{\bigcap_i F(H_i)} G_i\right) \neq \emptyset$, it can be easily seen that Φ is geodesic convex and closed and $F(G_i)$ is also geodesic convex and closed. Therefore Π is also geodesic convex and closed, indicating that $P_\Pi(\zeta_1)$ is well defined. Now we prove that the set $\mathcal{B}_n$ is closed and geodesic convex subset of Γ for all $n \geq 1$. We prove this by mathematical induction. If $\mathcal{B}_1 = \Gamma$, then it is geodesic convex and closed. Now, let us assume that $\mathcal{B}_n$ is geodesic convex and closed subset in Γ for some $n \geq 2$. Then we need to prove that $\mathcal{B}_{n+1}$ is a geodesic convex and closed subset of Γ. Since $\nu \mapsto \rho(\zeta, \nu)$ is a convex geodesic function, we can easily say that $\mathcal{B}_{n+1}$ is a closed and geodesic convex subset of Γ.

Next we prove that $\Pi \subset \mathcal{B}_n$ for all $n \geq 1$. We can easily see that $\Pi \subset \mathcal{B}_1 = \Gamma$. Now we prove that $\Pi \subset \mathcal{B}_n$ for all $n \geq 2$. If $\zeta^\dagger \in \Pi$, then $\zeta^\dagger \in \Phi$ and $\zeta^\dagger \in \bigcap_i F(G_{\lambda_i})$. Now let for a fix $n \in \mathbb{N}$, $\Delta(G_{\lambda_i}(\zeta_n), H_{\lambda_i}(\zeta_n), \zeta^\dagger) \subseteq \Gamma$ be a geodesic triangle with vertices $G_{\lambda_i}(\zeta_n)$, $H_{\lambda_i}(\zeta_n)$ and $\zeta^\dagger$, and $\Delta(\overline{G_{\lambda_i}(\zeta_n)}, \overline{H_{\lambda_i}(\zeta_n)}, \overline{\zeta^\dagger}) \subseteq \mathbb{R}^2$ a corresponding comparison triangle. We have $\rho(G_{\lambda_i}(\zeta_n) - H_{\lambda_i}(\zeta_n)) = \left\|\overline{G_{\lambda_i}(\zeta_n)} - \overline{H_{\lambda_i}(\zeta_n)}\right\|$, $\rho(G_{\lambda_i}(\zeta_n), \zeta^\dagger) = \left\|\overline{G_{\lambda_i}(\zeta_n)} - \overline{\zeta^\dagger}\right\|$,

$\rho(H_{\lambda_i}(\zeta_n), \zeta^\dagger) = \left\|\overline{H_{\lambda_i}(\zeta_n)} - \overline{\zeta^\dagger}\right\|$. Suppose $\overline{\vartheta_n} = \delta_n \overline{G_{\lambda_i}(\zeta_n)} + (1-\delta_n)\overline{H_{\lambda_i}(\zeta_n)}$ is the comparison point of ϑ_n. Using the nonexpansiveness of $G_{\lambda_i}, H_{\lambda_i}$ and Lemma 3 (2), we get

$$
\begin{aligned}
\rho^2(\vartheta_n, \zeta^\dagger) &= \|\overline{\vartheta_n} - \overline{\zeta^\dagger}\|^2 \\
&\leq \delta_n \left\|\overline{G_{\lambda_i}(\zeta_n)} - \overline{\zeta^\dagger}\right\|^2 + (1-\delta_n)\left\|\overline{H_{\lambda_i}(\zeta_n)} - \overline{\zeta^\dagger}\right\| \\
&\quad - \delta_n(1-\delta_n)\left\|\overline{G_{\lambda_i}(\zeta_n)} - \overline{H_{\lambda_i}(\zeta_n)}\right\| \\
&= \delta_n \rho^2\left(G_{\lambda_i}(\zeta_n), \zeta^\dagger\right) + (1-\delta_n)\rho^2\left(H_{\lambda_i}(\zeta_n), \zeta^\dagger\right) \\
&\quad - \delta_n(1-\delta_n)\rho^2\left(G_{\lambda_i}(\zeta_n), H_{\lambda_i}(\zeta_n)\right) \\
&\leq \delta_n \rho^2(\zeta_n, \zeta^\dagger) + (1-\delta_n)\rho^2(\zeta_n, \zeta^\dagger) - \delta_n(1-\delta_n)\rho^2\left(G_{\lambda_i}(\zeta_n), H_{\lambda_i}(\zeta_n)\right) \\
&\leq \rho^2(\zeta_n, \zeta^\dagger) - \delta_n(1-\delta_n)\rho^2\left(G_{\lambda_i}(\zeta_n), H_{\lambda_i}(\zeta_n)\right) \qquad (10) \\
&\leq \rho^2(\zeta_n, \zeta^\dagger).
\end{aligned}
$$

We can say $\zeta^\dagger \in \mathcal{B}_{n+1}$ for all $\zeta^\dagger \in \Pi$ and hence $\Pi \subset \mathcal{B}_n$ for all $n \geq 1$. Thus $\mathcal{B}_n$ is a nonempty, geodesic convex and closed subset of Γ and $\Pi \subset \mathcal{B}_{n+1} \subset \mathcal{B}_n$ for all $n \geq 1$. Hence sequence $\{\zeta_n\}$ is well defined.

Now we prove that $\lim_{n\to+\infty} \rho(\zeta_n, \zeta_1)$ exists and the sequence $\{\zeta_n\}$ is bounded. Since we have $\zeta_n = P_{\mathcal{B}_n}(\zeta_1)$ and using the fact that $\Pi \subset \mathcal{B}_{n+1} \subset \mathcal{B}_n$ we get

$$
\rho(\zeta_n, \zeta_1) \leq \rho(\zeta^\dagger, \zeta_1), \text{ for all } \zeta^\dagger \in \Pi, n \geq 1. \qquad (11)
$$

Hence, we get the sequence $\{\zeta_n\}$ is bounded. Since $\zeta_{n+1} \in \mathcal{B}_{n+1} \subset \mathcal{B}_n$ and we get $\rho(\zeta_n, \zeta_1) \leq \rho(\zeta_{n+1}, \zeta_1)$ and hence $\lim_{n\to+\infty} \rho(\zeta_n, \zeta_1)$ exists.

Next using Proposition 2 (3) we get

$$
\rho^2(\zeta_n, \zeta_{n+1}) \leq \rho^2(\zeta_{n+1}, \zeta_1) - \rho^2(\zeta_n, \zeta_1). \qquad (12)
$$

Applying summation on (12) and using (11), we have

$$
\begin{aligned}
\sum_{n=1}^{N} \rho^2(\zeta_{n+1}, \zeta_n) &\leq \sum_{n=1}^{N} \left(\rho^2(\zeta_{n+1}, \zeta_1) - \rho^2(\zeta_n, \zeta_1)\right) \\
&\leq \rho^2(\zeta_{N+1}, \zeta_1) - \rho^2(\zeta_1, \zeta_1) \\
&\leq \rho^2(\zeta^\dagger, \zeta_1),
\end{aligned}
$$

and it gives us $\sum_{n=1}^{N} \rho^2(\zeta_{n+1}, \zeta_n)$ is convergent and hence

$$
\lim_{n\to+\infty} \rho(\zeta_{n+1}, \zeta_n) = 0. \qquad (13)
$$

Since from (9) we have $\zeta_{n+1} = P_{\mathcal{B}_{n+1}}(\zeta_1) \in \mathcal{B}_{n+1}$, so using the definition of $\mathcal{B}_{n+1}$ we have $\rho(\vartheta_n, \zeta_{n+1}) \le \rho(\zeta_n, \zeta_{n+1})$. Using the above equation, we have

$$\lim_{n\to+\infty} \rho(\vartheta_n, \zeta_{n+1}) = 0. \tag{14}$$

Now

$$\rho(\vartheta_n, \zeta_n) \le \rho(\vartheta_n, \zeta_{n+1}) + \rho(\zeta_{n+1}, \zeta_n),$$

applying limit $n \to +\infty$ and using (13) and (14), we get

$$\lim_{n\to+\infty} \rho(\vartheta_n, \zeta_n) = 0. \tag{15}$$

Using (10), we will have

$$\begin{aligned} \delta_1(1-\delta_2)\rho^2(G_{\lambda_i}(\zeta_n), H_{\lambda_i}(\zeta_n)) &\le \delta_n(1-\delta_n)\rho^2(G_{\lambda_i}(\zeta_n), H_{\lambda_i}(\zeta_n)) \\ &\le \rho^2(\zeta_n, \zeta^\dagger) - \rho^2(\vartheta_n, \zeta^\dagger) \\ &\le C_1\rho(\zeta_n, \vartheta_n). \end{aligned} \tag{16}$$

Where $C_1 = \sup_{n\ge1}\{\rho(\zeta_n, \zeta^\dagger) + \rho(\vartheta_n, \zeta^\dagger)\}$. Using (15), we get

$$\lim_{n\to+\infty} \rho(G_{\lambda_i}(\zeta_n), H_{\lambda_i}(\zeta_n)) = 0. \tag{17}$$

Next we prove that the $\{\zeta_n\}$ is a Cauchy sequence in Γ and $\{\zeta_n\}$ converges to $\zeta \in \Pi$. Since $P_{\mathcal{B}_n}$ is firmly nonexpansive, $\zeta_m = P_{\mathcal{B}_m}(\zeta_1) \in \mathcal{B}_m \subset \mathcal{B}_n$ for any $n, m \in \mathbb{N}$ and $m > n$. If we take $\mathcal{B} = \mathcal{B}_n$, $\zeta = \zeta_1$ and $\vartheta = \zeta_m$, we get

$$\rho^2\left(P_{\mathcal{B}_n}\zeta_1\right) = \rho^2(\zeta_n, \zeta_m) \le \rho^2(\zeta_1, \zeta_m) - \rho^2(\zeta_n, \zeta_1),$$

Applying limit $n, m \to +\infty$ and using proposition 2 (3), we get

$$\lim_{n,m\to+\infty} \rho(\zeta_n, \zeta_m) \to 0.$$

It gives us that the sequence $\{\zeta_n\}$ is a Cauchy sequence and since Γ is complete we can assume that $\lim_{n\to+\infty} \zeta_n = \zeta \in \Gamma$.

Now we prove that $\zeta \in \Pi$. From (9) we have

$$\rho(\vartheta_n, G_{\lambda_i}(\zeta_n)) = \delta_n\rho(G_{\lambda_i}(\zeta_n), H_{\lambda_i}(\zeta_n)) \le \delta_2\rho(G_{\lambda_i}(\zeta_n), H_{\lambda_i}(\zeta_n)).$$

Applying limit $n \to +\infty$ and using (17), we get

$$\lim_{n\to+\infty} \rho(\vartheta_n, G_{\lambda_i}(\zeta_n)) = 0. \tag{18}$$

And

$$\rho(G_{\lambda_i}(\zeta_n), \zeta_n) \le \rho(G_{\lambda_i}(\zeta_n), \vartheta_n) + \rho(\vartheta_n, \zeta_n).$$

Applying limit $n \to +\infty$ to the above equation and using (18) and (15), we get

$$\lim_{n\to+\infty} \rho(G_{\lambda_i}(\zeta_n), \zeta_n) = 0. \tag{19}$$

Similarly,

$$\rho(H_{\lambda_i}(\zeta_n), \zeta_n) \le \rho(H_{\lambda_i}(\zeta_n), G_{\lambda_i}(\zeta_n)) + \rho(G_{\lambda_i}(\zeta_n), \zeta_n).$$

Applying limit $n \to +\infty$ to the above equation and using (18) and (19), we get

$$\lim_{n\to+\infty} \rho(H_{\lambda_i}(\zeta_n), \zeta_n) = 0. \tag{20}$$

Since H_{λ_i} nonexpansive so it is demiclosed at 0 and hence $\zeta \in \bigcap_i H_i$, further from (19) we can also say $\zeta \in \bigcap_i G_i$. Next we prove that $\zeta \in \Phi$. Suppose $\zeta^* \in \bigcap_i H_i$ is arbitrary such that $\zeta \neq \zeta^*$. Let the triangles $\Delta(G_{\lambda_i}(\zeta_n), H_{\lambda_i}(\zeta_n), \zeta^*)$, $\Delta(G_{\lambda_i}(\zeta^*), H_{\lambda_i}(\zeta_n), \zeta^*)$ and $\Delta(G_{\lambda_i}(\zeta_n), G_{\lambda_i}(\zeta^*), H_{\lambda_i}(\zeta_n))$ then there exist comparison triangles $\Delta(\overline{G_{\lambda_i}(\zeta_n)}, \overline{H_{\lambda_i}(\zeta_n)}, \overline{\zeta^*})$, $\Delta(\overline{G_{\lambda_i}(\zeta^*)}, \overline{H_{\lambda_i}(\zeta_n)}, \overline{\zeta^*})$ and $\Delta(\overline{G_{\lambda_i}(\zeta_n)}, \overline{G_{\lambda_i}(\zeta^*)}, \overline{H_{\lambda_i}(\zeta_n)})$ such that $\rho(G_{\lambda_i}(\zeta_n), H_{\lambda_i}(\zeta_n)) = \left\|\overline{G_{\lambda_i}(\zeta_n)} - \overline{H_{\lambda_i}(\zeta_n)}\right\|$, $\rho(G_{\lambda_i}(\zeta_n), \zeta^*) = \left\|\overline{G_{\lambda_i}(\zeta_n)} - \overline{\zeta^*}\right\|$, $\rho(H_{\lambda_i}(\zeta_n), \zeta^*) = \left\|\overline{H_{\lambda_i}(\zeta_n)} - \overline{\zeta^*}\right\|$, $\rho(G_{\lambda_i}(\zeta^*), \zeta^*) = \left\|\overline{G_{\lambda_i}(\zeta^*)} - \overline{\zeta^*}\right\|$, $\rho(H_{\lambda_i}(\zeta_n), \zeta^*) = \left\|\overline{H_{\lambda_i}(\zeta_n)} - \overline{\zeta^*}\right\|$, and $\rho(G_{\lambda_i}(\zeta_n), G_{\lambda_i}(\zeta^*)) = \left\|\overline{G_{\lambda_i}(\zeta_n)} - \overline{G_{\lambda_i}(\zeta^*)}\right\|$. Suppose θ be the angle at ζ^* in the triangle $\Delta(G_{\lambda_i}(\zeta^*), H_{\lambda_i}(\zeta_n), \zeta^*)$ and $\overline{\theta}$ be the angle at $\overline{\zeta^*}$ in the comparison triangle $\Delta(\overline{G_{\lambda_i}(\zeta^*)}, \overline{H_{\lambda_i}(\zeta_n)}, \overline{\zeta^*})$. Then $\theta \le \overline{\theta}$, and thus, $\cos(\overline{\theta}) \le \cos(\theta)$. Since $\overline{\vartheta_n} = \delta_n \overline{G_{\lambda_i}(\zeta_n)} + (1-\delta_n)\overline{H_{\lambda_i}(\zeta_n)}$ is the comparison point of ϑ_n. Using Proposition 3 and Lemma 3 (2), we have

$$\begin{aligned}
\rho^2(\vartheta_n, \zeta^*) &\le \left\|\overline{\vartheta_n} - \overline{\zeta^*}\right\|^2 \\
&= \left\|\delta_n \overline{G_{\lambda_i}(\zeta_n)} + (1-\delta_n)\overline{H_{\lambda_i}(\zeta_n)} - \overline{\zeta^*}\right\| \\
&= \delta_n^2 \left\|\overline{G_{\lambda_i}(\zeta_n)} - \overline{\zeta^*}\right\|^2 + (1-\delta_n)^2 \left\|\overline{H_{\lambda_i}(\zeta_n)} - \overline{\zeta^*}\right\|^2 \\
&+ 2\delta_n(1-\delta_n)\left\langle \overline{G_{\lambda_i}(\zeta_n)} - \overline{\zeta^*}, \overline{H_{\lambda_i}(\zeta_n)} - \overline{\zeta^*} \right\rangle \\
&= \delta_n^2 \left\|\overline{G_{\lambda_i}(\zeta_n)} - \overline{\zeta^*}\right\|^2 + (1-\delta_n)^2 \left\|\overline{H_{\lambda_i}(\zeta_n)} - \overline{\zeta^*}\right\|^2 \\
&+ 2\delta_n(1-\delta_n)\left(\left\langle \overline{G_{\lambda_i}(\zeta_n)} - \overline{G_{\lambda_i}(\zeta^*)}, \overline{H_{\lambda_i}(\zeta_n)} - \overline{\zeta^*} \right\rangle + \left\langle \overline{G_{\lambda_i}(\zeta^*)} - \overline{\zeta^*}, \overline{H_{\lambda_i}(\zeta_n)} - \overline{\zeta^*} \right\rangle\right) \\
&\le \delta_n^2 \left\|\overline{G_{\lambda_i}(\zeta_n)} - \overline{\zeta^*}\right\|^2 + (1-\delta_n)^2 \left\|\overline{H_{\lambda_i}(\zeta_n)} - \overline{\zeta^*}\right\|^2 \\
&+ 2\delta_n(1-\delta_n)\left(\left\|\overline{G_{\lambda_i}(\zeta_n)} - \overline{G_{\lambda_i}(\zeta^*)}\right\| \left\|\overline{H_{\lambda_i}(\zeta_n)} - \overline{\zeta^*}\right\| \right. \\
&\left. + \left\|\overline{G_{\lambda_i}(\zeta^*)} - \overline{\zeta^*}\right\| \left\|\overline{H_{\lambda_i}(\zeta_n)} - \overline{\zeta^*}\right\| \cos(\overline{\theta})\right) \\
&\le \delta_n^2 \rho^2\left(G_{\lambda_i}(\zeta_n), \zeta^*\right) + (1-\delta_n)^2 \rho^2\left(H_{\lambda_i}(\zeta_n), \zeta^*\right)
\end{aligned}$$

$$+ 2\delta_n(1-\delta_n)\rho\left(G_{\lambda_i}(\zeta_n), G_{\lambda_i}(\zeta^*)\right)\rho(H_{\lambda_i}(\zeta_n), \zeta^*)$$
$$+ 2\delta_n(1-\delta_n)\rho\left(G_{\lambda_i}(\zeta^*), \zeta^*\right)\rho(H_{\lambda_i}(\zeta_n), \zeta^*)\cos(\theta)$$

$$\begin{aligned}
\rho^2(\vartheta_n, \zeta^*) &\leq \delta_n^2\rho^2(G_{\lambda_i}(\zeta_n), \zeta^*) + (1-\delta_n)^2\rho^2(\zeta_n, \zeta^*) + 2\delta_n(1-\delta_n)\rho^2(\zeta_n, \zeta^*) \\
&\quad + 2\delta_n(1-\delta_n)\mathcal{R}\left(\exp_{\zeta^*}^{-1} G_{\lambda_i}(\zeta^*), \exp_{\zeta^*}^{-1} H_{\lambda_i}(\zeta_n)\right) \\
&= \delta_n^2\rho^2(G_{\lambda_i}(\zeta_n), \zeta^*) + (1-\delta_n^2)\rho^2(\zeta_n, \zeta^*) \\
&\quad + 2\delta_n(1-\delta_n)\mathcal{R}\left(\exp_{\zeta^*}^{-1} G_{\lambda_i}(\zeta^*), \exp_{\zeta^*}^{-1} H_{\lambda_i}(\zeta_n)\right) \\
&= \delta_n^2\rho^2(G_{\lambda_i}(\zeta_n), \zeta^*) + \rho^2(\zeta_n, \zeta^*) \\
&\quad + 2\delta_n(1-\delta_n)\mathcal{R}\left(\exp_{\zeta^*}^{-1} G_{\lambda_i}(\zeta^*), \exp_{\zeta^*}^{-1} H_{\lambda_i}(\zeta_n)\right),
\end{aligned}$$

and we get

$$\begin{aligned}
&0 \leq \delta_n^2\rho^2(G_{\lambda_i}, \zeta^*) + \rho^2(\zeta_n, \zeta^*) - \rho^2(\vartheta_n, \zeta^*) \\
&\quad + 2\delta_n(1-\delta_n)\mathcal{R}\left(\exp_{\zeta^*}^{-1} G_{\lambda_i}(\zeta^*), \exp_{\zeta^*}^{-1} H_{\lambda_i}(\zeta_n)\right) \\
&\quad \leq \delta_n^2\rho^2(G_{\lambda_i}(\zeta_n), \zeta^*) + C_2\rho(\zeta_n, \vartheta_n) + 2\delta_n(1-\delta_n)\mathcal{R}\left(\exp_{\zeta^*}^{-1} G_{\lambda_i}(\zeta^*), \exp_{\zeta^*}^{-1} H_{\lambda_i}(\zeta_n)\right).
\end{aligned}$$

Here $C_2 = \sup_{n\geq 1}\{\rho(\zeta_n, \zeta^*) + \rho(\vartheta_n, \zeta^*)\}$. Since $0 < \delta_1 \leq \delta_n \leq \delta_2 < 1$ we can have $\frac{1}{\delta_n} \leq \frac{1}{\delta_1}$. Therefore above equation becomes

$$\begin{aligned}
0 &\leq \frac{\delta_n}{2}\rho^2(G_{\lambda_i}(\zeta_n), \zeta^*) + \frac{C_2}{2\delta_n}\rho(\zeta_n, \vartheta_n) + (1-\delta_n)\mathcal{R}\left(\exp_{\zeta^*}^{-1} G_{\lambda_i}(\zeta^*), \exp_{\zeta^*}^{-1} H_{\lambda_i}(\zeta_n)\right) \\
&\leq \frac{\delta_n}{2}\rho^2(G_{\lambda_i}(\zeta_n), \zeta^*) + \frac{C_2}{2\delta_1}\rho(\zeta_n, \vartheta_n) + (1-\delta_n)\mathcal{R}\left(\exp_{\zeta^*}^{-1} G_{\lambda_i}(\zeta^*), \exp_{\zeta^*}^{-1} H_{\lambda_i}(\zeta_n)\right)
\end{aligned}$$

Since $\zeta \in \bigcap_i F(H_{\lambda_i})$, $\lim_{n\to+\infty} \zeta_n = \zeta$ and $\lim_{n\to+\infty} \delta_n = 0$, and using (14), we get

$$\mathcal{R}\left(\exp_{\zeta^*}^{-1} G_{\lambda_i}(\zeta^*), -\exp_{\zeta^*}^{-1} \zeta\right) \leq 0,$$

i.e.

$$\mathcal{R}\left(\exp_{\zeta^*}^{-1} G_{\lambda_i}(\zeta^*), P_{\zeta^*,\zeta}\exp_{\zeta}^{-1} \zeta^*\right) \leq 0, \quad \text{for all } \zeta^* \in \bigcap_i F(H_{\lambda_i}). \tag{21}$$

Since $\zeta \in \bigcap_i F(H_{\lambda_i})$, $\bigcap_i F(H_{\lambda_i})$ is geodesic convex, we get $\exp_\zeta t\exp_\zeta^{-1}\mu \in \bigcap_i F(H_{\lambda_i})$ for any $\mu \in \bigcap_i F(H_{\lambda_i})$ and $t \in (0,1)$. Replacing ζ^* by $u_t = \exp_\zeta t\exp_\zeta^{-1}\mu \in \bigcap_i F(H_{\lambda_i})$ where $\mu \in \bigcap_i F(H_{\lambda_i})$ and $t \in (0,1)$, in the equation (21), we get

$$\mathcal{R}\left(\exp_{\exp_\zeta t\exp_\zeta^{-1}\mu}^{-1} G_{\lambda_i}(\exp_\zeta t\exp_\zeta^{-1}\mu), P_{u_t,\zeta}\exp_\zeta^{-1}\exp_\zeta t\exp_\zeta^{-1}\mu\right) \leq 0,$$

which gives us

$$\mathcal{R}\left(\exp^{-1}_{\exp_\zeta t\exp^{-1}_\zeta \mu} G_{\lambda_i}(\exp_\zeta t\exp^{-1}_\zeta \mu), P_{u_t,\zeta} t\exp^{-1}_\zeta \mu\right) \leq 0.$$

Since the map P is linear, we get

$$\mathcal{R}\left(\exp^{-1}_{\exp_\zeta t\exp^{-1}_\zeta \mu} G_{\lambda_i}(\exp_\zeta t\exp^{-1}_\zeta \mu), P_{u_t,\zeta} \exp^{-1}_\zeta \mu\right) \leq 0.$$

Applying $t \to 0$ then $u_t \to \zeta$. Since $\exp_\zeta 0 = \zeta$ for all $\zeta \in \Gamma$, we get

$$\mathcal{R}\left(\exp^{-1}_\zeta G_{\lambda_i}(\zeta), P_{\zeta,\zeta} \exp^{-1}_\zeta \mu\right) \leq 0.$$

Thus

$$\mathcal{R}\left(\exp^{-1}_\zeta G_{\lambda_i}(\zeta), \exp^{-1}_\zeta \mu\right) \leq 0, \text{ for all } \mu \in \bigcap_i F(H_{\lambda_i}). \tag{22}$$

Hence, $\zeta \in \Phi$, and we have $\zeta \in \Pi$. Now finally we prove that the sequence $\{\zeta_n\}$ converges to $\zeta = P_\Pi(\zeta_1)$. From $\zeta_n = P_{\mathcal{B}_n}(\zeta_1)$, using Proposition 2 (1), we get

$$\mathcal{R}\left(\exp^{-1}_{\zeta_n} \zeta_1, \exp^{-1}_{\zeta_n} \nu\right) \leq 0, \quad \text{for all } \nu \in \mathcal{B}_n.$$

Using the fact that $\Pi \subset \mathcal{B}_n$, we get

$$\mathcal{R}\left(\exp^{-1}_{\zeta_n} \zeta_1, \exp^{-1}_{\zeta_n} \nu\right) \leq 0, \quad \text{for all } \nu \in \Pi. \tag{23}$$

Applying limit $n \to +\infty$ in the above equation, we get

$$\mathcal{R}\left(\exp^{-1}_{\zeta} \zeta_1, \exp^{-1}_{\zeta} \nu\right) \leq 0, \quad \text{for all } \nu \in \Pi,$$

i.e. $\lim_{n\to+\infty} \zeta_n = \zeta = P_\Pi(\zeta_1)$. Thus the proof is complete.

4. Examples

In this section, we present a couple of examples which are not nonexpansive but does satisfy strict pseudo-contraction. Later, we also present a nontrivial example which illustrates the Theorem 2.

Example 1. [24] *Let $\Gamma = \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{R}$ be a mapping defined as*

$$G(\zeta) = -\left(\tan^{-1}(\zeta) + \frac{\sin(\zeta) + \cos(\zeta)}{4}\right), \quad \text{for all } \zeta \in \mathbb{R}.$$

The, mapping G is a β-strict pseudocontractive mapping with $\beta = \frac{5}{9}$, but it is not a nonexpansive mapping.

Example 2. [24] *Let* $\Gamma = \mathbb{R}^2$ *and* $G : \Gamma \to \Gamma$ *be a mapping defined as*

$$G(\zeta_1, \zeta_2) = \left(-2\tan^{-1}(\zeta_1), 2\cot^{-1}(\zeta_2)\right), \text{ for all } (\zeta_1, \zeta_2) \in \mathbb{R}^2.$$

The mapping G *is a* β*-strict pseudocontractive mapping with* $\beta = \frac{3}{4}$ *but it is not a nonexpansive mapping.*

Example 3. *Define the inner product on* $\mathbb{R}$ *as*

$$\langle \vartheta, \zeta \rangle = -\vartheta\zeta \text{ for all } \vartheta, \zeta \in \mathbb{R}.$$

Define

$$\mathbb{H} = \{\zeta \in \mathbb{R} : \langle \zeta, \zeta \rangle = -1\}.$$

Then this inner product induces Riemannian metric ρ*, on tangent space* $T_p\mathbb{H} \subset T_p\mathbb{R}$ *for* $p \in \mathbb{H}$ *defined by*

$$\rho(\vartheta, \zeta) = \cosh^{-1}(-\langle \vartheta, \zeta \rangle) \text{ for all } \vartheta, \zeta \in \mathbb{H}.$$

Then $(\mathbb{H}, \rho)$ *is the Hadamard manifold with sectional curvature* -1 *at any point.*

Now, let $\Gamma = \mathbb{H}$*,* $G_1(\zeta) = -2\zeta, G_2(\zeta) = -5\zeta$ *and* $H_1(\zeta) = \frac{1}{3}\zeta + \frac{1}{2}\sin\zeta, H_2(\zeta) = -6\zeta$*. Here mappings* G_1, G_2 *are* $\frac{1}{3}, \frac{2}{3}$ *strict pseudocontractive mappings, respectively. And the mappings* H_1, H_2 *are* $\frac{1}{4}, \frac{5}{7}$ *strict pseudocontractive mappings, respectively. Also* $\bigcap_{i=1}^{2} F(G_i) \bigcap_{i=1}^{2} F(H_i) \bigcap \Phi = \{0\}$*. It satisfies all the conditions of the Theorem 2, hence for* $i = 1, 2$ *the sequence generated by* (9) *converges to* $P_\Pi(\zeta_1)$*.*

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Conflicts of Interest

The authors declare that they have no conflicts of interests.

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Data Availability

There is no any availability of data.

References

[1] P. Hartman and G. Stampacchia. On some non-linear elliptic differential-functional equations. *Acta Math.*, 115:271–310, 1966.

[2] P. Patel and R. Pant. Viscosity approximation methods for quasi-nonexpansive mappings in Banach spaces. *Filomat*, 35(9):3113–3126, 2021.

[3] A. Moudafi. Viscosity approximation methods for fixed-points problems. *J. Math. Anal. Appl.*, 241(1):46–55, 2000.

[4] H. K. Xu. Viscosity approximation methods for nonexpansive mappings. *J. Math. Anal. Appl.*, 298(1):279–291, 2004.

[5] V. Berinde and M. Păcurar. Krasnoselskij-type algorithms for fixed point problems and variational inequality problems in Banach spaces. *Topology Appl.*, 340:Paper No. 108708, 15, 2023.

[6] P. Patel and R. Shukla. Viscosity approximation methods for generalized modification of the system of equilibrium problem and fixed point problems of an infinite family of nonexpansive mappings. *Mathematics and Statistics*, 12(4):339–347, 2024.

[7] P. Patel and R. Shukla. Mann-Dotson's algorithm for a countable family of non-self strict pseudo-contractive mappings. *Rend. Circ. Mat. Palermo (2)*, 73(1):225–240, 2024.

[8] S. Z. Németh. Variational inequalities on Hadamard manifolds. *Nonlinear Anal.*, 52(5):1491–1498, 2003.

[9] A. Moudafi and P.-E. Maingé. Towards viscosity approximations of hierarchical fixed-point problems. *Fixed Point Theory Appl.*, pages Art. ID 95453, 10, 2006.

[10] H. K. Xu. Viscosity method for hierarchical fixed point approach to variational inequalities. *Taiwanese J. Math.*, 14(2):463–478, 2010.

[11] P. E. Maingé and A. Moudafi. Strong convergence of an iterative method for hierarchical fixed-point problems. *Pac. J. Optim.*, 3(3):529–538, 2007.

[12] Y. Zhao, X. Liu, and R. Sun. Iterative algorithms of common solutions for a hierarchical fixed point problem, a system of variational inequalities, and a split equilibrium problem in Hilbert spaces. *J. Inequal. Appl.*, pages Paper No. 111, 22, 2021.

[13] Y. Yao, Y. J. Cho, and Y. C. Liou. Iterative algorithms for hierarchical fixed points problems and variational inequalities. *Math. Comput. Modelling*, 52(9-10):1697–1705, 2010.

[14] L. Ćirić, A. Rafiq, S. Radenović, M. Rajović, and J. S. Ume. On Mann implicit iterations for strongly accretive and strongly pseudo-contractive mappings. *Appl. Math. Comput.*, 198(1):128–137, 2008.

[15] D. Filali, M. Dilshad, M. Akram, F. Babu, and I. Ahmad. Viscosity method for hierarchical variational inequalities and variational inclusions on Hadamard manifolds. *J. Inequal. Appl.*, pages Paper No. 66, 20, 2021.

[16] P. Patel and R. Shukla. Common solution for a finite family of equilibrium problems, inclusion problems and fixed points of a finite family of nonexpansive mappings in hadamard manifolds. *Sahand Communications in Mathematical Analysis*, 21(1):255–271, 2024.

[17] S. Chang, J. C. Yao, L. Yang, C. F. Wen, and D. P. Wu. Convergence analysis for variational inclusion problems equilibrium problems and fixed point in Hadamard manifolds. *Numer. Funct. Anal. Optim.*, 42(5):567–582, 2021.

[18] T. Sakai. *Riemannian geometry*, volume 149 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1996. Translated from the 1992 Japanese original by the author.

[19] C. Li, G. López, V. Martín-Márquez, and J. H. Wang. Resolvents of set-valued monotone vector fields in Hadamard manifolds. *Set-Valued Var. Anal.*, 19(3):361–383, 2011.

[20] M. R. Bridson and A Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[21] Q. H. Ansari, F. Babu, and D. R. Sahu. Iterative algorithms for system of variational inclusions in Hadamard manifolds. *Acta Math. Sci. Ser. B (Engl. Ed.)*, 42(4):1333–1356, 2022.

[22] S. Z. Németh. Monotonicity of the complementary vector field of a nonexpansive map. *Acta Math. Hungar.*, 84(3):189–197, 1999.

[23] J. X. da Cruz Neto, O. P. Ferreira, and L. R. Lucambio Pérez. Monotone point-to-set vector fields. *Balkan Journal of Geometry and its Applications*, 5(1):69–80, 2000.

[24] S. Baiya and K. Ungchittrakool. Modified inertial Mann's algorithm and inertial hybrid algorithm for k-strict pseudo-contractive mappings. *Carpathian J. Math.*, 39(1):27–43, 2023.