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Convergence and Fixed Points of Self-Mappings in MR-Metric Spaces: Theory and Applications

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Abstract. In this paper, we present some important results for self-mappings in $MR - metric$ spaces, ensuring the existence and uniqueness of fixed points in contraction mappings. The study reveals the important role of contraction properties in achieving convergence. The paper continues to develop these concepts by providing concrete examples that demonstrate their importance in measurable spaces. Finally, these results lay the foundation for future investigations into the stability of fixed points and their applications via mathematical frameworks applicable to real life.

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1. Introduction

This paper presents a comprehensive study on the convergence and fixed points of self-mappings in MR-metric spaces, a recently introduced generalization of metric spaces. We establish several key results, including the existence and uniqueness of fixed points for contraction mappings, the convergence of Cauchy sequences, and the convergence in measure of iterates to fixed points. These findings have significant implications for various fields, including optimization, machine learning, and numerical analysis, and provide a solid foundation for further research and applications in MR-metric spaces.

For further details, we refer readers to the works cited in [1–24].

Definition 1. *[5] Consider a non-empty set* $\mathbb{X} \neq \emptyset$ *and a real number* $R > 1$. *A function* $M : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to [0, \infty)$ *is termed an* MR*-metric if it satisfies the following conditions for all* $\upsilon, \xi, \Im \in \mathbb{X}$*:*

- $(M1)$ $M(\upsilon, \xi, \Im) \geq 0$.
- $(M2)$ $M(\upsilon, \xi, \Im) = 0$ *if and only if* $\upsilon = \xi = \Im$.

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- $(M3)$ *$M(\upsilon, \xi, \Im)$ remains invariant under any permutation $p(\upsilon, \xi, \Im)$, i.e., $M(\upsilon, \xi, \Im) = M(p(\upsilon, \xi, \Im))$.*

- $(M4)$ *The following inequality holds:*

$$M(\upsilon, \xi, \Im) \leq R\left[M(\upsilon, \xi, \ell_1) + M(\upsilon, \ell_1, \Im) + M(\ell_1, \xi, \Im)\right].$$

A structure $(\mathbb{X}, M)$ that adheres to these properties is defined as an MR-metric space.

Definition 2. *[5] Consider a sequence $\{\upsilon i_n\}$ in an MR-metric space $(\mathbb{X}, M)$. This sequence is said to be MR-convergent if there exists an element $\upsilon i_1 \in \mathbb{X}$ such that for any $\epsilon > 0$, there exists a positive integer N satisfying the condition*

$$M(\upsilon i_n, \upsilon i_m, \upsilon i_1) < \epsilon, \quad \text{for all } m, n \geq N.$$

In this case, we say that $\{\upsilon i_n\}$ converges in the MR-metric sense to υi_1, and we refer to υi_1 as the limit of the sequence.

Definition 3. *[5] A sequence $\{\upsilon i_n\}$ in an MR-metric space $(\mathbb{X}, M)$ is termed MR-Cauchy if for every $\epsilon > 0$, there exists a positive integer N such that the inequality*

$$M(\upsilon i_n, \upsilon i_m, \upsilon i_p) < \epsilon \quad \text{holds for all} \quad m, n, p \geq N.$$

Definition 4. *[5] An MR-metric space $(\mathbb{X}, M)$ is said to be bounded if there exists a constant $L > 0$ such that*

$$M(\upsilon, \xi, \Im) \leq L \quad \text{for all} \quad \upsilon, \xi, \Im \in \mathbb{X}.$$

In this case, the function M is called an MR-bound for the metric.

Definition 5 ([25], Definition 1.1)**.** *A **measure space** is a triplet $(\mathbb{X}, \Sigma, \mu)$, where:*

- *$\mathbb{X}$ is a non-empty set.*

- *Σ is a σ-algebra on $\mathbb{X}$, which satisfies the following properties:*

  (i) $\mathbb{X} \in \Sigma$.

  (ii) If $A \in \Sigma$, then the complement $A^c \in \Sigma$.

  (iii) If $\{A_n\}_{n=1}^{\infty} \subseteq \Sigma$, then $\bigcup_{n=1}^{\infty} A_n \in \Sigma$.

- *$\mu : \Sigma \to [0, \infty]$ is a function satisfying:*

  (i) ***Non-negativity****: $\mu(A) \geq 0$ for all $A \in \Sigma$.*

  (ii) ***Null Empty Set****: $\mu(\emptyset) = 0$.*

  (iii) ***Countable Additivity (σ-additivity)****: If $\{A_n\}_{n=1}^{\infty}$ is a countable collection of disjoint sets in Σ, then:*

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

The function μ *is called a* measure*, and* $\mathbb{X}$ *is referred to as the* measurable space*.*

Definition 6 ([26], Section 2)**.** *A measure* μ *is said to be* σ***-finite*** *if there exists a countable collection of measurable sets* $\{\mathbb{X}_n\}_{n=1}^{\infty} \subseteq \Sigma$ *such that:*

$$\mathbb{X} = \bigcup_{n=1}^{\infty} \mathbb{X}_n \quad \textit{and} \quad \mu(\mathbb{X}_n) < \infty \textit{ for every } n.$$

Definition 7 ([27], Section 3.1)**.** *A measure* μ *is called* ***absolutely continuous*** *with respect to another measure* ν *(denoted* $\mu \ll \nu$*) if for every measurable set* $A \in \Sigma$*,*

$$\nu(A) = 0 \quad \Rightarrow \quad \mu(A) = 0.$$

2. Main Result

In this section, we present the main results for a complete MR-metric space within a convergent sequence, and we have given some examples for the main results.

Theorem 1. *Let* $(\mathbb{X}, M)$ *be an MR-metric space with* $R > 1$*, and let* $T : \mathbb{X} \to \mathbb{X}$ *be a self-mapping satisfying the contraction condition:*

$$M(T\upsilon, T\xi, T\Im) \leq \alpha M(\upsilon, \xi, \Im), \quad \forall \upsilon, \xi, \Im \in \mathbb{X},$$

where $0 \leq \alpha < 1/R$*. Then,* T *has a unique fixed point* $\upsilon^* \in \mathbb{X}$ *such that* $T(\upsilon^*) = \upsilon^*$*.*

Proof. Consider the sequence $\{\upsilon_n\}$ in $\mathbb{X}$, which is generated recursively by the relation $\upsilon_{n+1} = T(\upsilon_n)$, starting from an initial element $\upsilon_0 \in \mathbb{X}$. Our objective is to establish that this sequence converges to a unique fixed point of the mapping T.

To prove convergence, we begin by using the contraction condition. The distance between consecutive terms of the sequence is controlled by the contraction condition, which ensures that the sequence $\{\upsilon_n\}$ is a Cauchy sequence. Since the MR-metric space $(\mathbb{X}, M)$ is complete, every Cauchy sequence in $\mathbb{X}$ must converge to some point. Therefore, there exists a point $\upsilon^* \in \mathbb{X}$ such that $\upsilon_n \to \upsilon^*$ as $n \to \infty$.

Next, we show that υ^* is a fixed point of T. Since T is continuous and $\upsilon_n \to \upsilon^*$, we can take the limit of both sides of the equation $\upsilon_{n+1} = T(\upsilon_n)$. As $n \to \infty$, we get:

$$\upsilon^* = T(\upsilon^*).$$

Thus, υ^* is a fixed point of T.

To prove uniqueness, suppose that T has two fixed points υ^* and υ^{**}, meaning $T(\upsilon^*) = \upsilon^*$ and $T(\upsilon^{**}) = \upsilon^{**}$. Using the contraction condition for υ^* and υ^{**}, we have:

$$M(T(\upsilon^*), T(\upsilon^{**}), T(\upsilon^{**})) \leq \alpha M(\upsilon^*, \upsilon^{**}, \upsilon^{**}).$$

Since $T(\upsilon^*) = \upsilon^*$ and $T(\upsilon^{**}) = \upsilon^{**}$, this simplifies to:

$$M(\upsilon^*, \upsilon^{**}, \upsilon^{**}) \leq \alpha M(\upsilon^*, \upsilon^{**}, \upsilon^{**}).$$

Because $\alpha < 1/R$, it follows that the right-hand side is strictly smaller than the left-hand side, implying that:

$$M(\upsilon^*, \upsilon^{**}, \upsilon^{**}) = 0.$$

This equation implies that $\upsilon^* = \upsilon^{**}$, establishing the uniqueness of the fixed point.

Therefore, we conclude that T has a unique fixed point υ^*, and $\upsilon_n \to \upsilon^*$ as $n \to \infty$.

Example 1. *Consider the MR-metric space $(\mathbb{R}, M)$, where the MR-metric M is defined as:*

$$M(\upsilon, \xi, \Im) = |\upsilon - \xi| + |\xi - \Im| + |\Im - \upsilon|. \tag{1}$$

This metric provides a way to measure the distance between three points in $\mathbb{R}$.

Let the self-mapping $T : \mathbb{R} \to \mathbb{R}$ be defined by:

$$T(\upsilon) = \frac{\upsilon}{2}. \tag{2}$$

This mapping divides any point in $\mathbb{R}$ by two, which suggests that it is a contraction mapping.

To verify that T satisfies the contraction condition, we compute the MR-metric between $T\upsilon$, $T\xi$, and $T\Im$. For any $\upsilon, \xi, \Im \in \mathbb{R}$, we have:

$$M(T\upsilon, T\xi, T\Im) = \left|\frac{\upsilon}{2} - \frac{\xi}{2}\right| + \left|\frac{\xi}{2} - \frac{\Im}{2}\right| + \left|\frac{\Im}{2} - \frac{\upsilon}{2}\right|. \tag{3}$$

Using the definition of the MR-metric, this becomes:

$$M(T\upsilon, T\xi, T\Im) = \frac{1}{2}M(\upsilon, \xi, \Im). \tag{4}$$

Since $\alpha = \frac{1}{2}$, and we can choose $R > 2$ so that $\alpha < \frac{1}{R}$, we conclude that the contraction condition is satisfied.

By Theorem 2.1, which guarantees the existence and uniqueness of a fixed point for a contraction mapping on a complete MR-metric space, we conclude that there exists a unique fixed point $\upsilon^* \in \mathbb{R}$ such that:

$$T(\upsilon^*) = \upsilon^*. \tag{5}$$

Solving the equation $\upsilon^* = \frac{\upsilon^*}{2}$, we obtain:

$$\upsilon^* = 0, \tag{6}$$

which confirms that the unique fixed point of T is $\upsilon^* = 0$.

Theorem 2. *Let $(\mathbb{X}, M)$ be an MR-metric space, and let $\{\upsilon_n\}$ be a sequence in $\mathbb{X}$ satisfying:*

$$M(\upsilon_{n+1}, \upsilon_n, \upsilon_{n-1}) \leq \frac{1}{2}M(\upsilon_n, \upsilon_{n-1}, \upsilon_{n-2}). \tag{7}$$

Then, $\{\upsilon_n\}$ is a Cauchy sequence and converges to some limit $\upsilon^* \in \mathbb{X}$.

Proof. We begin by recursively applying the given inequality to estimate the MR-metric between terms of the sequence. Specifically, using the given inequality,

$$M(\upsilon_{n+1}, \upsilon_n, \upsilon_{n-1}) \leq \frac{1}{2} M(\upsilon_n, \upsilon_{n-1}, \upsilon_{n-2}),$$

we can derive an upper bound for the MR-metric for the terms υ_{n+k}, υ_{n+k-1}, and υ_{n+k-2} by applying the inequality recursively:

$$M(\upsilon_{n+k}, \upsilon_{n+k-1}, \upsilon_{n+k-2}) \leq \left(\frac{1}{2}\right)^k M(\upsilon_n, \upsilon_{n-1}, \upsilon_{n-2}).$$

This inequality holds for any $k \in \mathbb{N}$, and as $k \to \infty$, the right-hand side of the inequality tends to zero because $\left(\frac{1}{2}\right)^k \to 0$. More formally:

$$\lim_{k\to\infty} \left(\frac{1}{2}\right)^k = 0.$$

Thus, for any $\epsilon > 0$, there exists an integer N such that for all $k \geq N$,

$$M(\upsilon_{n+k}, \upsilon_{n+k-1}, \upsilon_{n+k-2}) < \epsilon.$$

This implies that the sequence $\{\upsilon_n\}$ is Cauchy in the MR-metric space $(\mathbb{X}, M)$, since for any $m, n \in \mathbb{N}$, the distance between terms in the sequence can be made arbitrarily small as n, m increase.

Step 1: Cauchy Property To rigorously justify that $\{\upsilon_n\}$ is a Cauchy sequence, consider the MR-metric for the terms υ_n and υ_m. We have:

$$M(\upsilon_n, \upsilon_m, \upsilon_{m-1}) \leq \left(\frac{1}{2}\right)^{|n-m|} M(\upsilon_m, \upsilon_{m-1}, \upsilon_{m-2}).$$

Since $\left(\frac{1}{2}\right)^{|n-m|} \to 0$ as $|n-m| \to \infty$, we conclude that

$$M(\upsilon_n, \upsilon_m, \upsilon_{m-1}) \to 0 \quad \text{as} \quad |n-m| \to \infty.$$

This shows that $\{\upsilon_n\}$ is Cauchy, meaning that the terms of the sequence become arbitrarily close to each other as n and m increase.

Step 2: Convergence to a Limit Since $(\mathbb{X}, M)$ is a complete MR-metric space, every Cauchy sequence converges to a limit. Therefore, the sequence $\{\upsilon_n\}$ converges to some point $\upsilon^* \in \mathbb{X}$. In other words, there exists $\upsilon^* \in \mathbb{X}$ such that:

$$\upsilon_n \to \upsilon^* \quad \text{as} \quad n \to \infty.$$

Thus, we have shown that the sequence $\{\upsilon_n\}$ converges to a limit υ^*.

Conclusion: The sequence $\{\upsilon_n\}$ is Cauchy, and by the completeness of the MR-metric space $(\mathbb{X}, M)$, it converges to a limit $\upsilon^* \in \mathbb{X}$. Therefore, the theorem is proved.

Example 2. *Consider the MR-metric space $(\mathbb{R}, M)$, where the MR-metric M is defined as:*

$$M(\upsilon, \xi, \Im) = |\upsilon - \xi| + |\xi - \Im| + |\Im - \upsilon|. \tag{8}$$

Define the sequence $\{\upsilon_n\}$ in $\mathbb{R}$ by:

$$\upsilon_n = \frac{1}{2^n}. \tag{9}$$

We need to verify that the sequence satisfies the condition of the theorem. First, we compute the MR-metric $M(\upsilon_{n+1}, \upsilon_n, \upsilon_{n-1})$:

$$M(\upsilon_{n+1}, \upsilon_n, \upsilon_{n-1}) = \left|\frac{1}{2^{n+1}} - \frac{1}{2^n}\right| + \left|\frac{1}{2^n} - \frac{1}{2^{n-1}}\right| + \left|\frac{1}{2^{n-1}} - \frac{1}{2^{n+1}}\right|. \tag{10}$$

Since

$$\left|\frac{1}{2^{n+1}} - \frac{1}{2^n}\right| = \frac{1}{2^n} - \frac{1}{2^{n+1}} = \frac{1}{2^{n+1}}, \tag{11}$$

$$\left|\frac{1}{2^n} - \frac{1}{2^{n-1}}\right| = \frac{1}{2^{n-1}} - \frac{1}{2^n} = \frac{1}{2^n}, \tag{12}$$

$$\left|\frac{1}{2^{n-1}} - \frac{1}{2^{n+1}}\right| = \frac{1}{2^{n-1}} - \frac{1}{2^{n+1}} = \frac{1}{2^{n+1}} + \frac{1}{2^n}. \tag{13}$$

Summing these terms, we get:

$$M(\upsilon_{n+1}, \upsilon_n, \upsilon_{n-1}) = \frac{1}{2^{n+1}} + \frac{1}{2^n} + \frac{1}{2^{n+1}} + \frac{1}{2^n} = \frac{3}{2^{n+1}}. \tag{14}$$

Next, we compute $M(\upsilon_n, \upsilon_{n-1}, \upsilon_{n-2})$:

$$M(\upsilon_n, \upsilon_{n-1}, \upsilon_{n-2}) = \left|\frac{1}{2^n} - \frac{1}{2^{n-1}}\right| + \left|\frac{1}{2^{n-1}} - \frac{1}{2^{n-2}}\right| + \left|\frac{1}{2^{n-2}} - \frac{1}{2^n}\right|. \tag{15}$$

This simplifies to:

$$M(\upsilon_n, \upsilon_{n-1}, \upsilon_{n-2}) = \frac{3}{2^n}. \tag{16}$$

Thus, we observe that:

$$M(\upsilon_{n+1}, \upsilon_n, \upsilon_{n-1}) = \frac{1}{2} M(\upsilon_n, \upsilon_{n-1}, \upsilon_{n-2}), \tag{17}$$

which satisfies the condition of the theorem.

Since the sequence $M(\upsilon_{n+1}, \upsilon_n, \upsilon_{n-1}) = \frac{3}{2^{n+1}}$ converges to 0 as $n \to \infty$, the sequence $\{\upsilon_n\}$ is Cauchy. Since $\mathbb{R}$ is complete, it follows that there exists a limit $\upsilon^* \in \mathbb{R}$ such that:

$$\lim_{n\to\infty} \upsilon_n = \upsilon^*. \tag{18}$$

Since $\upsilon_n = \frac{1}{2^n}$, we have:

$$\upsilon^* = 0. \tag{19}$$

Thus, the sequence $\{\upsilon_n\}$ converges to $\upsilon^* = 0$, and we have verified that the sequence satisfies the conditions of the Theorem 2.2 and converges to the limit.

Theorem 3. *Let $(\mathbb{X}, M)$ be a complete MR-metric space with a σ-finite measure μ. Suppose that $T : \mathbb{X} \to \mathbb{X}$ is a measurable self-mapping satisfying the contraction condition:*

$$M(T\upsilon, T\xi, T\Im) \leq \alpha M(\upsilon, \xi, \Im), \quad \forall \upsilon, \xi, \Im \in \mathbb{X}, \tag{20}$$

where $0 \leq \alpha < \frac{1}{R}$. Then, there exists a unique fixed point $\upsilon^* \in \mathbb{X}$ such that $T(\upsilon^*) = \upsilon^*$, and $\upsilon_n \to \upsilon^*$ in measure.

Proof. Step 1: Defining the Recursive Sequence We define a sequence $\{\upsilon_n\}$ in $\mathbb{X}$ recursively by setting:

$$\upsilon_{n+1} = T\upsilon_n, \quad \text{for some initial point} \quad \upsilon_0 \in \mathbb{X}.$$

Our goal is to show that this sequence $\{\upsilon_n\}$ converges to a fixed point of T and that the convergence is in measure.

Step 2: Applying the Contraction Condition Using the contraction condition, we obtain the following inequality for all $n \geq 1$:

$$M(\upsilon_{n+1}, \upsilon_n, \upsilon_{n-1}) \leq \alpha M(\upsilon_n, \upsilon_{n-1}, \upsilon_{n-2}).$$

This contraction condition, when applied repeatedly, leads to:

$$M(\upsilon_{n+k}, \upsilon_{n+k-1}, \upsilon_{n+k-2}) \leq \alpha^k M(\upsilon_n, \upsilon_{n-1}, \upsilon_{n-2}).$$

By induction, it can be shown that this inequality holds for all $k \geq 1$, and since $0 \leq \alpha < 1/R$, the right-hand side tends to zero as $k \to \infty$. Hence, the distances between the terms of the sequence shrink exponentially, implying that the sequence $\{\upsilon_n\}$ is a Cauchy sequence.

Step 3: Convergence of the Sequence Since $(\mathbb{X}, M)$ is a complete MR-metric space, we can conclude that the sequence $\{\upsilon_n\}$ converges to some point $\upsilon^* \in \mathbb{X}$. Therefore, we have:

$$\upsilon_n \to \upsilon^* \quad \text{as} \quad n \to \infty.$$

Step 4: Showing that υ^* is a Fixed Point To prove that υ^* is a fixed point of T, we take the limit of the contraction condition as $n \to \infty$. By continuity of T and the fact that $\upsilon_n \to \upsilon^*$, we obtain:

$$M(T\upsilon^*, T\upsilon^*, T\upsilon^*) = \lim_{n\to\infty} M(T\upsilon_n, T\upsilon_{n-1}, T\upsilon_{n-2}).$$

Since $\upsilon_n \to \upsilon^*$, it follows that:

$$M(T\upsilon^*, T\upsilon^*, T\upsilon^*) \leq \alpha M(\upsilon^*, \upsilon^*, \upsilon^*) = 0.$$

Therefore, we have:

$$M(T\upsilon^*, T\upsilon^*, T\upsilon^*) = 0,$$

which implies that $T\upsilon^* = \upsilon^*$. Thus, υ^* is a fixed point of T.

Step 5: Showing Convergence in Measure Now, we establish that $\upsilon_n \to \upsilon^*$ in measure. For this, we consider the sets:

$$A_n = \{\upsilon \in \mathbb{X} : M(\upsilon_n, \upsilon^*, \upsilon^*) \geq \epsilon\},$$

where $\epsilon > 0$. Since $M(\upsilon_n, \upsilon^*, \upsilon^*) \to 0$ as $n \to \infty$, it follows that for sufficiently large n, the measure of A_n becomes arbitrarily small. Specifically:

$$\mu(A_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Thus, $\upsilon_n \to \upsilon^*$ in measure.

Step 6: Uniqueness of the Fixed Point Finally, we show the uniqueness of the fixed point. Suppose that there are two fixed points υ^* and υ^{**} such that $T\upsilon^* = \upsilon^*$ and $T\upsilon^{**} = \upsilon^{**}$. From the contraction condition, we have:

$$M(T\upsilon^*, T\upsilon^{**}, T\upsilon^{**}) \leq \alpha M(\upsilon^*, \upsilon^{**}, \upsilon^{**}).$$

Since $T\upsilon^* = \upsilon^*$ and $T\upsilon^{**} = \upsilon^{**}$, we obtain:

$$M(\upsilon^*, \upsilon^{**}, \upsilon^{**}) \leq \alpha M(\upsilon^*, \upsilon^{**}, \upsilon^{**}).$$

Since $\alpha < 1/R$, this forces the right-hand side to be strictly smaller than the left-hand side, leading to:

$$M(\upsilon^*, \upsilon^{**}, \upsilon^{**}) = 0.$$

Thus, $\upsilon^* = \upsilon^{**}$, which proves the uniqueness of the fixed point.

Conclusion We have shown that there exists a unique fixed point υ^* such that $T(\upsilon^*) = \upsilon^*$, and that the sequence υ_n converges to υ^* in measure. This completes the proof.

Example 3. *Let $\mathbb{X} = [0,1]$ be a metric space equipped with the MR-metric:*

$$M(\upsilon, \xi, \Im) = |\upsilon - \xi| + |\xi - \Im| + |\Im - \upsilon|. \tag{21}$$

This function satisfies the properties of an MR-metric and provides a measure of the three-point distance in the space $\mathbb{X}$.

Now, consider the mapping $T : \mathbb{X} \to \mathbb{X}$ defined by:

$$T\upsilon = \frac{\upsilon}{2}. \tag{22}$$

This function maps each point in $\mathbb{X}$ to half its value, ensuring that the sequence $\{\upsilon_n\}$ is non-increasing and convergent.

To verify that T satisfies the contraction condition, we compute:

$$M(T\upsilon, T\xi, T\Im) = \left|\frac{\upsilon}{2} - \frac{\xi}{2}\right| + \left|\frac{\xi}{2} - \frac{\Im}{2}\right| + \left|\frac{\Im}{2} - \frac{\upsilon}{2}\right| \tag{23}$$

$$= \frac{1}{2}\left(|\upsilon - \xi| + |\xi - \Im| + |\Im - \upsilon|\right) \tag{24}$$

$$= \frac{1}{2} M(\upsilon, \xi, \Im). \tag{25}$$

Thus, T is a contraction with contraction coefficient $\alpha = \frac{1}{2}$. Assuming that $R > 2$, we satisfy the requirement that $\alpha < \frac{1}{R}$.

Next, we define the sequence recursively by:

$$\upsilon_{n+1} = T\upsilon_n, \tag{26}$$

with an initial point $\upsilon_0 \in \mathbb{X}$. We get:

$$\upsilon_n = \frac{\upsilon_0}{2^n}. \tag{27}$$

Since $0 \leq \upsilon_0 \leq 1$, it follows that:

$$\lim_{n\to\infty} \upsilon_n = 0. \tag{28}$$

This shows that the sequence $\{\upsilon_n\}$ is convergent.

To confirm that $\{\upsilon_n\}$ is Cauchy, we check that for any $\epsilon > 0$, there exists an integer N such that for all $m, n \geq N$:

$$M(\upsilon_m, \upsilon_n, \upsilon_{n-1}) < \epsilon. \tag{29}$$

Since:

$$M(\upsilon_m, \upsilon_n, \upsilon_{n-1}) = \frac{1}{2^n} M(\upsilon_0, \upsilon_0, \upsilon_0), \tag{30}$$

and $M(\upsilon_0, \upsilon_0, \upsilon_0)$ is bounded, the right-hand side goes to zero as $n \to \infty$. Hence, $\{\upsilon_n\}$ is a Cauchy sequence.

Since $(\mathbb{X}, M)$ is complete, there exists a limit $\upsilon^* \in \mathbb{X}$ such that:

$$\lim_{n\to\infty} \upsilon_n = \upsilon^*. \tag{31}$$

Taking the limit in the recursive relation $\upsilon_{n+1} = T\upsilon_n$, we obtain:

$$T\upsilon^* = \upsilon^*. \tag{32}$$

Thus, $\upsilon^* = 0$ is a fixed point of T.

Finally, since the measure μ is σ-finite, and $\upsilon_n \to \upsilon^*$ in the MR-metric, we conclude that $\upsilon_n \to \upsilon^*$ in measure.

3. Applications

In this section we present some important applications of our main results.

Example 4. *Consider the MR-metric space $(\mathbb{R}^n, M)$, where the MR-metric M is defined by:*

$$M(\upsilon, \xi, \Im) = \|\upsilon - \xi\| + \|\xi - \Im\| + \|\Im - \upsilon\|.$$

Here, $\|\cdot\|$ represents the Euclidean norm in $\mathbb{R}^n$. The MR-metric is useful for distance measurement between three points, which is relevant in analyzing iterative methods.

Next, consider the self-mapping $T : \mathbb{R}^n \to \mathbb{R}^n$ given by:

$$T(\upsilon) = \frac{\upsilon}{2}.$$

This is a contraction mapping because it scales any point $\upsilon \in \mathbb{R}^n$ by $\frac{1}{2}$, and such transformations are frequently used in optimization algorithms like gradient descent, where the goal is to iteratively improve estimates toward an optimal solution.

We now check if T satisfies the contraction condition from Theorem 2.1:

$$M(T\upsilon, T\xi, T\Im) \leq \alpha M(\upsilon, \xi, \Im), \quad \forall \upsilon, \xi, \Im \in \mathbb{R}^n.$$

Substituting $T(\upsilon) = \frac{\upsilon}{2}$, we get:

$$M(T\upsilon, T\xi, T\Im) = \left\| \frac{\upsilon}{2} - \frac{\xi}{2} \right\| + \left\| \frac{\xi}{2} - \frac{\Im}{2} \right\| + \left\| \frac{\Im}{2} - \frac{\upsilon}{2} \right\|.$$

Since the Euclidean norm scales linearly, this simplifies to:

$$M(T\upsilon, T\xi, T\Im) = \frac{1}{2} M(\upsilon, \xi, \Im).$$

Thus, T is a contraction with contraction factor $\alpha = \frac{1}{2}$, which satisfies $\alpha < 1$. According to Theorem 2.1, this guarantees the existence of a unique fixed point υ^* such that:

$$T(\upsilon^*) = \upsilon^*.$$

Solving $T(\upsilon^*) = \frac{\upsilon^*}{2} = \upsilon^*$, we find $\upsilon^* = 0$. Therefore, the sequence generated by iteratively applying T converges to the fixed point $\upsilon^* = 0$.

Practical Significance: *The MR-metric provides a reliable tool for proving the convergence of iterative algorithms, ensuring that they reach a unique fixed point. This is particularly important in optimization, as it supports the stability and efficiency of algorithms used in fields like machine learning and numerical analysis.*

Example 5. *In the context of signal processing, let $\mathbb{X}$ represent a space consisting of signals defined over the time interval $[0, T]$. To quantify the variation among three signals, we define the MR-metric as follows:*

$$M(\upsilon, \xi, \Im) = \int_0^T |\upsilon(t) - \xi(t)| + |\xi(t) - \Im(t)| + |\Im(t) - \upsilon(t)| \, dt,$$

where $\upsilon(t)$, $\xi(t)$, and $\Im(t)$ are real-valued functions over the interval $[0, T]$. This metric captures the total deviation among the three signals throughout the specified time domain, making it a useful tool for comparing signal differences.

A significant application of this MR-metric is in various signal processing techniques, such as denoising, compression, and energy optimization. These methods often rely on *iterative procedures aimed at improving signal quality, whether by reducing noise, enhancing representation, or minimizing energy usage. The iterative transformations involved in these processes can be effectively represented using self-mappings.*

Consider the self-mapping $T : \mathbb{X} \to \mathbb{X}$ *given by:*

$$T(\upsilon) = \frac{\upsilon}{2}. \tag{33}$$

This operator scales the amplitude of a signal by a factor of $1/2$*. Such an operation is commonly employed in signal processing algorithms where gradual attenuation of a signal is required, for instance, to suppress noise or reduce energy levels.*

A practical interpretation of this transformation is in the context of iterative noise suppression. In such scenarios, unwanted signal fluctuations (noise) are progressively diminished through repeated applications of T*. Mathematically, applying* T *iteratively to an initial signal* υ_0 *generates the sequence:*

$$\upsilon_n = \frac{\upsilon_0}{2^n}.$$

As $n \to \infty$*, this sequence converges to:*

$$\upsilon^* = 0.$$

This result indicates that, given enough iterations, the signal will converge to the zero signal, representing complete noise suppression or a fully attenuated signal.

From a theoretical standpoint, verifying the contraction condition is crucial. We check whether T *satisfies the MR-metric contraction property:*

$$M(T\upsilon, T\xi, T\Im) \leq \alpha M(\upsilon, \xi, \Im), \quad \forall \upsilon, \xi, \Im \in \mathbb{X}.$$

Substituting $T(\upsilon) = \upsilon/2$ *into the MR-metric, we compute:*

$$M(T\upsilon, T\xi, T\Im) = \int_0^T \left|\frac{\upsilon(t)}{2} - \frac{\xi(t)}{2}\right| + \left|\frac{\xi(t)}{2} - \frac{\Im(t)}{2}\right| + \left|\frac{\Im(t)}{2} - \frac{\upsilon(t)}{2}\right| dt.$$

Factoring out the common term $1/2$*, we obtain:*

$$M(T\upsilon, T\xi, T\Im) = \frac{1}{2} M(\upsilon, \xi, \Im).$$

Since $\frac{1}{2} < 1/R$ *for any* $R > 1$*, the contraction condition is satisfied, ensuring that* T *has a unique fixed point,* $\upsilon^* = 0$*, which is reached as the number of iterations increases.*

This result has significant implications in signal processing. For instance, in denoising applications, an iterative algorithm that successively applies T *can remove high-frequency noise components while preserving the main structure of the signal. Similarly, in compression algorithms, gradual amplitude reduction can lead to a more compact and efficient signal representation.*

The role of MR-metrics in this context is to provide a rigorous framework for analyzing the convergence of iterative signal transformations. By ensuring that the sequence of processed signals converges reliably to a well-defined state, MR-metrics help in designing stable and effective signal processing algorithms. Whether in noise reduction, compression, or energy optimization, the application of MR-metrics guarantees that iterative methods lead to meaningful and predictable outcomes.

4. Conclusion

In this paper, we have explored the existence and uniqueness of fixed points for self-mappings in MR-metric spaces, a generalized framework extending traditional metric spaces. By establishing key contraction principles and convergence criteria, we have demonstrated that iterative sequences in MR-metric spaces exhibit stable and predictable behavior, leading to unique fixed points under well-defined conditions.

Our results provide a solid foundation for further studies on the stability and applications of MR-metric spaces, particularly in optimization, numerical analysis, and machine learning. Additionally, the examples presented validate the theoretical findings, highlighting their practical significance. Future research can explore the extension of these results to more complex structures, such as probabilistic or fuzzy MR-metric spaces, and investigate their applications in real-world problems.

The theoretical advancements presented in this study contribute to the broader field of fixed point theory and its applications, offering new insights into the convergence behavior of mappings in generalized metric frameworks.

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