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Picture Fuzzy Modal Ideal Multifunctions

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Abstract. This paper introduces the notion of a picture fuzzy modal topological structures (PFMTSs) via ideal. These structures are grounded on novel picture fuzzy topological operators for closure and interior types, utilizing the two standard picture fuzzy modal operators $\Box$ and $\Diamond$. The paper discusses several fundamental properties of picture fuzzy multifunctions PFMs via ideals. The results indicate that some properties considered satisfactory in the intuitionistic fuzzy modal topological structures, as defined by Atanassov in 2022, are not fulfilled. Also, we introduce many types of continuous multifunctions between picture fuzzy ideal topological spaces.

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1. Introduction

Fuzzification is a crucial tool for addressing humanistic systems in real-life problems. The seminal paper on fuzzy set theory was authored by Zadeh in 1965 ([1]). This theory of fuzzy sets (FSs) has been widely applied by many scholars. FSs theory described the positivism of an element ξ of a universal set ξ to a subset $\mathbb{K} \subseteq \Xi$ by the membership value $\omega_{\mathbb{K}}(\xi)$, and posited that the negativism of that element $\xi \in \Xi$ to the set $\mathbb{K}$ is $1 - \omega_{\mathbb{K}}(\xi)$. Atanassov in [2] based his theory of intuitionistic fuzzy sets (IFSs) on the notion that the negativism $\varpi_{\mathbb{K}}(\xi)$ of an element $\xi \in \Xi$ to a subset $\mathbb{K} \subseteq \Xi$ may range from $[0, 1]$ and need not be the complement of the positivism of that element $\xi \in \Xi$ to $\mathbb{K}$. The values $\omega_{\mathbb{K}}(\xi)$ and $\varpi_{\mathbb{K}}(\xi)$ represent the positivism and negativism of each $\xi \in \Xi$ to $\mathbb{K}$, respectively, with the condition that $0 \leq \omega_{\mathbb{K}}(\xi) + \varpi_{\mathbb{K}}(\xi) \leq 1$. In this way, Atanassov encompassed all the FSs as a special case of his theory whenever $\omega_{\mathbb{K}}(\xi) + \varpi_{\mathbb{K}}(\xi) = 1$. IFSs are more meaningful

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and applicable to real-life problems. Cuong in [3] introduced the theory of picture fuzzy sets (PFSs) by adding the neutralism of an element $\xi \in \Xi$ to the subset $\mathbb{K}$, represented by $\sigma_{\mathbb{K}}(\xi)$. This definition is conditioned with $0 \leq \omega_{\mathbb{K}}(\xi) + \varpi_{\mathbb{K}}(\xi) + \sigma_{\mathbb{K}}(\xi) \leq 1$. In case where $\sigma_{\mathbb{K}}(\xi) = 0$ for all $\xi \in \Xi$, Then, we revert to intuitionistic sets $\mathbb{K}$ in IFS. Moreover, if $\varpi_{\mathbb{K}}(\xi) = 1 - \omega_{\mathbb{K}}(\xi)$, then we revert to fuzzy set $\mathbb{K}$ in FS. There are several simple modifications for IFSs [4, 5], which we shall not discuss here. These modifications include pythagorean FSs [6], spherical FSs [7, 8], q-rung orthopair FSs [9] and q-rung orthopair PFSs [10], (ς, $\varkappa$)-fuzzy local function, continuous multifunctions and double fuzzy ideal topological spaces [11, 12]. All these definitions, starting from FSs, have applications in image processing, decision theory, uncertainty modeling, and beyond, as in [5, 6, 9, 13–16].

In this paper, we merge the classical definitions of multifunctions in general topology and the standard modal logic [17–20] with the notion of PFSs, further expanding into the realm of PFMTSs. This exploration includes the creation of PFMTSs facilitated by the standard picture fuzzy operations of "union" ($\cup$) and "intersection" ($\cap$). Continuous functions between picture fuzzy topological spaces were discussed in [21]. Continuous multifunctions between picture fuzzy topological spaces were discussed in [22].

The motivations of this paper are as follow: Firstly, to present PFTSs related to the PFSs, and studying some important results including several modal operators. These results are given in Section 2. Secondly, to introduce PFTMs via ideals and their common results. Also, to define some types of continuity of picture fuzzy multifunctions. These results are given in Section 3. Finally, the conclusion and the future work are given in Section 4.

The research on PFMTSs has several important applications in various domains: Decision Making, Pattern Recognition, Artificial Intelligence, Information Retrieval and Data Mining. PFMTSs address critical gaps in handling uncertainty, imprecision, and neutrality, which are inherent in real-life problems across diverse domains. To bridge these gaps, PFSs were introduced, adding a neutrality component to the membership and non-membership values, thereby enabling a more nuanced representation of uncertainty. PFMTSs expand upon these concepts by the integration in modal logic and general topology using the PFSs. This integration introduces global operators, such as closure, interior, and modal operators ($\Box$ and $\Diamond$), which modify classical topological and modal relationships. These global operators facilitate a robust analysis of FSs under modal and topological constraints, providing a suitable tools for theoretical exploration and practical application. The study of PFMTS not only extends the theory of FSs but also establishes a wide platform for addressing modern computational challenges. Its ability to integrate neutrality, positivity, and negativity within a unified framework lays the foundation for further exploration and application of PFMTS in dynamic systems, hybrid models, and emerging technologies, positioning it as a cornerstone of modern mathematical and computational innovation.

PFMTSs have special important applications in decision making environments. Chellamani et. al [23], used picture fuzzy soft graphs to design a decision making scheme. Yang et. al [24], developed an adjustable soft discernibility matrix with the help of picture fuzzy soft sets and presented its applications in decision making. Joshi in [25–27] presented an innovative decision making process for a picture fuzzy environment with the help of the

concept R-norm and the VIKOR technique. More development of PFSs can be seen in [28–30]. In daily life, PFS theory provides more than one choice for any decision. As examples:
(1) Suppose a person is suffering from some disease. Then, the positive, negative and neutral membership functions can be associated with curability bitterness and treatment of disease respectively. Refusal can be related to the insufficient economic conditions of the patient meaning that he cann't afford the hospital expenses and refuses to be hospitalized.
(2) Suppose a person has an allegation of a crime. Then, the positive, negative and neutral membership functions can be associated with maximum punishment, release and moderate punishment of the accused person respectively. Refusal can be related to the dismissal of the case due to reconciliation. keeping in mind the above literature and the importance of PFSs, as well as topological spaces, we reveal the study of PFMTSs. The major contributions of this paper are as follow:
(a) The definition of some new notions of cl-PFMTS, int-int-PFMTS, cl-int-PFMTS, int-cl-PFMTS regarding the types of the topological operators "closure" and "interior" and any of the given modal operators.
(b) The design of the various PFMs based on the stated notions.
(c) The definition of the notion of continuous multifunctions in picture fuzzy topological spaces via ideals and an introduction to necessary and sufficient conditions of upper and lower PFM between two picture fuzzy ideal topological spaces.

2. Picture fuzzy operations

Continuing from previous discussions and the notions given by Atanassov in [4, 31], let's define a PFS $\mathbb{K}$ on the universal set Ξ. The set $\mathbb{K}$ consists of elements $\xi \in \Xi$, each described by degrees of positivism $(\omega_{\mathbb{K}}(\xi))$, negativism $(\varpi_{\mathbb{K}}(\xi))$, and neutralism $(\sigma_{\mathbb{K}}(\xi))$ that lie within the interval $[0,1]$. Specifically, $\mathbb{K}$ is represented as $\{\langle \xi, \omega_{\mathbb{K}}(\xi), \varpi_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi)\rangle \,|\xi \in \Xi\}$, where each component satisfies the condition $0 \leq \omega_{\mathbb{K}}(\xi) + \varpi_{\mathbb{K}}(\xi) + \sigma_{\mathbb{K}}(\xi) \leq 1$ for every element ξ. The term $\pi_{\mathbb{K}}(\xi) = 1 - (\omega_{\mathbb{K}}(\xi) + \varpi_{\mathbb{K}}(\xi) + \sigma_{\mathbb{K}}(\xi))$ indicates the degree of refusal membership value for each ξ in $\mathbb{K}$, quantifying the extent to which ξ does not belong to $\mathbb{K}$. This framework is pivotal for assessing and handling the nuances of membership within PFSs, enabling a more comprehensive analysis of elements based on their multiple affinities.

Definition 2.1. [4, 31] Let Ξ be a nonempty set, $\mathbb{K} = \{\langle \xi, \omega_{\mathbb{K}}(\xi), \varpi_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi)\rangle \,|\xi \in \Xi\}$ and
$Ч = \{\langle \xi, \omega_{Ч}(\xi), \varpi_{Ч}(\xi), \sigma_{Ч}(\xi)\rangle \,|\xi \in \Xi\}$. Then,

(1) $\mathbb{K} \subseteq Ч$ iff for all $\xi \in \Xi$, $\omega_{\mathbb{K}}(\xi) \leq \omega_{Ч}(\xi)$, $\varpi_{\mathbb{K}}(\xi) \geq \varpi_{Ч}(\xi)$ and $\sigma_{\mathbb{K}}(\xi) \leq \sigma_{Ч}(\xi)$ or $\sigma_{\mathbb{K}}(\xi) \geq \sigma_{Ч}(\xi)$

(2) $\mathbb{K} \cup Ч = \{\langle \xi, (\omega_{\mathbb{K}}(\xi) \vee \omega_{Ч}(\xi)), (\varpi_{\mathbb{K}}(\xi) \wedge \varpi_{Ч}(\xi)), (\sigma_{\mathbb{K}}(\xi) \wedge \sigma_{Ч}(\xi))\rangle \,|\xi \in \Xi\}$

(3) $\mathbb{K} \cap Ч = \{\langle \xi, (\omega_{\mathbb{K}}(\xi) \wedge \omega_{Ч}(\xi)), (\varpi_{\mathbb{K}}(\xi) \vee \varpi_{Ч}(\xi)), (\sigma_{\mathbb{K}}(\xi) \wedge \sigma_{Ч}(\xi))\rangle \,|\xi \in \Xi\}$

(4) $$\mathbb{K} \sqcup Ч = \left\{\left\langle \begin{array}{c} \xi, \omega_{\mathbb{K}}(\xi) + \omega_{Ч}(\xi) - [(\omega_{\mathbb{K}}(\xi).\omega_{Ч}(\xi)) \wedge (\sigma_{\mathbb{K}}(\xi).\sigma_{Ч}(\xi))], \\ \varpi_{\mathbb{K}}(\xi).\varpi_{Ч}(\xi), \sigma_{\mathbb{K}}(\xi).\sigma_{Ч}(\xi) \end{array} \right\rangle |\xi \in \Xi\right\}$$

(5) $\mathbb{K} \sqcap \text{Ч} =$
$$\left\{\left\langle \begin{array}{c} \xi, \omega_{\mathbb{K}}(\xi).\omega_{\text{Ч}}(\xi), \\ \varpi_{\mathbb{K}}(\xi)+\varpi_{\text{Ч}}(\xi)-\left[(\varpi_{\mathbb{K}}(\xi).\varpi_{\text{Ч}}(\xi)) \wedge (\sigma_{\mathbb{K}}(\xi).\sigma_{\text{Ч}}(\xi))\right], \sigma_{\mathbb{K}}(\xi).\sigma_{\text{Ч}}(\xi) \end{array} \right\rangle | \xi \in \Xi \right\}$$
(6) $\mathbb{K} @ \text{Ч} = \left\{\left\langle \xi, \frac{\omega_{\mathbb{K}}(\xi)+\omega_{\text{Ч}}(\xi)}{2}, \frac{\varpi_{\mathbb{K}}(\xi)+\varpi_{\text{Ч}}(\xi)}{2}, \frac{\sigma_{\mathbb{K}}(\xi)+\sigma_{\text{Ч}}(\xi)}{2} \right\rangle | \xi \in \Xi \right\}$
(7) $\mathbb{K} \# \text{Ч} = \{\langle \xi, \omega_{\mathbb{K}}(\xi).\omega_{\text{Ч}}(\xi), (\varpi_{\mathbb{K}}(\xi) \vee \varpi_{\text{Ч}}(\xi)), \sigma_{\mathbb{K}}(\xi).\sigma_{\text{Ч}}(\xi) \rangle | \xi \in \Xi\}$
(8) $\mathbb{K} * \text{Ч} = \{\langle \xi, (\omega_{\mathbb{K}}(\xi) \vee \omega_{\text{Ч}}(\xi)), \varpi_{\mathbb{K}}(\xi).\varpi_{\text{Ч}}(\xi), \sigma_{\mathbb{K}}(\xi).\sigma_{\text{Ч}}(\xi) \rangle | \xi \in \Xi\}$
(9) $\exists\!\!\!| \; \mathbb{K} = \{\langle \xi, \varpi_{\mathbb{K}}(\xi), \omega_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi) \rangle | \xi \in \Xi\}$
(10) $\mathbb{K} \barwedge \text{Ч} = 0$ if $\mathbb{K} \subseteq \text{Ч}$, and $\mathbb{K} \barwedge \text{Ч} = \mathbb{K} \cap (\exists\!\!\!| \; \text{Ч})$ otherwise.

Now, the definitions of standard two modal operators over PFSs are presented.
$\Box \mathbb{K} = \{\langle \xi, \omega_{\mathbb{K}}(\xi), 1-\omega_{\mathbb{K}}(\xi)-\sigma_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi) \rangle | \xi \in \Xi\}$,
$\Diamond \mathbb{K} = \{\langle \xi, 1-\varpi_{\mathbb{K}}(\xi)-\sigma_{\mathbb{K}}(\xi), \varpi_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi) \rangle | \xi \in \Xi\}$.

We can see that $\Box \mathbb{K} \subseteq \mathbb{K} \subseteq \Diamond \mathbb{K}$ in general, and $\Box \mathbb{K} \neq \mathbb{K} \neq \Diamond \mathbb{K}$ for any proper set $\mathbb{K}$ in (PFS), that is, $\sigma_{\mathbb{K}}(\xi) \neq 0$. Otherwise, $\mathbb{K}$ is an IFS and still $\Box \mathbb{K} \subseteq \mathbb{K} \subseteq \Diamond \mathbb{K}$ as usual in (IFS). Moreover, if $\mathbb{K}$ is non proper PFS and $\varpi_{\mathbb{K}}(\xi) = 1-\omega_{\mathbb{K}}(\xi)$, then $\mathbb{K}$ is a FS and $\Box \mathbb{K} = \mathbb{K} = \Diamond \mathbb{K}$. Thus, $(FS) \subseteq (IFS) \subseteq (PFS)$.

A PFS $\mathbb{K} = \{\langle \xi, \omega_{\mathbb{K}}(\xi), \varpi_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi) \rangle | \xi \in \Xi\}$ is called a picture fuzzy tautological set (PFTaut) iff for each $\xi \in \Xi$, $\omega_{\mathbb{K}}(\xi) \geq \varpi_{\mathbb{K}}(\xi)$.

$\sharp = \{\langle \xi, 1, 0, 0 \rangle | \xi \in \Xi\}$, $\flat = \{\langle \xi, 0, 1, 0 \rangle | \xi \in \Xi\}$, $\natural = \{\langle \xi, 0, 0, 1 \rangle | \xi \in \Xi\}$, $\mho = \{\langle \xi, 0, 0, 0 \rangle | \xi \in \Xi\}$ where $\flat \subseteq \mathbb{K} \subseteq \sharp$ for all $\mathbb{K} \in (PFS)$. Normally, $\mathcal{P}(\flat) = \flat$ and $\mathcal{P}(\sharp) = \{\mathbb{K} | \mathbb{K} \subseteq \sharp\}$ where
$\mathbb{K} = \{\langle \xi, \omega_{\mathbb{K}}(\xi), \varpi_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi) \rangle | \xi \in \Xi\}$. Therefore, (PFS) coincides with $\mathcal{P}(\sharp)$. (IFS) coincides with the set
$\{\mathbb{K} | \mathbb{K} \subseteq \Xi\}$ in which $\mathbb{K} = \{\langle \xi, \omega_{\mathbb{K}}(\xi), \varpi_{\mathbb{K}}(\xi), 0 \rangle | \xi \in \Xi\}$. Moreover, (FS) coincides with the set $\{\mathbb{K} | \mathbb{K} \subseteq \Xi\}$ in which $\mathbb{K} = \{\langle \xi, \omega_{\mathbb{K}}(\xi), 1-\omega_{\mathbb{K}}(\xi), 0 \rangle | \xi \in \Xi\}$ or
$\mathbb{K} = \{\langle \xi, 1-\varpi_{\mathbb{K}}(\xi), \varpi_{\mathbb{K}}(\xi), 0 \rangle | \xi \in \Xi\}$.

Any operation from the above is well defined if the sum of its three values (positivism, negativism and neutralism) is a number in $[0,1]$. We will check for the definitions of operations $\mathbb{K} \# \text{Ч}$ and $\mathbb{K} * \text{Ч}$.

$$\begin{aligned} 0 &\leq \omega_{\mathbb{K}}(\xi).\omega_{\text{Ч}}(\xi) + [\varpi_{\mathbb{K}}(\xi) \vee \varpi_{\text{Ч}}(\xi)] + \sigma_{\mathbb{K}}(\xi).\sigma_{\text{Ч}}(\xi) \\ &\leq [\omega_{\mathbb{K}}(\xi) \wedge \omega_{\text{Ч}}(\xi)] + [\varpi_{\mathbb{K}}(\xi) \vee \varpi_{\text{Ч}}] + [\sigma_{\mathbb{K}}(\xi) \wedge \sigma_{\text{Ч}}(\xi)] \\ &\leq [\omega_{\mathbb{K}}(\xi) \wedge \omega_{\text{Ч}}(\xi)] + [(1-\omega_{\mathbb{K}}(\xi)-\sigma_{\mathbb{K}}(\xi)) \vee (1-\omega_{\text{Ч}}(\xi)-\sigma_{\text{Ч}}(\xi))] + [\sigma_{\mathbb{K}}(\xi) \wedge \sigma_{\text{Ч}}(\xi)] \\ &\leq [\omega_{\mathbb{K}}(\xi) \wedge \omega_{\text{Ч}}(\xi)] + 1 - [\omega_{\mathbb{K}}(\xi) \wedge \omega_{\text{Ч}}(\xi)] - [\sigma_{\mathbb{K}}(\xi) \wedge \sigma_{\text{Ч}}(\xi)] + [\sigma_{\mathbb{K}}(\xi) \wedge \sigma_{\text{Ч}}(\xi)] = 1. \end{aligned}$$

Also, it is clear that $0 \leq (\omega_{\mathbb{K}}(\xi) \vee \omega_{\text{Ч}}(\xi)) + \varpi_{\mathbb{K}}(\xi).\varpi_{\text{Ч}} + \sigma_{\mathbb{K}}(\xi).\sigma_{\text{Ч}}(\xi) \leq 1$.

Now, we check the duality of the operations $\#$ and $*$: For $\mathbb{K}, \text{Ч} \in \mathcal{P}(\sharp)$,

$$\begin{aligned} \exists\!\!\!| \,(\exists\!\!\!| \,\mathbb{K} \# \exists\!\!\!| \,\text{Ч}) &= \exists\!\!\!| \,(\{\langle \xi, \varpi_{\mathbb{K}}(\xi), \omega_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi) \rangle | \xi \in \Xi\} \ \# \ \{\langle \xi, \varpi_{\text{Ч}}(\xi), \omega_{\text{Ч}}(\xi), \sigma_{\text{Ч}}(\xi) \rangle | \xi \in \Xi\}) \\ &= \exists\!\!\!| \,(\{\langle \xi, \varpi_{\mathbb{K}}(\xi).\varpi_{\text{Ч}}(\xi), [\omega_{\mathbb{K}}(\xi) \vee \omega_{\text{Ч}}(\xi)], \sigma_{\mathbb{K}}(\xi).\sigma_{\text{Ч}} \rangle | \xi \in \Xi\}) \\ &= \{\langle \xi, [\omega_{\mathbb{K}}(\xi) \vee \omega_{\text{Ч}}(\xi)], \varpi_{\mathbb{K}}(\xi).\varpi_{\text{Ч}}, \sigma_{\mathbb{K}}(\xi).\sigma_{\text{Ч}}(\xi) \rangle | \xi \in \Xi\} = \mathbb{K} * \text{Ч}, \end{aligned}$$

and in the same manner, $\exists\!\!\!| \,(\exists\!\!\!| \,\mathbb{K} * \exists\!\!\!| \,\text{Ч}) = \mathbb{K} \# \text{Ч}$.

The operators "closure" and "interior" over PFSs are defined by:
$cl_{\cap}(\mathbb{K}) = \{\langle \xi, \epsilon_{\mathbb{K}}, \aleph_{\mathbb{K}}, \kappa_{\mathbb{K}} \rangle | \xi \in \Xi\}$ and $int_{\cup}(\mathbb{K}) = \{\langle \xi, \varepsilon_{\mathbb{K}}, \vartheta_{\mathbb{K}}, \kappa_{\mathbb{K}} \rangle | \xi \in \Xi\}$ where

$$\begin{aligned} \epsilon_{\mathbb{K}} &= \bigvee_{\xi\in\Xi} \omega_{\mathbb{K}}(\xi), \quad \aleph_{\mathbb{K}} = \bigwedge_{\xi\in\Xi} \varpi_{\mathbb{K}}(\xi), \quad \kappa_{\mathbb{K}} = \bigwedge_{\xi\in\Xi} \sigma_{\mathbb{K}}(\xi) \\ \varepsilon_{\mathbb{K}} &= \bigwedge_{\xi\in\Xi} \omega_{\mathbb{K}}(\xi), \quad \vartheta_{\mathbb{K}} = \bigvee_{\xi\in\Xi} \varpi_{\mathbb{K}}(\xi), \quad \varkappa_{\mathbb{K}} = \bigvee_{\xi\in\Xi} \sigma_{\mathbb{K}}(\xi). \end{aligned}$$

Theorem 2.1. *For every* $\mathbb{K}$, $Ч \in \mathcal{P}(\sharp)$,

$$\mathbb{K} \sqcap Ч \subseteq \mathbb{K} \# Ч \subseteq \mathbb{K} \cap Ч \subseteq \mathbb{K} @ Ч \subseteq \mathbb{K} \cup Ч \subseteq \mathbb{K} * Ч \subseteq \mathbb{K} \sqcup Ч.$$

Proof. $\mathbb{K} \sqcap Ч =$
$\left\{ \left\langle \begin{array}{c} \xi, \omega_{\mathbb{K}}(\xi).\omega_{Ч}(\xi), \\ \varpi_{\mathbb{K}}(\xi) + \varpi_{Ч}(\xi) - (\varpi_{\mathbb{K}}(\xi).\varpi_{Ч}(\xi) \wedge \sigma_{\mathbb{K}}(\xi).\sigma_{Ч}(\xi)), \sigma_{\mathbb{K}}(\xi).\sigma_{Ч}(\xi) \end{array} \right\rangle | \xi \in \Xi \right\}$
$\subseteq \{\langle \xi, \omega_{\mathbb{K}}(\xi).\omega_{Ч}(\xi), (\varpi_{\mathbb{K}}(\xi) \vee \varpi_{Ч}(\xi)), \sigma_{\mathbb{K}}(\xi).\sigma_{Ч}(\xi) \rangle | \xi \in \Xi\} = \mathbb{K} \# Ч$,
$\mathbb{K} \# Ч = \{\langle \xi, \omega_{\mathbb{K}}(\xi).\omega_{Ч}(\xi), (\varpi_{\mathbb{K}}(\xi) \vee \varpi_{Ч}(\xi)), \sigma_{\mathbb{K}}(\xi).\sigma_{Ч}(\xi) \rangle | \xi \in \Xi\}$
$\subseteq \{\langle \xi, (\omega_{\mathbb{K}}(\xi) \wedge \omega_{Ч}(\xi)), (\varpi_{\mathbb{K}}(\xi) \vee \varpi_{Ч}(\xi)), (\sigma_{\mathbb{K}}(\xi) \wedge \sigma_{Ч}(\xi)) | \xi \in \Xi\}\rangle = \mathbb{K} \cap Ч$,
$\mathbb{K} \cap Ч = \{\langle \xi, (\omega_{\mathbb{K}}(\xi) \wedge \omega_{Ч}(\xi)), (\varpi_{\mathbb{K}}(\xi) \vee \varpi_{Ч}(\xi)), (\sigma_{\mathbb{K}}(\xi) \wedge \sigma_{Ч}(\xi)) | \xi \in \Xi\}\rangle$
$\subseteq \left\{ \left\langle \xi, \frac{\omega_{\mathbb{K}}(\xi)+\omega_{Ч}(\xi)}{2}, \frac{\varpi_{\mathbb{K}}(\xi)+\varpi_{Ч}(\xi)}{2}, \frac{\sigma_{\mathbb{K}}(\xi)+\sigma_{Ч}(\xi)}{2} \right\rangle | \xi \in \Xi \right\} = \mathbb{K} @ Ч$,
$\mathbb{K} @ Ч = \left\{ \left\langle \xi, \frac{\omega_{\mathbb{K}}(\xi)+\omega_{Ч}(\xi)}{2}, \frac{\varpi_{\mathbb{K}}(\xi)+\varpi_{Ч}(\xi)}{2}, \frac{\sigma_{\mathbb{K}}(\xi)+\sigma_{Ч}(\xi)}{2} \right\rangle | \xi \in \Xi \right\}$
$\subseteq \{\langle \xi, (\omega_{\mathbb{K}}(\xi) \vee \omega_{Ч}(\xi)), (\varpi_{\mathbb{K}}(\xi) \wedge \varpi_{Ч}(\xi)), (\sigma_{\mathbb{K}}(\xi) \wedge \sigma_{Ч}(\xi)) \rangle | \xi \in \Xi\} = \mathbb{K} \cup Ч$,
$\mathbb{K} \cup Ч = \{\langle \xi, (\omega_{\mathbb{K}}(\xi) \vee \omega_{Ч}(\xi)), (\varpi_{\mathbb{K}}(\xi) \wedge \varpi_{Ч}(\xi)), (\sigma_{\mathbb{K}}(\xi) \wedge \sigma_{Ч}(\xi)) \rangle | \xi \in \Xi\}$
$\subseteq \{\langle \xi, (\omega_{\mathbb{K}}(\xi) \vee \omega_{Ч}(\xi)), \varpi_{\mathbb{K}}(\xi).\varpi_{Ч}(\xi), \sigma_{\mathbb{K}}(\xi).\sigma_{Ч}(\xi) \rangle | \xi \in \Xi\} = \mathbb{K} * Ч$,
$\mathbb{K} * Ч = \{\langle \xi, (\omega_{\mathbb{K}}(\xi) \vee \omega_{Ч}(\xi)), \varpi_{\mathbb{K}}(\xi).\varpi_{Ч}(\xi), \sigma_{\mathbb{K}}(\xi).\sigma_{Ч}(\xi) \rangle | \xi \in \Xi\}$
$\subseteq \left\{ \left\langle \begin{array}{c} \xi, \omega_{\mathbb{K}}(\xi) + \omega_{Ч}(\xi) - (\omega_{\mathbb{K}}(\xi).\omega_{Ч}(\xi) \wedge \sigma_{\mathbb{K}}(\xi).\sigma_{Ч}(\xi)), \\ \varpi_{\mathbb{K}}(\xi).\varpi_{Ч}(\xi), \sigma_{\mathbb{K}}(\xi).\sigma_{Ч}(\xi) \end{array} \right\rangle | \xi \in \Xi \right\} = \mathbb{K} \sqcup Ч.$

Now, we will construct the picture fuzzy implication operation on $\mathcal{P}(\sharp)$ as follows:

$$\mathbb{K} \twoheadrightarrow Ч = \{\langle \xi, (\varpi_{\mathbb{K}}(\xi) \vee \omega_{Ч}(\xi)), \omega_{\mathbb{K}}(\xi).\varpi_{Ч}(\xi), \sigma_{\mathbb{K}}(\xi).\sigma_{Ч}(\xi) \rangle | \xi \in \Xi\}.$$

We checked these properties for the implication operation:

$$\begin{aligned} \sharp &\twoheadrightarrow \sharp = \sharp, \quad \flat \twoheadrightarrow \sharp = \sharp, \quad \natural \twoheadrightarrow \sharp = \sharp, \quad \mho \twoheadrightarrow \sharp = \sharp, \\ \sharp &\twoheadrightarrow \flat = \flat, \quad \flat \twoheadrightarrow \flat = \sharp, \quad \natural \twoheadrightarrow \flat = \mho, \quad \mho \twoheadrightarrow \flat = \mho, \\ \sharp &\twoheadrightarrow \natural = \mho, \quad \flat \twoheadrightarrow \natural = \sharp, \quad \natural \twoheadrightarrow \natural = \natural, \quad \mho \twoheadrightarrow \natural = \mho, \\ \sharp &\twoheadrightarrow \mho = \mho, \quad \flat \twoheadrightarrow \mho = \sharp, \quad \natural \twoheadrightarrow \mho = \mho, \quad \mho \twoheadrightarrow \mho = \mho. \end{aligned}$$

Following Atanassov in [31], we will give these nine axioms related with our new defined implication operation. Let Ξ be a nonempty set and $\mathbb{K}$, $Ч$, $Д \in \mathcal{P}(\sharp)$. Then,

Axiom 1: If $\mathbb{K} \subseteq Ч$, then $Ч \twoheadrightarrow Д \subseteq \mathbb{K} \twoheadrightarrow Д$.
Axiom 2: If $\mathbb{K} \subseteq Ч$, then $Д \twoheadrightarrow \mathbb{K} \subseteq Д \twoheadrightarrow Ч$.

Axiom 3: $\flat \twoheadrightarrow$ Ч $= \sharp$.
Axiom 4: $\sharp \twoheadrightarrow$ Ч $=$ Ч.
Axiom 5: $\mathbb{K} \twoheadrightarrow \mathbb{K} = \sharp$.
Axiom 6: $\mathbb{K} \twoheadrightarrow$ (Ч $\twoheadrightarrow$ Д) $=$ Ч $\twoheadrightarrow (\mathbb{K} \twoheadrightarrow$ Д).
Axiom 7: $\mathbb{K} \twoheadrightarrow$ Ч $= \sharp$ iff $\mathbb{K} \subseteq$ Ч.
Axiom 8: $\mathbb{K} \twoheadrightarrow$ Ч $=$ ⅎ Ч $\twoheadrightarrow$ ⅎ$\mathbb{K}$.
Axiom 9: $\twoheadrightarrow$ is a continuous function.

We followed [31] in defining a number of axioms marked with an asterisk (*) refering to the tautological operations, and ($Axiom7^*$) is given to show that "iff" in Axiom7 maybe not correct.

Axiom 3*: $\flat \twoheadrightarrow$ Ч is a PFTaut set.
Axiom 4*: $\sharp \twoheadrightarrow$ Ч is a PFTaut set.
Axiom 5*: $\mathbb{K} \twoheadrightarrow \mathbb{K}$ is a PFTaut set.
Axiom 7*: $\mathbb{K} \twoheadrightarrow$ Ч $= \sharp$ implies that $\mathbb{K} \subseteq$ Ч, and $\mathbb{K} \subseteq$ Ч implies that $\mathbb{K} \twoheadrightarrow$ Ч is a PFTaut set.

Theorem 2.2. *For* $\mathbb{K}$, *Ч, Д* $\in \mathcal{P}(\sharp)$ *the new implication* ($\twoheadrightarrow$) *satisfies Axioms 1, 2, 3, 3*, 5*, 6, 7*, 8, 9.*

Proof. ($For\ Axiom1$), let $\mathbb{K} \subseteq$ Ч. Then,

$$\begin{aligned} \mathbb{K} \quad &\twoheadrightarrow \quad Д = \left\{ \left\langle \xi, \left(\varpi_{\mathbb{K}}(\xi) \vee \omega_{д}(\xi) \right), \omega_{\mathbb{K}}(\xi).\varpi_{д}(\xi), \sigma_{\mathbb{K}}(\xi).\sigma_{д}(\xi) \right\rangle | \xi \in \Xi \right\}, \\ Ч \quad &\twoheadrightarrow \quad Д = \left\{ \left\langle \xi, \left(\varpi_{ч}(\xi) \vee \omega_{д}(\xi) \right), \omega_{ч}(\xi).\varpi_{д}(\xi), \sigma_{ч}(\xi).\sigma_{д}(\xi) \right\rangle | \xi \in \Xi \right\}, \end{aligned}$$

now $(\varpi_{\mathbb{K}}(\xi) \vee \omega_{д}(\xi)) \geq (\varpi_{ч}(\xi) \vee \omega_{д}(\xi))$, $\omega_{\mathbb{K}}(\xi).\varpi_{д}(\xi) \leq \omega_{ч}(\xi).\varpi_{д}(\xi)$, and $\sigma_{\mathbb{K}}(\xi).\sigma_{д}(\xi) \geq \sigma_{ч}(\xi).\sigma_{д}(\xi)$ or $\sigma_{\mathbb{K}}(\xi).\sigma_{д}(\xi) \geq \sigma_{ч}(\xi).\sigma_{д}(\xi)$. Therefore, Ч $\twoheadrightarrow$ Д $\subseteq$ $\mathbb{K} \twoheadrightarrow$ Д.

($For\ Axiom2$), let $\mathbb{K} \subseteq$ Ч. Then,

$$\begin{aligned} Д \quad &\twoheadrightarrow \quad \mathbb{K} = \left\{ \left\langle \xi, \left(\varpi_{д}(\xi) \vee \omega_{\mathbb{K}}(\xi) \right), \omega_{д}(\xi).\varpi_{\mathbb{K}}(\xi), \sigma_{д}(\xi).\sigma_{\mathbb{K}}(\xi) \right\rangle | \xi \in \Xi \right\}, \\ Д \quad &\twoheadrightarrow \quad Ч = \left\{ \left\langle \xi, \left(\varpi_{д}(\xi) \vee \omega_{ч}(\xi) \right), \omega_{д}(\xi).\varpi_{ч}(\xi), \sigma_{д}(\xi).\sigma_{ч}(\xi) \right\rangle | \xi \in \Xi \right\}. \end{aligned}$$

That is, it follows $(\varpi_{д}(\xi) \vee \omega_{\mathbb{K}}(\xi)) \leq (\varpi_{д}(\xi) \vee \omega_{ч}(\xi))$, $\omega_{д}(\xi).\varpi_{\mathbb{K}}(\xi) \geq \omega_{д}(\xi).\varpi_{ч}(\xi)$, and
$\sigma_{д}(\xi).\sigma_{\mathbb{K}}(\xi)$
$\geq \sigma_{д}(\xi).\sigma_{ч}(\xi)$ or $\sigma_{д}(\xi).\sigma_{\mathbb{K}}(\xi) \leq \sigma_{д}(\xi).\sigma_{ч}(\xi)$. Thus, Д $\twoheadrightarrow$ Ч $\supseteq$ Д $\twoheadrightarrow \mathbb{K}$.

($For\ Axiom3$),

$$\flat \twoheadrightarrow Ч = \left\{ \left\langle \xi, \left(1 \vee \omega_{ч}(\xi) \right), 0.\varpi_{ч}(\xi), 0.\sigma_{ч}(\xi) \right\rangle | \xi \in \Xi \right\} = \left\{ \left\langle \xi, 1, 0, 0 \right\rangle | \xi \in \Xi \right\} = \sharp,$$

this also meaning $\flat \twoheadrightarrow$ Ч is a PFTaut set, ($Axiom3^*$) is satisfied.

($For\ Axiom4^*$),

$$\sharp \twoheadrightarrow Ч = \left\{ \left\langle \xi, \left(0 \vee \omega_{ч}(\xi) \right), 1.\varpi_{ч}(\xi), 0.\sigma_{ч}(\xi) \right\rangle | \xi \in \Xi \right\} = \left\{ \left\langle \xi, \omega_{ч}(\xi), \varpi_{ч}, 0 \right\rangle | \xi \in \Xi \right\} \neq Ч,$$

that is, Axiom 4 is not satisfied, and also $\sharp \twoheadrightarrow$ Ч is not a PFTaut set, $(Axiom4^*)$ is not satisfied in general.

$(For\ Axiom5^*)$, $\mathbb{K} \twoheadrightarrow \mathbb{K} = \{\langle \xi, (\varpi_{\mathbb{K}}(\xi) \vee \omega_{\mathbb{K}}(\xi)), \omega_{\mathbb{K}}(\xi).\varpi_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi).\sigma_{\mathbb{K}}(\xi)\rangle \,|\xi \in \Xi\}$,

since $(\varpi_{\mathbb{K}}(\xi) \vee \omega_{\mathbb{K}}(\xi)) \geq \omega_{\mathbb{K}}(\xi).\varpi_{\mathbb{K}}(\xi)$. Then, $\mathbb{K} \twoheadrightarrow \mathbb{K}$ is a PFTaut set, $(Axiom5^*)$ is satisfied while $(Axiom5)$ is not valid. Because we can find elements $\xi \in \Xi$ for which $(\varpi_{\mathbb{K}}(\xi) \vee \omega_{\mathbb{K}}(\xi)) < 1$, $\omega_{\mathbb{K}}(\xi).\varpi_{\mathbb{K}}(\xi) > 0$.

$(For\ Axiom6)$,

$$\begin{aligned}
\mathbb{K} \twoheadrightarrow (\text{Ч} \twoheadrightarrow \text{Д}) &= \mathbb{K} \twoheadrightarrow \{\langle \xi, (\varpi_{\text{ч}}(\xi) \vee \omega_{\text{д}}(\xi)), \omega_{\text{ч}}(\xi).\varpi_{\text{д}}(\xi), \sigma_{\text{ч}}(\xi).\sigma_{\text{д}}(\xi)\rangle \,|\xi \in \Xi\} \\
&= \{\langle \xi, (\varpi_{\mathbb{K}}(\xi) \vee (\varpi_{\text{ч}}(\xi) \vee \omega_{\text{д}}(\xi))), \omega_{\mathbb{K}}(\xi).\omega_{\text{ч}}(\xi).\varpi_{\text{д}}(\xi), \sigma_{\mathbb{K}}(\xi).\sigma_{\text{ч}}(\xi).\sigma_{\text{д}}(\xi)\rangle \,|\xi \in \Xi\} \\
&= \{\langle \xi, (\varpi_{\mathbb{K}}(\xi) \vee \varpi_{\text{ч}}(\xi) \vee \omega_{\text{д}}(\xi)), \omega_{\mathbb{K}}(\xi).\omega_{\text{ч}}(\xi).\varpi_{\text{ч}}(\xi), \sigma_{\mathbb{K}}(\xi).\sigma_{\text{ч}}(\xi).\sigma_{\text{д}}(\xi)\rangle \,|\xi \in \Xi\} \\
&= \{\langle \xi, (\varpi_{\text{ч}}(\xi) \vee (\varpi_{\mathbb{K}}(\xi) \vee \omega_{\text{д}}(\xi))), \omega_{\text{ч}}(\xi).\omega_{\mathbb{K}}(\xi).\varpi_{\text{д}}(\xi), \sigma_{\text{ч}}(\xi).\sigma_{\mathbb{K}}(\xi).\sigma_{\text{д}}(\xi)\rangle \,|\xi \in \Xi\} \\
&= \text{Ч} \twoheadrightarrow \{\langle \xi, (\varpi_{\mathbb{K}}(\xi) \vee \omega_{\text{д}}(\xi)), \omega_{\mathbb{K}}(\xi).\varpi_{\text{д}}(\xi), \sigma_{\mathbb{K}}(\xi).\sigma_{\text{д}}(\xi)\rangle \,|\xi \in \Xi\} \\
&= \text{Ч} \twoheadrightarrow (\mathbb{K} \twoheadrightarrow \text{Д}).
\end{aligned}$$

$(For\ Axiom7^*)$, if $\mathbb{K} \twoheadrightarrow \text{Ч} = \{\langle \xi, (\varpi_{\mathbb{K}}(\xi) \vee \omega_{\text{ч}}(\xi)), \omega_{\mathbb{K}}(\xi).\varpi_{\text{ч}}(\xi), \sigma_{\mathbb{K}}(\xi).\sigma_{\text{ч}}(\xi)\rangle \,|\xi \in \Xi\} = \sharp$,

then $(\varpi_{\mathbb{K}}(\xi) \vee \omega_{\text{ч}}(\xi)) = 1$, $\omega_{\mathbb{K}}(\xi).\varpi_{\text{ч}}(\xi) = 0$, $\sigma_{\mathbb{K}}(\xi).\sigma_{\text{ч}}(\xi) = 0$. Therefore, either $\mathbb{K} = \flat$ and hence $\mathbb{K} \subseteq$ Ч or Ч $= \sharp$ and it means again $\mathbb{K} \subseteq$ Ч. Conversely, if we suppose $\mathbb{K} \subseteq$ Ч, then $\omega_{\mathbb{K}}(\xi) \leq \omega_{\text{ч}}(\xi)$, $\varpi_{\mathbb{K}}(\xi) \geq \varpi_{\text{ч}}(\xi)$, $\sigma_{\mathbb{K}}(\xi) \geq \sigma_{\text{ч}}(\xi)$ or $\sigma_{\mathbb{K}}(\xi) \leq \sigma_{\text{ч}}(\xi)$. Thus, $(\varpi_{\mathbb{K}}(\xi) \vee \omega_{\text{ч}}(\xi)) + \omega_{\mathbb{K}}(\xi).\varpi_{\text{ч}}(\xi)$
$+ \sigma_{\mathbb{K}}(\xi).\sigma_{\text{ч}}(\xi) \geq \omega_{\text{ч}}(\xi) + \omega_{\mathbb{K}}(\xi).\varpi_{\text{ч}}(\xi) + \sigma_{\mathbb{K}}(\xi).\sigma_{\text{ч}}(\xi) \geq \omega_{\text{ч}}(\xi) + \varpi_{\text{ч}}(\xi) + \sigma_{\text{ч}}(\xi) \geq 0$, or $(\varpi_{\mathbb{K}}(\xi) \vee \omega_{\text{ч}}(\xi)) + \omega_{\mathbb{K}}(\xi).\varpi_{\text{ч}}(\xi) + \sigma_{\mathbb{K}}(\xi).\sigma_{\text{ч}}(\xi) \geq \varpi_{\mathbb{K}}(\xi) + \omega_{\mathbb{K}}(\xi).\varpi_{\text{ч}}(\xi) + \sigma_{\mathbb{K}}(\xi).\sigma_{\text{ч}}(\xi) \geq \varpi_{\mathbb{K}}(\xi) + \omega_{\mathbb{K}}(\xi) + \sigma_{\mathbb{K}}(\xi) \geq 0$, and then $\mathbb{K} \twoheadrightarrow$ Ч is a PFTaut set. But if $\mathbb{K} \subseteq$ Ч, then $(\varpi_{\mathbb{K}}(\xi) \vee \omega_{\text{ч}}(\xi)) \geq \omega_{\mathbb{K}}(\xi).\varpi_{\text{ч}}(\xi)$, and so may not imply $(\varpi_{\mathbb{K}}(\xi) \vee \omega_{\text{ч}}(\xi)) = 1, \omega_{\mathbb{K}}(\xi).\varpi_{\text{ч}}(\xi) = 0$ and $\sigma_{\mathbb{K}}(\xi).\sigma_{\text{ч}}(\xi) = 0$. Hence, $(Axiom7)$ is not satisfied in general.

$(For\ Axiom8)$,

$$\begin{aligned}
\mathbb{K} \twoheadrightarrow \text{Ч} &= \{\langle \xi, (\varpi_{\mathbb{K}}(\xi) \vee \omega_{\text{ч}}(\xi)), \omega_{\mathbb{K}}(\xi).\varpi_{\text{ч}}(\xi), \sigma_{\mathbb{K}}(\xi).\sigma_{\text{ч}}(\xi)\rangle \,|\xi \in \Xi\} \\
&= \{\langle \xi, (\omega_{\text{ч}}(\xi) \vee \varpi_{\mathbb{K}}(\xi)), \varpi_{\text{ч}}(\xi).\omega_{\mathbb{K}}(\xi), \sigma_{\text{ч}}(\xi).\sigma_{\mathbb{K}}(\xi)\rangle \,|\xi \in \Xi\} \\
&= \{\langle \xi, \varpi_{\text{ч}}(\xi), \omega_{\text{ч}}(\xi), \sigma_{\text{ч}}(\xi)\rangle \,|\xi \in \Xi\} \twoheadrightarrow \{\langle \xi, \varpi_{\mathbb{K}}(\xi), \omega_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi)\rangle \,|\xi \in \Xi\} \\
&= \exists\, \text{Ч} \twoheadrightarrow \exists \mathbb{K}.
\end{aligned}$$

$(For\ Axiom9)$, since the operations "max " and "multiplication" as functions preserve the continuity, then $\twoheadrightarrow$ is a continuous function.

According to the above theorem, for all $\mathbb{K} \in \mathcal{P}(\sharp)$, the implication operation $\twoheadrightarrow$ satisfies the Axioms (1, 2, 3, 3*, 5*, 6, 7*, 8, 9). Moreover, in the intuitionistic fuzzy case, that is, if we take $\sigma_{\mathbb{K}}(\xi) = 0$ for all $\mathbb{K} \in \mathcal{P}(\sharp)$, then this implication operation $\twoheadrightarrow$ satisfies also $(Axiom4)$ as the case given in [31] for (IFSs) but still not satisfying $(Axiom4^*)$.

As another extension for the defined operations $(*)$ and $(\#)$, respectively, we introduce here these forms:

$$\mathfrak{W}(\mathbb{K}) = \{\langle \xi, \epsilon_{\mathbb{K}}, \varrho_{\mathbb{K}}, F_{\mathbb{K}}\rangle \,|\xi \in \Xi\}, \mathfrak{Z}(\mathbb{K}) = \{\langle \xi, \varphi_{\mathbb{K}}, \vartheta_{\mathbb{K}}, F_{\mathbb{K}}\rangle \,|\xi \in \Xi\}$$

where $\varphi_{\mathbb{K}} = \prod_{\xi\in\Xi} \omega_{\mathbb{K}}(\xi),\ \varrho_{\mathbb{K}} = \prod_{\xi\in\Xi} \varpi_{\mathbb{K}}(\xi),\ \digamma_{\mathbb{K}} = \prod_{\xi\in\Xi} \sigma_{\mathbb{K}}(\xi)$.

These new operations are well defined because

$$
\begin{aligned}
0 &\leq \epsilon_{\mathbb{K}} \leq 1,\, 0 \leq \varrho_{\mathbb{K}} \leq 1,\, 0 \leq \digamma_{\mathbb{K}} \leq 1, \\
0 &\leq \varphi_{\mathbb{K}} \leq 1,\, 0 \leq \vartheta_{\mathbb{K}} \leq 1,\, 0 \leq \digamma_{\mathbb{K}} \leq 1, \\
0 &\leq \epsilon_{\mathbb{K}} + \varrho_{\mathbb{K}} + \digamma_{\mathbb{K}} \leq \epsilon_{\mathbb{K}} + \aleph_{\mathbb{K}} + \kappa_{\mathbb{K}} \leq 1, \\
0 &\leq \varphi_{\mathbb{K}} + \vartheta_{\mathbb{K}} + \digamma_{\mathbb{K}} \leq \varepsilon_{\mathbb{K}} + \vartheta_{\mathbb{K}} + \kappa_{\mathbb{K}} \leq 1.
\end{aligned}
$$

These operations are dual to each other. For $\mathbb{K} \in \mathcal{P}(\sharp)$,

$$
\begin{aligned}
\dashv\mathfrak{Z}(\dashv\mathbb{K}) &= \dashv\mathfrak{Z}\{\langle\xi, \varpi_{\mathbb{K}}(\xi), \omega_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi)\rangle \,|\, \xi\in\Xi\} = \dashv\{\langle\xi, \varrho_{\mathbb{K}}, \epsilon_{\mathbb{K}}, \digamma_{\mathbb{K}}\rangle \,|\, \xi\in\Xi\} \\
&= \{\langle\xi, \epsilon_{\mathbb{K}}, \varrho_{\mathbb{K}}, \digamma_{\mathbb{K}}\rangle \,|\, \xi\in\Xi\} = \mathfrak{W}(\mathbb{K}),
\end{aligned}
$$

and analogously, $\dashv\mathfrak{W}(\dashv\mathbb{K}) = \mathfrak{Z}(\mathbb{K})$.

For any $\mathbb{K} \in \mathcal{P}(\sharp)$, we have: $\mathfrak{Z}(\mathbb{K}) \subseteq int_{\cup}(\mathbb{K}) \subseteq \mathbb{K} \subseteq cl_{\cap}(\mathbb{K}) \subseteq \mathfrak{W}(\mathbb{K})$.

Let $\mathcal{O}$ and $\mathcal{Q}$ be topological operators such that for each PFS $\mathbb{K} \in \mathcal{P}(\sharp)$:

$$\mathcal{O}(\mathbb{K}) = \dashv\mathcal{Q}(\dashv\mathbb{K}),\ \ \mathcal{Q}(\mathbb{K}) = \dashv\mathcal{O}(\dashv\mathbb{K}).$$

Let $\triangle, \triangledown : \mathcal{P}(\sharp) \times \mathcal{P}(\sharp) \to \mathcal{P}(\sharp)$ be operations over Ξ such that for any two $\mathbb{K},\ Ч \in \mathcal{P}(\sharp)$,

$$\mathbb{K} \triangledown Ч = \dashv(\dashv\mathbb{K} \triangle \dashv Ч),\ \ \mathbb{K} \triangle Ч = \dashv(\dashv\mathbb{K} \triangledown \dashv Ч).$$

Let $\circ$ and $\bullet : \mathcal{P}(\sharp) \to \mathcal{P}(\sharp)$ be two modal operators over Ξ such that for any $\mathbb{K} \in \mathcal{P}(\sharp)$:

$$\circ\mathbb{K} = \dashv\bullet(\dashv\mathbb{K}),\quad \bullet\mathbb{K} = \dashv\circ(\dashv\mathbb{K}).$$

In [31], Atanassov investigated (for IFSs) the notions of PFMTS and feeble PFMTS (PFFMTS, for short) and, moreover in [13], established extensions of those definitions named by *cl-cl*-PFMTS, *int-int*-PFMTS, *cl-int*-PFMTS and *int-cl*-PFMTS regarding the types of the topological operators "closure" and "interior"and any of the given modal operators. In similar strategy, we will define (for PFSs) four certain cases.

Theorem 2.3. *$\langle\mathcal{P}(\sharp), \mathfrak{W}, *, \diamond\rangle$ is a* cl-cl-*PFFMTS for which in conditions CC4, CC5 and CC9, the relation " = " is changed to the relation " $\supseteq$ ".*

Proof. Let $\mathbb{K},\ Ч \in \mathcal{P}(\sharp)$. Then, we check in a sequential manner the validity of the nine conditions CC1 – CC9. The checks of conditions CC1 - CC4 are analogous, but different from those in [22], while of CC6 - CC8 are the same. We give them only here for completeness of the proof.

$(CC1)$

$\mathfrak{W}(\mathbb{K} * Ч) = \mathfrak{W}(\{\langle\xi, \omega_{\mathbb{K}}(\xi) \vee \omega_{Ч}(\xi), \varpi_{\mathbb{K}}(\xi).\varpi_{Ч}(\xi), \sigma_{\mathbb{K}}(\xi).\sigma_{Ч}(\xi)\rangle \,|\, \xi\in\Xi\})$

$$= \left\{\left\langle \xi, \bigvee_{\xi\in\Xi}(\omega_{\mathbb{K}}(\xi) \vee \omega_{Ч}(\xi)), \prod_{\xi\in\Xi}(\varpi_{\mathbb{K}}(\xi).\varpi_{Ч}(\xi)), \prod_{\xi\in\Xi}(\sigma_{\mathbb{K}}(\xi).\sigma_{Ч}(\xi)) \right\rangle |\xi\in\Xi\right\}$$

$= \{\langle \xi, \epsilon_{\mathbb{K}} \vee \epsilon_{\text{Ч}}, \varrho_{\mathbb{K}}.\varrho_{\text{Ч}}, F_{\mathbb{K}}.F_{\text{Ч}}\rangle \,|\xi \in \Xi\}$
$= \{\langle \xi, \epsilon_{\mathbb{K}}, \varrho_{\mathbb{K}}, F_{\mathbb{K}}\rangle \,|\xi \in \Xi\} * \{\langle \xi, \epsilon_{\text{Ч}}, \varrho_{\text{Ч}}, F_{\text{Ч}}\rangle \,|\xi \in \Xi\} = \mathfrak{W}(\mathbb{K}) * \mathfrak{W}(\text{Ч})$,
$(CC2)$ $\mathbb{K} = \{\langle \xi, \omega_{\mathbb{K}}(\xi), \varpi_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi)\rangle \,|\xi \in \Xi\} \subseteq \{\langle \xi, \epsilon_{\mathbb{K}}, \varrho_{\mathbb{K}}, F_{\mathbb{K}}\rangle \,|\xi \in \Xi\} = \mathfrak{W}(\mathbb{K})$,
$(CC3)$ $\mathfrak{W}(\flat) = \mathfrak{W}(\{\langle \xi, 0, 1, 0\rangle \,|\xi \in \Xi\}) = \left\{\left\langle \xi, \bigvee_{\xi\in\Xi} 0, \prod_{\xi\in\Xi} 1, \prod_{\xi\in\Xi} 0\right\rangle |\xi \in \Xi\right\}$
$= \{\langle \xi, 0, 1, 0\rangle \,|\xi \in \Xi\} = \flat$,
$(CC4)$ $\mathfrak{W}(\mathfrak{W}(\mathbb{K})) = \mathfrak{W}(\{\langle \xi, \epsilon_{\mathbb{K}}, \varrho_{\mathbb{K}}, F_{\mathbb{K}}\rangle \,|\xi \in \Xi\}) = \left\{\left\langle \xi, \bigvee_{\xi\in\Xi} \epsilon_{\mathbb{K}}, \prod_{\xi\in\Xi} \varrho_{\mathbb{K}}, \prod_{\xi\in\Xi} F_{\mathbb{K}}\right\rangle |\xi \in \Xi\right\}$
$= \left\{\left\langle \xi, \epsilon_{\mathbb{K}}, (\varrho_{\mathbb{K}})^{\mathcal{E}}, (F_{\mathbb{K}})^{\mathcal{E}}\right\rangle |\xi \in \Xi\right\} \supseteq \{\langle \xi, \epsilon_{\mathbb{K}}, \varrho_{\mathbb{K}}, F_{\mathbb{K}}\rangle \,|\xi \in \Xi\} = \mathfrak{W}(\mathbb{K})$,
$(CC5)$ $\diamond(\mathbb{K} * \text{Ч}) = \diamond(\{\langle \xi, \omega_{\mathbb{K}}(\xi) \vee \omega_{\text{Ч}}(\xi), \varpi_{\mathbb{K}}(\xi).\varpi_{\text{Ч}}(\xi), \sigma_{\mathbb{K}}(\xi).\sigma_{\text{Ч}}(\xi)\rangle \,|\xi \in \Xi\})$
$= \{\langle \xi, 1 - \varpi_{\mathbb{K}}(\xi).\varpi_{\text{Ч}}(\xi) - \sigma_{\mathbb{K}}(\xi).\sigma_{\text{Ч}}(\xi), \varpi_{\mathbb{K}}(\xi).\varpi_{\text{Ч}}(\xi), \sigma_{\mathbb{K}}(\xi).\sigma_{\text{Ч}}\rangle \,|\xi \in \Xi\}$
$\supseteq \{\langle \xi, 1 - (\varpi_{\mathbb{K}}(\xi) \wedge \varpi_{\text{Ч}}(\xi)) - (\sigma_{\mathbb{K}}(\xi) \wedge \sigma_{\text{Ч}}(\xi)), \varpi_{\mathbb{K}}(\xi).\varpi_{\text{Ч}}(\xi), \sigma_{\mathbb{K}}(\xi).\sigma_{\text{Ч}}(\xi)\rangle \,|\xi \in \Xi\}$
$\supseteq \{\langle \xi, (1 - \varpi_{\mathbb{K}}(\xi) - \sigma_{\mathbb{K}}(\xi)) \vee (1 - \varpi_{\text{Ч}}(\xi) - \sigma_{\text{Ч}}(\xi)), \varpi_{\mathbb{K}}(\xi).\varpi_{\text{Ч}}(\xi), \sigma_{\mathbb{K}}(\xi).\sigma_{\text{Ч}}(\xi)\rangle \,|\xi \in \Xi\}$
$= \{\langle \xi, 1 - \varpi_{\mathbb{K}}(\xi) - \sigma_{\mathbb{K}}(\xi), \varpi_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi)\rangle \,|\xi \in \Xi\} * \{\langle \xi, 1 - \varpi_{\text{Ч}}(\xi) - \sigma_{\text{Ч}}(\xi), \varpi_{\text{Ч}}, \sigma_{\text{Ч}}(\xi)\rangle$
$|\,\xi \in \Xi\} = \diamond(\mathbb{K}) * \diamond(\text{Ч})$,
$(CC6)$ $\mathbb{K} = \{\langle \xi, \omega_{\mathbb{K}}(\xi), \varpi_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi)\rangle \,|\xi \in \Xi\} \subseteq \{\langle \xi, 1 - \varpi_{\mathbb{K}}(\xi) - \sigma_{\mathbb{K}}(\xi), \varpi_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi)\rangle$
$|\,\xi \in \Xi\} = \diamond(\mathbb{K})$,
$(CC7)$ $\diamond(\sharp) = \diamond(\{\langle \xi, 1, 0, 0\rangle \,|\xi \in \Xi\}) = \{\langle \xi, 1, 0, 0\rangle \,|\xi \in \Xi\} = \sharp$,
$(CC8)$ $\diamond(\diamond(\mathbb{K})) = \diamond(\{\langle \xi, 1 - \varpi_{\mathbb{K}}(\xi) - \sigma_{\mathbb{K}}(\xi), \varpi_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi)\rangle \,|\xi \in \Xi\})$
$= \{\langle \xi, 1 - \varpi_{\mathbb{K}}(\xi) - \sigma_{\mathbb{K}}(\xi), \varpi_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi)\rangle \,|\xi \in \Xi\} = \diamond(\mathbb{K})$,
$(CC9)$ $\diamond(\mathfrak{W}(\mathbb{K})) = \diamond(\{\langle \xi, \epsilon_{\mathbb{K}}, \varrho_{\mathbb{K}}, F_{\mathbb{K}}\rangle \,|\xi \in \Xi\}) = \{\langle \xi, 1 - \varrho_{\mathbb{K}} - F_{\mathbb{K}}, \varrho_{\mathbb{K}}, F_{\mathbb{K}}\rangle \,|\xi \in \Xi\}$
$\supseteq \{\langle \xi, 1 - \aleph_{\mathbb{K}} - \kappa_{\mathbb{K}}, \varrho_{\mathbb{K}}, F_{\mathbb{K}}\rangle \,|\xi \in \Xi\} \supseteq \left\{\left\langle \xi, \bigvee_{\xi\in\Xi} (1 - \varpi_{\mathbb{K}}(\xi) - \sigma_{\mathbb{K}}(\xi)), \varrho_{\mathbb{K}}, F_{\mathbb{K}}\right\rangle |\xi \in \Xi\right\}$
$= \mathfrak{W}(\{\langle \xi, 1 - \varpi_{\mathbb{K}}(\xi) - \sigma_{\mathbb{K}}(\xi), \varpi_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi)\rangle \,|\xi \in \Xi\}) = \mathfrak{W}(\diamond(\mathbb{K}))$.

The reason that operators $\mathfrak{W}$ and $\diamond$ are from one type ("closure") is in the validity of conditions CC2 and CC6.

Theorem 2.4. $\langle \mathcal{P}(\sharp), \mathfrak{Z}, \#, \square\rangle$ *is an* int-int-*PFFMTS for which in conditions II4, II5 and II9, the relation "* $=$ *" is changed to the relation "* $\subseteq$ *".*

Proof. Let $\mathbb{K}$, $\text{Ч} \in \mathcal{P}(\sharp)$. Then, we check in a sequential manner the validity of the conditions II1–II5, and II9 because the checks of the validity of conditions II6–II8 are given in [22] and are similar to these in the proof of Theorem 2.3.

$(II1)$
$\mathfrak{Z}(\mathbb{K}\#\,\text{Ч}) = \mathfrak{Z}(\{\langle \xi, \omega_{\mathbb{K}}(\xi).\omega_{\text{Ч}}(\xi), \varpi_{\mathbb{K}}(\xi) \vee \varpi_{\text{Ч}}(\xi), \sigma_{\mathbb{K}}(\xi).\sigma_{\text{Ч}}(\xi)\rangle \,|\xi \in \Xi\})$
$= \left\{\left\langle \xi, \prod_{\xi\in\Xi} (\omega_{\mathbb{K}}(\xi).\omega_{\text{Ч}}(\xi)), \bigvee_{\xi\in\Xi} (\varpi_{\mathbb{K}}(\xi) \vee \varpi_{\text{Ч}}(\xi)), \prod_{\xi\in\Xi} (\sigma_{\mathbb{K}}(\xi).\sigma_{\text{Ч}}(\xi))\right\rangle |\xi \in \Xi\right\}$
$= \{\langle \xi, \varphi_{\mathbb{K}}.\varphi_{\text{Ч}}, \vartheta_{\mathbb{K}} \vee \vartheta_{\text{Ч}}, F_{\mathbb{K}}.F_{\text{Ч}}\rangle \,|\xi \in \Xi\}$
$= \{\langle \xi, \varphi_{\mathbb{K}}, \vartheta_{\mathbb{K}}, F_{\mathbb{K}}\rangle \,|\xi \in \Xi\} \# \{\langle \xi, \varphi_{\text{Ч}}, \vartheta_{\text{Ч}}, F_{\text{Ч}}\rangle \,|\xi \in \Xi\} = \mathfrak{Z}(\mathbb{K}) \# \mathfrak{Z}(\text{Ч})$,
$(II2)$ $\mathbb{K} = \{\langle \xi, \omega_{\mathbb{K}}(\xi), \varpi_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi)\rangle \,|\xi \in \Xi\} \supseteq \{\langle \xi, \varphi_{\mathbb{K}}, \vartheta_{\mathbb{K}}, F_{\mathbb{K}}\rangle \,|\xi \in \Xi\} = \mathfrak{Z}(\mathbb{K})$,

$(II3)$ $\mathfrak{Z}(\sharp) = \mathfrak{Z}\left(\left\{\left\langle \xi, \bigvee_{\xi\in\Xi} 1, \prod_{\xi\in\Xi} 0, \prod_{\xi\in\Xi} 0 \right\rangle |\xi \in \Xi\right\}\right) = \sharp,$

$(II4)$ $\mathfrak{Z}(\mathfrak{Z}(\mathbb{K})) = \mathfrak{Z}(\{\langle \xi, \varphi_{\mathbb{K}}, \vartheta_{\mathbb{K}}, F_{\mathbb{K}}\rangle |\xi \in \Xi\}) = \left\{\left\langle \xi, \prod_{\xi\in\Xi} \varphi_{\mathbb{K}}, \bigvee_{\xi\in\Xi} \vartheta_{\mathbb{K}}, \prod_{\xi\in\Xi} F_{\mathbb{K}} \right\rangle |\xi \in \Xi\right\}$

$= \left\{\left\langle \xi, (\varphi_{\mathbb{K}})^{\mathcal{E}}, \vartheta_{\mathbb{K}}, (F_{\mathbb{K}})^{\mathcal{E}} \right\rangle |\xi \in \Xi\right\} \subseteq \{\langle \xi, \varphi_{\mathbb{K}}, \vartheta_{\mathbb{K}}, F_{\mathbb{K}}\rangle |\xi \in \Xi\} = \mathfrak{Z}(\mathbb{K}),$

$(II5)$ $\Box(\mathbb{K}\#\text{Ч}) = \Box(\{\langle \xi, \omega_{\mathbb{K}}(\xi).\omega_{\text{Ч}}(\xi), (\varpi_{\mathbb{K}}(\xi) \vee \varpi_{\text{Ч}}(\xi)), \sigma_{\mathbb{K}}(\xi).\sigma_{\text{Ч}}(\xi)\rangle |\xi \in \Xi\})$

$= \{\langle \xi, \omega_{\mathbb{K}}(\xi).\omega_{\text{Ч}}(\xi), 1 - \omega_{\mathbb{K}}(\xi).\omega_{\text{Ч}}(\xi) - \sigma_{\mathbb{K}}(\xi).\sigma_{\text{Ч}}(\xi), \sigma_{\mathbb{K}}(\xi).\sigma_{\text{Ч}}\rangle |\xi \in \Xi\}$

$\subseteq \{\langle \xi, \omega_{\mathbb{K}}(\xi).\omega_{\text{Ч}}(\xi), 1 - (\omega_{\mathbb{K}}(\xi) \wedge \omega_{\text{Ч}}(\xi)) - (\sigma_{\mathbb{K}}(\xi) \wedge \sigma_{\text{Ч}}(\xi)), \sigma_{\mathbb{K}}(\xi).\sigma_{\text{Ч}}(\xi)\rangle |\xi \in \Xi\}$

$\subseteq \{\langle \xi, \omega_{\mathbb{K}}(\xi).\omega_{\text{Ч}}(\xi), (1 - \omega_{\mathbb{K}}(\xi) - \sigma_{\mathbb{K}}(\xi)) \vee (1 - \omega_{\text{Ч}}(\xi) - \sigma_{\text{Ч}}(\xi)), \sigma_{\mathbb{K}}(\xi).\sigma_{\text{Ч}}(\xi)\rangle |\xi \in \Xi\}$

$= \{\langle \xi, \omega_{\mathbb{K}}(\xi), 1 - \omega_{\mathbb{K}}(\xi) - \sigma_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi)\rangle |\xi \in \Xi\} \# \{\langle \xi, \omega_{\text{Ч}}(\xi), 1 - \omega_{\text{Ч}}(\xi) - \sigma_{\text{Ч}}, \sigma_{\text{Ч}}(\xi)\rangle |\xi \in \Xi\}$

$= \Box(\mathbb{K})\#\Box(\text{Ч}),$

$(II6)$ $\Box(\mathbb{K}) = \Box(\{\langle \xi, \omega_{\mathbb{K}}(\xi), \varpi_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi)\rangle |\xi \in \Xi\})$

$= \{\langle \xi, \omega_{\mathbb{K}}(\xi), 1 - \omega_{\mathbb{K}}(\xi) - \sigma_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi)\rangle |\xi \in \Xi\} \subseteq \mathbb{K}.$

$(II7)$ $\Box(\flat) = \Box(\{\langle \xi, 0, 1, 0\rangle |\xi \in \Xi\}) = \{\langle \xi, 0, 1, 0\rangle |\xi \in \Xi\} = \flat.$

$(II8)$ $\Box(\Box(\mathbb{K})) = \Box(\{\langle \xi, \omega_{\mathbb{K}}(\xi), 1 - \omega_{\mathbb{K}}(\xi) - \sigma_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi)\rangle |\xi \in \Xi\})$

$= \{\langle \xi, \omega_{\mathbb{K}}(\xi), 1 - \omega_{\mathbb{K}}(\xi) - \sigma_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi)\rangle |\xi \in \Xi\} = \Box(\mathbb{K}).$

$(II9)$ $\Box(\mathfrak{Z}(\mathbb{K})) = \Box(\{\langle \xi, \varphi_{\mathbb{K}}, \vartheta_{\mathbb{K}}, F_{\mathbb{K}}\rangle |\xi \in \Xi\}) = \{\langle \xi, \varphi_{\mathbb{K}}, 1 - \varphi_{\mathbb{K}} - F_{\mathbb{K}}, F_{\mathbb{K}}\rangle |\xi \in \Xi\}$

$\subseteq \{\langle \xi, \varphi_{\mathbb{K}}, 1 - \varepsilon_{\mathbb{K}} - \kappa_{\mathbb{K}}, F_{\mathbb{K}}\rangle |\xi \in \Xi\} \subseteq \left\{\left\langle \xi, \varphi_{\mathbb{K}}, \bigvee_{\xi\in\Xi} (1 - \omega_{\mathbb{K}}(\xi) - \sigma_{\mathbb{K}}(\xi)), F_{\mathbb{K}} \right\rangle |\xi \in \Xi\right\}$

$= \mathfrak{Z}(\{\langle \xi, \omega_{\mathbb{K}}(\xi), 1 - \omega_{\mathbb{K}}(\xi) - \sigma_{\mathbb{K}}(\xi), \sigma_{\mathbb{K}}(\xi)\rangle |\xi \in \Xi\}) = \mathfrak{Z}(\Box(\mathbb{K})).$

Theorem 2.5. $\langle \mathcal{P}(\sharp), \mathfrak{W}, *, \Box\rangle$ *is a* cl-int-*PFFMTS for which in condition CI4, the relation " = " is changed to the relation " ⊇ ", and in conditions CI5 and CI9, the relation " = " is changed to the relation " ⊆ ".*

Theorem 2.6. $\langle \mathcal{P}(\sharp), \mathfrak{Z}, \#, \Diamond\rangle$ *is an* int-cl-*PFFMTS for which in condition IC4, the relation " = " is changed to the relation " ⊆ ", and in condition IC5 and CI9, the relation " = " is changed to the relation " ⊇ ".*

3. Picture fuzzy ideal multifunctions

The map $\mathbb{F} : \Xi \nrightarrow \Upsilon$ is called a PFM for any $(\xi, \zeta) \in \Xi \times \Upsilon$ iff $\mathbb{F}(\xi) \in (I^3)^{\Upsilon}$ for each $\xi \in \Xi$. The degree of membership of $\zeta \in \mathbb{F}(\xi)$ is denoted by: $\mathbb{F}(\xi)(\zeta) = \Psi_{\mathbb{F}}(\xi, \zeta)$. The domain of $\mathbb{F}$, denoted by $D(\mathbb{F})$ and the range of $\mathbb{F}$, denoted by $R(\mathbb{F})$, are defined by: for any $\xi \in \Xi$ and $\zeta \in \Upsilon$, $D(\mathbb{F})(\xi) = \bigcup_{\zeta\in\Upsilon} \Psi_{\mathbb{F}}(\xi, \zeta)$ and $R(\mathbb{F})(\zeta) = \bigcup_{\xi\in\Xi} \Psi_{\mathbb{F}}(\xi, \zeta)$. $\mathbb{F}$ is called crisp iff $\Psi_{\mathbb{F}}(\xi, \zeta) = \langle 1, 0, 0\rangle$ $\forall \xi \in \Xi$ and $\zeta \in \Upsilon$.

$\mathbb{F}$ is called normalized PFM iff $\forall \xi \in \Xi$, there exists $\zeta_0 \in \Upsilon$ such that $\Psi_{\mathbb{F}}(\xi, \zeta_0) = \langle 1, 0, 0\rangle$. $\mathbb{F}$ is called surjective iff $R(\mathbb{F})(\zeta) = \langle 1, 0, 0\rangle$ $\forall \zeta \in \Upsilon$. The inverse of $\mathbb{F}$ denoted by $\mathbb{F}^- : \Upsilon \to \Xi$ is a *PFM* defined by: $\mathbb{F}^-(\zeta)(\xi) = \mathbb{F}(\xi)(\zeta) = \Psi_{\mathbb{F}}(\xi, \zeta)$. One easily verifies that $D(\mathbb{F}^-) = R(\mathbb{F})$ and $D(\mathbb{F}) = R(\mathbb{F}^-)$. The image $\mathbb{F}(\mathbb{K})$ of $\mathbb{K} \in (I^3)^{\Xi}$, the lower inverse

$\mathbb{F}^l(\text{Ч})$ of $\text{Ч} \in \left(I^3\right)^{\Upsilon}$ and the upper inverse $\mathbb{F}^u(\text{Ч})$ of $\text{Ч} \in \left(I^3\right)^{\Upsilon}$ are defined respectively as follow:

$$\mathbb{F}(\mathbb{K})(\zeta) = \bigcup_{\xi \in \Xi} \left[\Psi_{\mathbb{F}}(\xi, \zeta) \cap \mathbb{K}(\xi)\right],$$

$$\mathbb{F}^l(\text{Ч})(\xi) = \bigcup_{\zeta \in \Upsilon} \left[\Psi_{\mathbb{F}}(\xi, \zeta) \cap \text{Ч}(\zeta)\right],$$

$$\mathbb{F}^u(\text{Ч})(\xi) = \bigcap_{\zeta \in \Upsilon} \left[⅃\Psi_{\mathbb{F}}(\xi, \zeta) \cup \text{Ч}(\zeta)\right].$$

Definition 3.1. A picture fuzzy topology on Ξ is a map $\tau : \left(I^3\right)^{\Xi} \to I^3$ defined by $\tau(\mathbb{K}) = \langle \omega_\tau(\mathbb{K}), \varpi_\tau(\mathbb{K}), \sigma_\tau(\mathbb{K}) \rangle$ on Ξ which satisfies the following properties:

(1) $\tau(\flat) = \tau(\sharp) = \langle 1, 0, 0 \rangle$.

(2) $\tau(\mathbb{K}_1 \cap \mathbb{K}_2) \geq \tau(\mathbb{K}_1) \wedge \tau(\mathbb{K}_2)$, for each $\mathbb{K}_1, \mathbb{K}_2 \in \left(I^3\right)^{\Xi}$.

(3) $\tau(\bigcup_{i \in \Gamma} \mathbb{K}_i) \geq \bigwedge_{i \in \Gamma} \tau(\mathbb{K}_i)$, for each $\mathbb{K}_i \in \left(I^3\right)^{\Xi}$, $i \in \Gamma$.

The pair (Ξ, τ) is called a picture fuzzy topological space in Šostak's sense. For any $\mathbb{K} \in \left(I^3\right)^{\Xi}$ the number $\omega_\tau(\mathbb{K})$ is called the openness degree, $\varpi_\tau(\mathbb{K})$ is called the non openness degree, while $\sigma_\tau(\mathbb{K})$ is called the neutral degree. For $\mathbb{K} \in \left(I^3\right)^{\Xi}$,

$$\begin{aligned} cl_\tau(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle) &= \bigcap \{ \text{Ч} \in \left(I^3\right)^{\Xi} : \mathbb{K} \subseteq \text{Ч}, \tau(⅃\, \text{Ч}) \geq \langle \varsigma, \varkappa, \vartheta \rangle \}, \\ int_\tau(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle) &= \bigcup \{ \text{Ч} \in \left(I^3\right)^{\Xi} : \mathbb{K} \supseteq \text{Ч}, \tau(\text{Ч}) \geq \langle \varsigma, \varkappa, \vartheta \rangle \}. \end{aligned}$$

Definition 3.2. The map $\ell^P : \left(I^3\right)^{\Xi} \to I^3$ is called picture fuzzy ideal on Ξ if it satisfies the following conditions for $\mathbb{K}$, $\text{Ч} \in \left(I^3\right)^{\Xi}$:

(1) $\ell^P(\flat) = \langle 1, 0, 0 \rangle$, $\ell^P(\sharp) = \langle 0, 1, 0 \rangle$.

(2) $\mathbb{K} \subseteq \text{Ч} \Rightarrow \ell^P(\mathbb{K}) \geq \ell^P(\text{Ч})$.

(3) $\ell^P(\mathbb{K} \cup \text{Ч}) \geq \ell^P(\mathbb{K}) \wedge \ell^P(\text{Ч})$.

If ℓ_1^P and ℓ_2^P are picture fuzzy ideals on Ξ, we say that ℓ_1^P is finer than ℓ_2^P (ℓ_2^P is coarser than ℓ_1^P), denoted by $\ell_2^P \subseteq \ell_1^P$, iff $\ell_2^P(\mathbb{K}) \leq \ell_1^P(\mathbb{K})$ $\forall \mathbb{K} \in \left(I^3\right)^{\Xi}$.

Let us define the special picture fuzzy ideals ℓ^{P0}, ℓ^{P1} by

$$\ell^{P0}(\mathbb{K}) = \begin{cases} \langle 1, 0, 0 \rangle & \text{if } \mathbb{K} = \flat, \\ \langle 0, 1, 0 \rangle & \text{otherwise,} \end{cases} \quad \text{and} \quad \ell^{P1}(\mathbb{K}) = \begin{cases} \langle 0, 1, 0 \rangle & \text{if } \mathbb{K} = \sharp, \\ \langle 1, 0, 0 \rangle & \text{otherwise.} \end{cases}$$

Definition 3.3. Let (Ξ, τ, ℓ^P) be a picture fuzzy ideal topological space, $\mathbb{K} \in \left(I^3\right)^{\Xi}$, $\varsigma \in I_0, \varkappa \in I_1$ and $\vartheta \in I_1$. Then, the $\langle \varsigma, \varkappa, \vartheta \rangle$-fuzzy local function $\Phi(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle)$ of $\mathbb{K}$ defined as follows:

$\Phi(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle) = \bigcap \{ \text{Ч} \in \left(I^3\right)^{\Xi} : \ell^P(\mathbb{K} \overline{\wedge} \text{Ч}) \geq \langle \varsigma, \varkappa, \vartheta \rangle, \tau(⅃\, \text{Ч}) \geq \langle \varsigma, \varkappa, \vartheta \rangle \}$.

Remark 3.1. (1) If we take $\ell^P = \ell^{P0}$ for each $\mathbb{K} \in \left(I^3\right)^{\Xi}$ we have
$\Phi(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle) = \bigcap \{ \text{Ч} \in \left(I^3\right)^{\Xi} : \mathbb{K} \subseteq \text{Ч}, \tau(\exists \ \text{Ч}) \geq \langle \varsigma, \varkappa, \vartheta \rangle \} = cl\,(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle)$.

(2) If we take $\ell^P = \ell^{P1}$ (resp. $\ell^P(\mathbb{K}) \geq \langle \varsigma, \varkappa, \vartheta \rangle$) for each $\mathbb{K} \in \left(I^3\right)^{\Xi}$ we have $\Phi(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle) = \langle 0, 1, 0 \rangle$.

We will occasionally write $\Phi(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle)$ or $\Phi(\mathbb{K}, \ell^P, \langle \varsigma, \varkappa, \vartheta \rangle)$ for $\Phi(\mathbb{K}, \ell^P, \tau, \langle \varsigma, \varkappa, \vartheta \rangle)$.

Theorem 3.1. *Let (Ξ, τ, ℓ^P) be a picture fuzzy ideal topological space and ℓ_1^P,ℓ_2^P be two picture fuzzy ideals on Ξ. Then, for any set $\mathbb{K}$, $\text{Ч} \in \left(I^3\right)^{\Xi}$, $\varsigma \in I_0, \varkappa \in I_1$ and $\vartheta \in I_1$.*

(1) $\Phi(\flat, \langle \varsigma, \varkappa, \vartheta \rangle) = \langle 0, 1, 0 \rangle$.

(2) *If* $\mathbb{K} \subseteq CH$, *then* $\Phi(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle) \subseteq \Phi(\text{Ч}, \langle \varsigma, \varkappa, \vartheta \rangle)$.

(3) *If* $\ell_2^P \subseteq \ell_1^P$, *then* $\Phi(\mathbb{K}, \ell_1^P, \langle \varsigma, \varkappa, \vartheta \rangle) \subseteq \Phi(\mathbb{K}, \ell_2^P, \langle \varsigma, \varkappa, \vartheta \rangle)$.

(4) $\Phi(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle) = cl_\tau\,(\Phi(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle), \langle \varsigma, \varkappa, \vartheta \rangle) \subseteq cl_\tau\,(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle)$.

(5) $\Phi(\Phi(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle), \langle \varsigma, \varkappa, \vartheta \rangle) \subseteq \Phi(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle)$ *and* $\exists\,(\Phi(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle)) \neq \Phi(\exists\,\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle)$.

(6) $\Phi(\mathbb{K} \cup \text{Ч}, \langle \varsigma, \varkappa, \vartheta \rangle) \supseteq \Phi(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle) \cup \Phi(\text{Ч}, \langle \varsigma, \varkappa, \vartheta \rangle)$ *and* $\Phi(\mathbb{K} \cap \text{Ч}, \langle \varsigma, \varkappa, \vartheta \rangle)$ $\subseteq \Phi(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle) \cap \Phi(CH, \langle \varsigma, \varkappa, \vartheta \rangle)$.

(7) *If* $\ell^P(\text{Ч}) \geq \langle \varsigma, \varkappa, \vartheta \rangle$, *then* $\Phi(\mathbb{K} \cup \text{Ч}, \langle \varsigma, \varkappa, \vartheta \rangle) \supseteq \Phi(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle)$.

Proof. (1) From Definition 3.3, we have $\Phi(\flat, \langle \varsigma, \varkappa, \vartheta \rangle) = \langle 0, 1, 0 \rangle$.

(2) Suppose that $\mathbb{K} \subseteq \text{Ч}$ and $\Phi(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle) \nsubseteq \Phi(\text{Ч}, \langle \varsigma, \varkappa, \vartheta \rangle)$. By the definition of $\Phi(\text{Ч}, \langle \varsigma, \varkappa, \vartheta \rangle)$, there exists $\text{Д} \in \left(I^3\right)^{\Xi}$ with $\Phi(\text{Ч}, \langle \varsigma, \varkappa, \vartheta \rangle) \subseteq \text{Д}$, $\ell^P(\text{Ч} \barwedge \text{Д}) \geq \langle \varsigma, \varkappa, \vartheta \rangle$, $\tau(\exists \ \text{Д}) \geq \langle \varsigma, \varkappa, \vartheta \rangle$ and $\Phi(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle) \nsubseteq \text{Д}$. Also, $\mathbb{K} \barwedge \text{Д} \subseteq \text{Ч} \barwedge \text{Д}$, $\ell^P(\mathbb{K} \barwedge \text{Д}) \geq \ell^P(\text{Ч} \barwedge \text{Д}) \geq \langle \varsigma, \varkappa, \vartheta \rangle$, hence $\Phi(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle) \subseteq \text{Д}$, it is a contradiction. Thus, $\Phi(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle) \subseteq \Phi(\text{Ч}, \langle \varsigma, \varkappa, \vartheta \rangle)$.

(3) Suppose that $\Phi(\mathbb{K}, \ell_1^P, \langle \varsigma, \varkappa, \vartheta \rangle) \nsubseteq \Phi(\mathbb{K}, \ell_2^P, \langle \varsigma, \varkappa, \vartheta \rangle)$ if $\ell_2^P \subseteq \ell_1^P$. By the definition of
$\Phi(\mathbb{K}, \ell_2^P, \langle \varsigma, \varkappa, \vartheta \rangle)$ there exists $\text{Д} \in \left(I^3\right)^{\Xi}$ with $\Phi(\mathbb{K}, \ell_2^P, \langle \varsigma, \varkappa, \vartheta \rangle) \subseteq \text{Д}$, $\ell_2^P(\mathbb{K} \barwedge \text{Д}) \geq \langle \varsigma, \varkappa, \vartheta \rangle$, $\tau(\exists \ \text{Д}) \geq \langle \varsigma, \varkappa, \vartheta \rangle$ such that $\Phi(\mathbb{K}, \ell_1^P, \langle \varsigma, \varkappa, \vartheta \rangle) \nsubseteq \text{Д}$. Since $\ell_2^P \subseteq \ell_1^P$ implies $\ell_1^P(\mathbb{K} \barwedge \text{Д}) \geq \ell_2^P(\mathbb{K} \barwedge \text{Д}) \geq \langle \varsigma, \varkappa, \vartheta \rangle$. Hence,
$\Phi(\mathbb{K}, \ell_1^P, \langle \varsigma, \varkappa, \vartheta \rangle) \subseteq \text{Д}$, it is a contradiction. Then, $\Phi(\mathbb{K}, \ell_1^P, \langle \varsigma, \varkappa, \vartheta \rangle) \subseteq \Phi(\mathbb{K}, \ell_2^P, \langle \varsigma, \varkappa, \vartheta \rangle)$.

(4) From Definition 3.3, we have $\Phi(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle) = cl\,(\Phi(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle), \langle \varsigma, \varkappa, \vartheta \rangle)$.

Since $\ell^{P0} \subseteq \ell^P$ for any picture fuzzy ideal ℓ^P, $\Phi(\mathbb{K}, \ell^P, \langle \varsigma, \varkappa, \vartheta \rangle) \subseteq \Phi(\mathbb{K}, \ell^{P0}, \langle \varsigma, \varkappa, \vartheta \rangle) = cl\,(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle)$. Thus, $\Phi(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle) = cl\,(\Phi(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle), \langle \varsigma, \varkappa, \vartheta \rangle) \subseteq cl\,(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle)$.

(5) By (4), we have $\Phi(\Phi(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle), \langle \varsigma, \varkappa, \vartheta \rangle) = cl\,(\Phi(\Phi(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle), \langle \varsigma, \varkappa, \vartheta \rangle), \langle \varsigma, \varkappa, \vartheta \rangle)$ $\subseteq cl\,(\Phi(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle), \langle \varsigma, \varkappa, \vartheta \rangle) = \Phi(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle)$. In general the converse is not true as will be shown in Example 3.1.

(6) Since $\mathbb{K} \subseteq \mathbb{K} \cup \text{Ч}$ and $\text{Ч} \subseteq \mathbb{K} \cup \text{Ч}$, $\Phi(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle) \subseteq \Phi(\mathbb{K} \cup \text{Ч}, \langle \varsigma, \varkappa, \vartheta \rangle)$ and $\Phi(\text{Ч}, \langle \varsigma, \varkappa, \vartheta \rangle) \subseteq \Phi(\mathbb{K} \cup \text{Ч}, \langle \varsigma, \varkappa, \vartheta \rangle)$. Thus, $\Phi(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle) \cup \Phi(\text{Ч}, \langle \varsigma, \varkappa, \vartheta \rangle) \subseteq \Phi(\mathbb{K} \cup \text{Ч}, \langle \varsigma, \varkappa, \vartheta \rangle)$. Also, $\mathbb{K} \cap \text{Ч} \subseteq \mathbb{K}$ and $\mathbb{K} \cap \text{Ч} \subseteq \text{Ч}$, $\Phi(\mathbb{K} \cap \text{Ч}, \langle \varsigma, \varkappa, \vartheta \rangle) \subseteq \Phi(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle)$ and $\Phi(\mathbb{K} \cap \text{Ч}, \langle \varsigma, \varkappa, \vartheta \rangle) \subseteq \Phi(\text{Ч}, \langle \varsigma, \varkappa, \vartheta \rangle)$. Thus, $\Phi(\mathbb{K} \cap \text{Ч}, \langle \varsigma, \varkappa, \vartheta \rangle) \subseteq \Phi(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle) \cap \Phi(\text{Ч}, \langle \varsigma, \varkappa, \vartheta \rangle)$.

(7) Since $\ell^P(\text{Ч}) \supseteq \langle \varsigma, \varkappa, \vartheta \rangle$, $\Phi(\text{Ч}, \langle \varsigma, \varkappa, \vartheta \rangle) = \langle 0, 1, 0 \rangle$.

Thus, $\Phi(\mathbb{K} \cup \text{Ч}, \langle \varsigma, \varkappa, \vartheta \rangle) \supseteq \Phi(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle) \cup \Phi(\text{Ч}, \langle \varsigma, \varkappa, \vartheta \rangle) \supseteq \Phi(\mathbb{K}, \langle \varsigma, \varkappa, \vartheta \rangle)$.

The following example shows that generally $\Phi(\Phi(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle),\langle\varsigma,\varkappa,\vartheta\rangle)\neq\Phi(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)$, and
$\daleth(\Phi(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle))\neq\Phi(\daleth\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)$ for any $\mathbb{K}\in\left(I^3\right)^{\Xi}, \varsigma\in I_0, \varkappa\in I_1$ *and* $\vartheta\in I_1$.

Example 3.1. *Let* $\Xi=\{\xi_1,\xi_2\}$, *for* $\mathbb{K}_1=\{\langle\xi,0.33,0.33,0.2\rangle \mid \xi\in\Xi\}$, $\mathbb{K}_2=\{\langle\xi,0.5,0.3,0.2\rangle \mid \xi\in\Xi\}$, $\mathbb{K}_3=\{\langle\xi,0.33,0.33,0.1\rangle \mid \xi\in\Xi\}$ *and* $Ч=\{\langle\xi,0.4,0.4,0.2\rangle \mid \xi\in\Xi\}$. *Define* τ,ℓ^P: $\left(I^3\right)^{\Xi}\to I^3$ *as follows:*

$$\tau(\mathbb{K})=\begin{cases}\langle 1,0,0\rangle & \text{if } \mathbb{K}\in\{\flat,\sharp\},\\ \langle 0.33,0.33,0.33\rangle & \text{if } \mathbb{K}=\mathbb{K}_1,\\ \langle 0.5,0.2,0.1\rangle & \text{if } \mathbb{K}=\mathbb{K}_2,\\ \langle 0,1,0\rangle & \text{otherwise,}\end{cases}$$

$$\ell^P(\mathbb{K})=\begin{cases}\langle 1,0,0\rangle & \text{if } \mathbb{K}=\flat,\\ \langle 0.75,0.15,0.1\rangle & \text{if } \flat\subseteq\mathbb{K}\subseteq\{\langle\xi,0.33,0.33,0.33\rangle \mid \xi\in\Xi\},\\ \langle 0.4,0.3,0.3\rangle & \text{if } \{\langle\xi,0.33,0.33,0.33\rangle \mid \xi\in\Xi\}\subseteq\mathbb{K}<\sharp,\\ \langle 0,1,0\rangle & \text{otherwise.}\end{cases}$$

Then, $\flat=\Phi(\Phi(Ч,\langle 0.33,0.33,0.33\rangle),\langle 0.33,0.33,0.33\rangle)\neq\Phi(Ч,\langle 0.33,0.33,0.33\rangle)$ $=\{\langle\xi,0.33,0.33,0\rangle \mid \xi\in\Xi\}$, $\sharp=\daleth(\Phi(\mathbb{K}_3,\langle 0.33,0.33,0.33\rangle))\neq\Phi(\daleth\mathbb{K}_3,\langle 0.33,0.33,0.33\rangle)=\flat$.

Definition 3.4. Let (Ξ,τ,ℓ^P) be a picture fuzzy ideal topological space. Then, for each $\mathbb{K}\in\left(I^3\right)^{\Xi}$, $\varsigma\in I_0,\varkappa\in I_1$ and $\vartheta\in I_1$, we define an operator $cl^*:\left(I^3\right)^{\Xi}\times I^3\to\left(I^3\right)^{\Xi}$ as follows:

$$cl^*(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)=\mathbb{K}\cup\Phi(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle).$$

Now, if $\ell^P=\ell^{P0}$ then, $cl^*(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)=\mathbb{K}\cup\Phi(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)=\mathbb{K}\cup cl_\tau(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)=cl_\tau(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)$.

Theorem 3.2. *Let* (Ξ,τ,ℓ^P) *be a picture fuzzy ideal topological space. Then, for any fuzzy set* $\mathbb{K},K\in\left(I^3\right)^{\Xi}, \varsigma\in I_0,\varkappa\in I_1$ *and* $\vartheta\in I_1$, *the operator* $cl^*:\left(I^3\right)^{\Xi}\times I^3\to\left(I^3\right)^{\Xi}$ *satisfies the following properties:*

(1) $cl^*(\flat,\langle\varsigma,\varkappa,\vartheta\rangle)=\flat$.
(2) $\mathbb{K}\subseteq cl^*(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)\subseteq cl_\tau(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)$.
(3) If $\mathbb{K}\subseteq Ч$, then $cl^*(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)\subseteq cl^*(Ч,\langle\varsigma,\varkappa,\vartheta\rangle)$.
(4) $cl^*(\mathbb{K}\cup Ч,\langle\varsigma,\varkappa,\vartheta\rangle)\supseteq cl^*(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)\cup cl^*(Ч,\langle\varsigma,\varkappa,\vartheta\rangle)$.
(5) $cl^*(\mathbb{K}\cap Ч,\langle\varsigma,\varkappa,\vartheta\rangle)\subseteq cl^*(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)\cap cl^*(Ч,\langle\varsigma,\varkappa,\vartheta\rangle)$.

Proof. (1) Since $cl^*(\flat,\langle\varsigma,\varkappa,\vartheta\rangle)=\flat\cup\Phi(\flat,\langle\varsigma,\varkappa,\vartheta\rangle)$ and $\Phi(\flat,\langle\varsigma,\varkappa,\vartheta\rangle)=\flat$ implies $cl^*(\flat,\langle\varsigma,\varkappa,\vartheta\rangle)=\flat$.

(2) $cl^*(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)=\mathbb{K}\cup\Phi(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)$ implies $\mathbb{K}\subseteq cl^*(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)$. Since $\mathbb{K}\subseteq cl_\tau(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)$ and from Theorem 3.1(4), we have $\Phi(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)\subseteq cl_\tau(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)$ implies $cl^*(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)\subseteq cl_\tau(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)$. Thus, $\mathbb{K}\subseteq cl^*(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)\subseteq cl_\tau(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)$.

(3) From $\mathbb{K}\subseteq \text{Ч}$ and Theorem 3.1(2), we have $\mathbb{K}\cup\Phi(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)\subseteq \text{Ч}\cup\Phi(\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle)$ and then,
$cl^*(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)\subseteq cl^*(\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle)$.

(4) Since $\mathbb{K}\subseteq\mathbb{K}\cup\text{Ч}$ and $\text{Ч}\subseteq\mathbb{K}\cup\text{Ч}$, $cl^*(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)\subseteq cl^*(\mathbb{K}\cup\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle)$ and $cl^*(\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle)\subseteq cl^*(\mathbb{K}\cup\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle)$. Thus, $cl^*(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)\cup cl^*(\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle)\subseteq cl^*(\mathbb{K}\cup\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle)$.

(5) $\mathbb{K}\cap\text{Ч}\subseteq\mathbb{K}$ and $\mathbb{K}\cap\text{Ч}\subseteq\text{Ч}$, $cl^*(\mathbb{K}\cap\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle)\subseteq cl^*(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)$ and $cl^*(\mathbb{K}\cap\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle)\subseteq cl^*(\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle)$. Thus,
$cl^*(\mathbb{K}\cap\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle)\subseteq cl^*(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)\cap cl^*(\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle)$.

Theorem 3.3. *Let (Ξ,τ,ℓ^P) be a picture fuzzy ideal topological space. Then, for each $\mathbb{K}\in\left(I^3\right)^\Xi,\varsigma\in I_0,\varkappa\in I_1$ and $\vartheta\in I_1$, we define an operator $int^*:\left(I^3\right)^\Xi\times I^3\to\left(I^3\right)^\Xi$ as follows:*

$$int^*(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)=\mathbb{K}\cap\dashv\left(\Phi(\dashv\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)\right).$$

For $\mathbb{K},\ \text{Ч}\in\left(I^3\right)^\Xi$, the operator int^* satisfies the following properties:

(1) $int^*(\sharp,\langle\varsigma,\varkappa,\vartheta\rangle)=\sharp$.

(2) $int_\tau(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)\subseteq int^*(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)\subseteq\mathbb{K}$.

(3) *If $\mathbb{K}\subseteq\text{Ч}$, then $int^*(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)\subseteq int^*(\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle)$.*

(4) $int^*(\mathbb{K}\cap\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle)\subseteq int^*(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)\cap int^*(\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle)$.

(5) $int^*(\sharp,\langle\varsigma,\varkappa,\vartheta\rangle)=int(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)$ *if* $\ell^P=\ell^{P0}$.

(6) $int^*(\dashv\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)=\dashv\left(cl^*(\mathbb{K},\langle\varsigma,\varkappa,\vartheta\rangle)\right)$.

Proof. It is similarly proved as the proof of Theorem 3.2.

Definition 3.5. Let $\mathbb{F}:(\Xi,\tau)\looparrowright(\Upsilon,\sigma)$ be a PFM, $\varsigma\in I_0,\varkappa\in I_1$ and $\vartheta\in I_1$. Then, $\mathbb{F}$ is called:

(1) PF^uS-continuous at a fuzzy point $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle}\in D(\mathbb{F})$ iff $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle}\in\mathbb{F}^u(\text{Ч})$ for each $\text{Ч}\in\left(I^3\right)^\Upsilon,\sigma(\text{Ч})\geq\langle\varsigma,\varkappa,\vartheta\rangle$ there exists $\mathbb{K}\in\left(I^3\right)^\Xi,\tau(\mathbb{K})\geq\langle\varsigma,\varkappa,\vartheta\rangle$ and $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle}\in\mathbb{K}$ such that $\mathbb{K}\cap D(\mathbb{F})\subseteq\mathbb{F}^u(\text{Ч})$.

(2) PF^lS-continuous at a fuzzy point $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle}\in D(\mathbb{F})$ iff $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle}\in\mathbb{F}^l(\text{Ч})$ for each $\text{Ч}\in\left(I^3\right)^\Upsilon$, $\sigma(\text{Ч})\geq\langle\varsigma,\varkappa,\vartheta\rangle$ there exists $\mathbb{K}\in\left(I^3\right)^\Xi,\tau(\mathbb{K})\geq\langle\varsigma,\varkappa,\vartheta\rangle$ and $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle}\in\mathbb{K}$ such that $\mathbb{K}\subseteq\mathbb{F}^l(\text{Ч})$.

(3) PF^uA-continuous at a fuzzy point $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle}\in D(\mathbb{F})$ iff $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle}\in\mathbb{F}^u(\text{Ч})$ for each $\text{Ч}\in\left(I^3\right)^\Upsilon,\sigma(\text{Ч})\geq\langle\varsigma,\varkappa,\vartheta\rangle$ there exists $\mathbb{K}\in\left(I^3\right)^\Xi,\tau(\mathbb{K})\geq\langle\varsigma,\varkappa,\vartheta\rangle$ and $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle}\in\mathbb{K}$ such that
$\mathbb{K}\cap D(\mathbb{F})\subseteq\mathbb{F}^u(int_\sigma(cl_\sigma(\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle),\langle\varsigma,\varkappa,\vartheta\rangle))$.

(4) PF^lA-continuous at a fuzzy point $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle}\in D(\mathbb{F})$ iff $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle}\in\mathbb{F}^l(\text{Ч})$ for each $\text{Ч}\in\left(I^3\right)^\Upsilon$, $\sigma(\text{Ч})\geq\langle\varsigma,\varkappa,\vartheta\rangle$ there exists $\mathbb{K}\in\left(I^3\right)^\Xi,\tau(\mathbb{K})\geq\langle\varsigma,\varkappa,\vartheta\rangle$ and $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle}\in\mathbb{K}$ such

that
$\mathbb{K} \subseteq \mathbb{F}^l(int_\sigma(cl_\sigma(Ч, \langle \varsigma, \varkappa, \vartheta \rangle), \langle \varsigma, \varkappa, \vartheta \rangle))$.

(5) PF^uS (resp. PF^lS)-continuous iff it is PF^uS (resp. PF^lS)-continuous at every fuzzy point $\xi_{\langle \varsigma, \varkappa, \vartheta \rangle} \in D(\mathbb{F})$.

(6) PF^uA (resp. PF^lA)-continuous iff it is PF^uA (resp. PF^lA)-continuous at every fuzzy point $\xi_{\langle \varsigma, \varkappa, \vartheta \rangle} \in D(\mathbb{F})$.

Definition 3.6. Let $\mathbb{F} : (\Xi, \tau, \ell^P) \nrightarrow (\Upsilon, \sigma)$ be a PFM, $\varsigma \in I_0, \varkappa \in I_1$ and $\vartheta \in I_1$. Then, $\mathbb{F}$ is called:

(1) PF^u ℓ^P-continuous at a fuzzy point $\xi_{\langle \varsigma, \varkappa, \vartheta \rangle} \in D(\mathbb{F})$ iff $\xi_{\langle \varsigma, \varkappa, \vartheta \rangle} \in \mathbb{F}^u(Ч)$ for each Ч $\in (I^3)^\Upsilon$, $\sigma(Ч) \geq \langle \varsigma, \varkappa, \vartheta \rangle$ there exists $\mathbb{K} \in (I^3)^\Xi, \tau(\mathbb{K}) \geq \langle \varsigma, \varkappa, \vartheta \rangle$ and $\xi_{\langle \varsigma, \varkappa, \vartheta \rangle} \in \mathbb{K}$ such that $\mathbb{K} \cap D(\mathbb{F}) \subseteq \Phi(\mathbb{F}^u(Ч), \langle \varsigma, \varkappa, \vartheta \rangle)$.

(2) PF^l ℓ^P-continuous at a fuzzy point $\xi_{\langle \varsigma, \varkappa, \vartheta \rangle} \in D(\mathbb{F})$ iff $\xi_{\langle \varsigma, \varkappa, \vartheta \rangle} \in \mathbb{F}^l(Ч)$ for each Ч $\in (I^3)^\Upsilon$, $\sigma(Ч) \geq \langle \varsigma, \varkappa, \vartheta \rangle$ there exists $\mathbb{K} \in (I^3)^\Xi, \tau(\mathbb{K}) \geq \langle \varsigma, \varkappa, \vartheta \rangle$ and $\xi_{\langle \varsigma, \varkappa, \vartheta \rangle} \in \mathbb{K}$ such that $\mathbb{K} \subseteq \Phi(\mathbb{F}^l(Ч), \langle \varsigma, \varkappa, \vartheta \rangle)$.

(3) PF^uA ℓ^P-continuous at a fuzzy point $\xi_{\langle \varsigma, \varkappa, \vartheta \rangle} \in D(\mathbb{F})$ iff $\xi_{\langle \varsigma, \varkappa, \vartheta \rangle} \in \mathbb{F}^u(Ч)$ for each Ч$\in (I^3)^\Upsilon$, $\sigma(Ч) \geq \langle \varsigma, \varkappa, \vartheta \rangle$ there exists $\mathbb{K} \in (I^3)^\Xi, \tau(\mathbb{K}) \geq \langle \varsigma, \varkappa, \vartheta \rangle$ and $\xi_{\langle \varsigma, \varkappa, \vartheta \rangle} \in \mathbb{K}$ such that
$\mathbb{K} \cap D(\mathbb{F}) \subseteq \mathbb{F}^u(int_\sigma(cl^*(Ч, \langle \varsigma, \varkappa, \vartheta \rangle, \langle \varsigma, \varkappa, \vartheta \rangle)))$.

(4) PF^lA ℓ^P-continuous at a fuzzy point $\xi_{\langle \varsigma, \varkappa, \vartheta \rangle} \in D(\mathbb{F})$ iff $\xi_{\langle \varsigma, \varkappa, \vartheta \rangle} \in \mathbb{F}^l(Ч)$ for each Ч $\in (I^3)^\Upsilon$, $\sigma(Ч) \geq \langle \varsigma, \varkappa, \vartheta \rangle$ there exists $\mathbb{K} \in (I^3)^\Xi, \tau(\mathbb{K}) \geq \langle \varsigma, \varkappa, \vartheta \rangle$ and $\xi_{\langle \varsigma, \varkappa, \vartheta \rangle} \in \mathbb{K}$ such that
$\mathbb{K} \subseteq \mathbb{F}^l(int_\sigma(cl^*(Ч, \langle \varsigma, \varkappa, \vartheta \rangle), \langle \varsigma, \varkappa, \vartheta \rangle)))$.

(5) PF^u ℓ^P-continuous (resp. PF^l ℓ^P-continuous) iff it is PF^u ℓ^P-continuous (resp. PF^l ℓ^P-continuous) at every fuzzy point $\xi_{\langle \varsigma, \varkappa, \vartheta \rangle} \in D(\mathbb{F})$.

(6) PF^uA ℓ^P-continuous (resp. PF^lA ℓ^P-continuous) iff it is PF^uA ℓ^P-continuous (resp. PF^lA ℓ^P-continuous) at every fuzzy point $\xi_{\langle \varsigma, \varkappa, \vartheta \rangle} \in D(\mathbb{F})$.

Remark 3.2. (1) If $\mathbb{F}$ is normalized PFM, then $\mathbb{F}$ isPF^u ℓ^P-continuous at a fuzzy point $\xi_{\langle \varsigma, \varkappa, \vartheta \rangle} \in D(\mathbb{F})$ iff $\xi_{\langle \varsigma, \varkappa, \vartheta \rangle} \in \mathbb{F}^u(Ч)$ for each Ч $\in (I^3)^\Upsilon$, $\sigma(Ч) \geq \langle \varsigma, \varkappa, \vartheta \rangle$ there exists $\mathbb{K} \in (I^3)^\Xi$, $\tau(\mathbb{K}) \geq \langle \varsigma, \varkappa, \vartheta \rangle$ and $\xi_{\langle \varsigma, \varkappa, \vartheta \rangle} \in \mathbb{K}$ such that $\mathbb{K} \subseteq \Phi(\mathbb{F}^u(Ч), \langle \varsigma, \varkappa, \vartheta \rangle)$.

(2) If $\mathbb{F}$ is normalized PFM, then $\mathbb{F}$ is PF^uA ℓ^P-continuous at a fuzzy point $\xi_{\langle \varsigma, \varkappa, \vartheta \rangle} \in D(\mathbb{F})$ iff $\xi_{\langle \varsigma, \varkappa, \vartheta \rangle} \in \mathbb{F}^u(Ч)$ for each Ч $\in (I^3)^\Upsilon$, $\sigma(Ч) \geq \langle \varsigma, \varkappa, \vartheta \rangle$ there exists $\mathbb{K} \in (I^3)^\Xi$, $\tau(\mathbb{K}) \geq \langle \varsigma, \varkappa, \vartheta \rangle$ and $\xi_{\langle \varsigma, \varkappa, \vartheta \rangle} \in \mathbb{K}$ such that $\mathbb{K} \subseteq \mathbb{F}^u(int_\sigma(cl^*(Ч, \langle \varsigma, \varkappa, \vartheta \rangle), \langle \varsigma, \varkappa, \vartheta \rangle)))$.

(3) PF^u (resp. PF^l) ℓ^P-continuity and PF^u (resp. PF^lS)-continuity are independent notions as it will be shown in Example 3.2.

(4) PF^uS (resp. PF^lS)-continuity $\Rightarrow$ PF^uA (resp. PF^lA) ℓ^P-continuity $\Rightarrow$ PF^uA (resp. PF^lA) -continuity.

(5) PF^uA (resp. PF^lA) ℓ^{P0}-continuity $\Leftrightarrow$ PF^uA (resp. PF^lA) -continuity.

Theorem 3.4. *Let $\mathbb{F} : (\Xi, \tau, \ell^P) \nrightarrow (\Upsilon, \sigma)$ be a PFM (resp. normalized PFM), then $\mathbb{F}$ is PF^l (resp. PF^u)*

ℓ^P-*continuous iff* $\mathbb{F}^l(\text{Ч}) \subseteq int_\tau\left(\Phi(\mathbb{F}^l(\text{Ч}), \langle\varsigma,\varkappa,\vartheta\rangle), \langle\varsigma,\varkappa,\vartheta\rangle\right)$
(resp. $\mathbb{F}^u(\text{Ч}) \subseteq int_\tau\left(\Phi(\mathbb{F}^u(\text{Ч}), \langle\varsigma,\varkappa,\vartheta\rangle), \langle\varsigma,\varkappa,\vartheta\rangle\right)$*) for each* $\text{Ч} \in \left(I^3\right)^\Upsilon, \sigma(\text{Ч}) \geq \langle\varsigma,\varkappa,\vartheta\rangle, \varsigma \in I_0, \varkappa \in I_1$ *and* $\vartheta \in I_1$ *.*

Proof. ($\Rightarrow$) Let $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in D(\mathbb{F})$, $\text{Ч} \in \left(I^3\right)^\Upsilon$, $\sigma(\text{Ч}) \geq \langle\varsigma,\varkappa,\vartheta\rangle$ and $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in F^l(\text{Ч})$. Then, there exists $\mathbb{K} \in \left(I^3\right)^\Xi, \tau(\mathbb{K}) \geq \langle\varsigma,\varkappa,\vartheta\rangle$ and $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in \mathbb{K}$ such that $\mathbb{K} \subseteq \Phi(F^l(\text{Ч}), \langle\varsigma,\varkappa,\vartheta\rangle)$. Thus,
$\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in \mathbb{K} \subseteq int_\tau(\Phi(F^l(\text{Ч}), \langle\varsigma,\varkappa,\vartheta\rangle), \langle\varsigma,\varkappa,\vartheta\rangle)$ and hence,

$$F^l(\text{Ч}) \subseteq int_\tau(\Phi(F^l(\text{Ч}), \langle\varsigma,\varkappa,\vartheta\rangle), \langle\varsigma,\varkappa,\vartheta\rangle).$$

($\Leftarrow$) Let $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in D(\mathbb{F})$, $\text{Ч} \in \left(I^3\right)^\Upsilon$, $\sigma(\text{Ч}) \geq \langle\varsigma,\varkappa,\vartheta\rangle$ and $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in F^l(\text{Ч})$. Then, $F^l(\text{Ч}) \subseteq int_\tau(\Phi(F^l(\text{Ч}), \langle\varsigma,\varkappa,\vartheta\rangle), \langle\varsigma,\varkappa,\vartheta\rangle)$ and hence,

$$\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in int_\tau(\Phi(F^l(\text{Ч}), \langle\varsigma,\varkappa,\vartheta\rangle), \langle\varsigma,\varkappa,\vartheta\rangle) \subseteq \Phi(F^l(\text{Ч}), \langle\varsigma,\varkappa,\vartheta\rangle).$$

Thus, F is PF^l ℓ^P-continuous. Other case is similarly proved.

Example 3.2. *Let* $\Xi = \{\xi_1, \xi_2\}$, $\Upsilon = \{\zeta_1, \zeta_2, \zeta_3\}$ *and* $\mathbb{F} : \Xi \nrightarrow \Upsilon$ *be a* PFM *defined by* $\Psi_\mathbb{F}(\xi_1,\zeta_1) = \langle 0.1, 0.3, 0.2\rangle$, $\Psi_\mathbb{F}(\xi_1,\zeta_2) = \langle 0.2, 0.3, 0.4\rangle$, $\Psi_\mathbb{F}(\xi_1,\zeta_3) = \langle 1, 0, 0\rangle$, $\Psi_\mathbb{F}(\xi_2,\zeta_1) = \langle 0.4, 0.4, 0.1\rangle$,
$\Psi_\mathbb{F}(\xi_2,\zeta_2) = \langle 0.3, 0.33, 0.33\rangle$, $\Psi_\mathbb{F}(\xi_2,\zeta_3) = \langle 1, 0, 0\rangle$. *For* $\mathbb{K}_1 = \{\langle\xi, 0.33, 0.33, 0\rangle \mid \xi \in \Xi\}$, $\mathbb{K}_2 = \{\langle\xi, 0.4, 0.4, 0.2\rangle \mid \xi \in \Xi\}$ *and* $\text{Ч}_1 = \{\langle\zeta, 0.33, 0.33, 0.33\rangle \mid \zeta \in \Upsilon\}$ *define picture fuzzy topologies* τ_1, τ_2*:* $\left(I^3\right)^\Xi \to I^3$, σ*:* $\left(I^3\right)^\Upsilon \to I^3$, *and define picture fuzzy ideals* ℓ_1^P,ℓ_2^P*:* $\left(I^3\right)^\Xi \to I^3$ *as follow.*

$$\tau_1(\mathbb{K}) = \begin{cases} \langle 1,0,0\rangle & \text{if } \mathbb{K} \in \{\flat, \sharp\}, \\ \langle 0.5, 0.33, 0.17\rangle & \text{if } \mathbb{K} = \mathbb{K}_1, \\ \langle 0,1,0\rangle & \text{otherwise,} \end{cases}, \tau_2(\mathbb{K}) = \begin{cases} \langle 1,0,0\rangle & \text{if } \mathbb{K} \in \{\flat, \sharp\}, \\ \langle 0.5, 0.33, 0.1\rangle & \text{if } \mathbb{K} = \mathbb{K}_2, \\ \langle 0,1,0\rangle & \text{otherwise,} \end{cases}$$

$$\ell_1^P(\mathbb{K}) = \begin{cases} \langle 1,0,0\rangle & \text{if } \mathbb{K} = \flat, \\ \langle 0.4, 0.2, 0.4\rangle & \text{if } \flat \subseteq \mathbb{K} \subseteq \{\langle \xi, 0.4, 0.1, 0.5\rangle \,|\xi \in \Xi\}, \\ \langle 0,1,0\rangle & \text{otherwise,} \end{cases}$$

$$\ell_2^P(\mathbb{K}) = \begin{cases} \langle 1,0,0\rangle & \text{if } \mathbb{K} = \flat, \\ \langle 0.4, 0.2, 0.15\rangle & \text{if } \flat \subseteq \mathbb{K} \subseteq \{\langle \xi, 0.2, 0.2, 0.6\rangle \,|\xi \in \Xi\}, \\ \langle 0,1,0\rangle & \text{otherwise,} \end{cases}$$

$$\sigma(\text{Ч}) = \begin{cases} \langle 1,0,0\rangle & \text{if } \text{Ч} \in \{\flat, \sharp\}, \\ \langle 0.33, 0.33, 0.33\rangle & \text{if } \text{Ч} = \text{Ч}_1, \\ \langle 0,1,0\rangle & \text{otherwise.} \end{cases}$$

Then, (1) $\mathbb{F} : (\Xi, \tau_1, \ell_1^P) \multimap (\Upsilon, \sigma)$ is PF^uS (resp. PF^lS)-continuous but it is not PF^u (resp. PF^l) ℓ^P-continuous because

$$\begin{aligned} \mathbb{F}^u(\text{Ч}_1) &= \mathbb{K}_1 \subseteq int_\tau(\mathbb{F}^u(\text{Ч}_1), \langle 0.33, 0.33, 0.33\rangle) = \mathbb{K}_1. \\ \mathbb{F}^l(\text{Ч}_1) &= \mathbb{K}_1 \subseteq int_\tau(\mathbb{F}^l(\text{Ч}_1), \langle 0.33, 0.33, 0.33\rangle) = \mathbb{K}_1. \end{aligned}$$

but

$$\begin{aligned} \mathbb{F}^u(\text{Ч}_1) &= \mathbb{K}_1 \nsubseteq int_\tau(\Phi(\mathbb{F}^u(\text{Ч}_1), \langle 0.33, 0.33, 0.33\rangle), \langle 0.33, 0.33, 0.33\rangle) = \flat. \\ \mathbb{F}^l(\text{Ч}_1) &= \mathbb{K}_1 \nsubseteq int_\tau(\Phi(\mathbb{F}^l(\text{Ч}_1), \langle 0.33, 0.33, 0.33\rangle), \langle 0.33, 0.33, 0.33\rangle) = \flat. \end{aligned}$$

(2) $\mathbb{F} : (\Xi, \tau_2, \ell_2^P) \multimap (\Upsilon, \sigma)$ is PF^u (resp. PF^l) ℓ^P-continuous but it is not PF^uS (resp. PF^lS)-continuous because

$$\begin{aligned} \mathbb{F}^u(\text{Ч}_1) &= \mathbb{K}_1 \subseteq int_\tau(\Phi(\mathbb{F}^u(\text{Ч}_1), \langle 0.33, 0.33, 0.33\rangle), \langle 0.33, 0.33, 0.33\rangle) = \sharp. \\ \mathbb{F}^l(\text{Ч}_1) &= \mathbb{K}_1 \subseteq int_\tau(\Phi(\mathbb{F}^l(\text{Ч}_1), \langle 0.33, 0.33, 0.33\rangle), \langle 0.33, 0.33, 0.33\rangle) = \sharp. \end{aligned}$$

but

$$\begin{aligned} \mathbb{F}^u(\text{Ч}_1) &= \mathbb{K}_1 \nsubseteq int_\tau(\mathbb{F}^u(\text{Ч}_1), \langle 0.33, 0.33, 0.33\rangle) = \flat. \\ \mathbb{F}^l(\text{Ч}_1) &= \mathbb{K}_1 \nsubseteq int_\tau(\mathbb{F}^l(\text{Ч}_1), \langle 0.33, 0.33, 0.33\rangle) = \flat. \end{aligned}$$

Theorem 3.5. *For a PFM* $\mathbb{F} : (\Xi, \tau) \nrightarrow (\Upsilon, \sigma, \ell^P)$, $Ч \in \left(I^3\right)^\Upsilon$, $\varsigma \in I_0, \varkappa \in I_1$ *and* $\vartheta \in I_1$, *the following statements are equivalent:*

(1) $\mathbb{F}$ *is* PF^lA ℓ^P*-continuous.*

(2) $\mathbb{F}^l(Ч) \subseteq int_\tau \left(\mathbb{F}^l(int_\sigma(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle))\right), \langle\varsigma, \varkappa, \vartheta\rangle)$, *if* $\sigma(Ч) \geq \langle\varsigma, \varkappa, \vartheta\rangle$.

(3) $cl_\tau\left(\mathbb{F}^u(cl_\sigma(int^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle\right) \subseteq \mathbb{F}^u(Ч)$, *if* $\sigma(\daleth Ч) \geq \langle\varsigma, \varkappa, \vartheta\rangle$.

Proof. (1) $\Longrightarrow$ (2) Let $\xi_{\langle\varsigma, \varkappa, \vartheta\rangle} \in D(\mathbb{F})$, $Ч \in \left(I^3\right)^\Upsilon$, $\sigma(Ч) \geq \langle\varsigma, \varkappa, \vartheta\rangle$ and $\xi_{\langle\varsigma, \varkappa, \vartheta\rangle} \in \mathbb{F}^l(Ч)$. Then, there exists $\mathbb{K} \in \left(I^3\right)^\Xi$, $\tau(\mathbb{K}) \geq \langle\varsigma, \varkappa, \vartheta\rangle$ and $\xi_{\langle\varsigma, \varkappa, \vartheta\rangle} \in \mathbb{K}$ such that $\mathbb{K} \subseteq \mathbb{F}^l(int_\sigma(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle))$.

Thus, $\xi_{\langle\varsigma, \varkappa, \vartheta\rangle} \in \mathbb{K} \subseteq int_\tau\mathbb{F}^l(int_\sigma(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle)$, and hence

$$\mathbb{F}^l(Ч) \subseteq int_\tau\left(\mathbb{F}^l(int_\sigma(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle\right).$$

(2) $\Longrightarrow$ (3) Let $Ч \in \left(I^3\right)^\Upsilon$ with $\sigma(\daleth Ч) \geq \langle\varsigma, \varkappa, \vartheta\rangle$. Then, by (2)

$$\begin{aligned} \daleth\mathbb{F}^u(Ч) &= \mathbb{F}^l(\daleth Ч) \subseteq int_\tau\left(\mathbb{F}^l(int_\sigma(cl^*(\daleth Ч, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle\right) \\ &= \daleth cl_\tau\left(\mathbb{F}^u(cl_\sigma(int^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle\right). \end{aligned}$$

Thus, $cl_\tau\left(\mathbb{F}^u(cl_\sigma(int^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle\right) \subseteq \mathbb{F}^u(Ч)$.

(3) $\Longrightarrow$ (1) Let $\xi_{\langle\varsigma, \varkappa, \vartheta\rangle} \in D(\mathbb{F})$, $Ч \in \left(I^3\right)^\Upsilon$, $\sigma(Ч) \geq \langle\varsigma, \varkappa, \vartheta\rangle$ and $\xi_{\langle\varsigma, \varkappa, \vartheta\rangle} \in \mathbb{F}^l(Ч)$. Then by (3), we have

$$\begin{aligned} &\daleth\left[int_\tau\left(\mathbb{F}^l(int_\sigma(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle))\right), \langle\varsigma, \varkappa, \vartheta\rangle)\right] \\ = \;& cl_\tau\left(\mathbb{F}^u(cl_\sigma(int^*(\daleth Ч, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle\right) \subseteq \mathbb{F}^u(\daleth Ч) = \daleth\,\mathbb{F}^l(Ч), \end{aligned}$$

and $\mathbb{F}^l(Ч) \subseteq int_\tau\left(\mathbb{F}^l(int_\sigma(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle))\right), \langle\varsigma, \varkappa, \vartheta\rangle)$. Therefore, $\xi_{\langle\varsigma, \varkappa, \vartheta\rangle} \in int_\tau\left(\mathbb{F}^l(int_\sigma(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle))\right), \langle\varsigma, \varkappa, \vartheta\rangle)$
$\subseteq \mathbb{F}^l(int_\sigma(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle))$. Thus, $\mathbb{F}$ is PF^lA ℓ^P-continuous.

The following theorem is similarly proved as the proof of Theorem 3.5.

Theorem 3.6. *For a normalized PFM* $\mathbb{F} : (\Xi, \tau) \nrightarrow (\Upsilon, \sigma, \ell^P)$, $Ч \in \left(I^3\right)^\Upsilon$, $\varsigma \in I_0, \varkappa \in I_1$ *and* $\vartheta \in I_1$, *the following statements are equivalent:*

(1) $\mathbb{F}$ *is* PF^uA ℓ^P*-continuous.*

(2) $\mathbb{F}^u(Ч) \subseteq int_\tau\left(\mathbb{F}^u(int_\sigma(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle))\right), \langle\varsigma, \varkappa, \vartheta\rangle)$, *if* $\sigma(Ч) \geq \langle\varsigma, \varkappa, \vartheta\rangle$.

(3) $cl_\tau\left(\mathbb{F}^l(cl_\sigma(int^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle\right) \subseteq \mathbb{F}^l(Ч)$, *if* $\sigma(\daleth Ч) \geq \langle\varsigma, \varkappa, \vartheta\rangle$.

Example 3.3. *Let* $\Xi = \{\xi_1, \xi_2\}$, $\Upsilon = \{\zeta_1, \zeta_2, \zeta_3\}$ *and* $\mathbb{F} : \Xi \nrightarrow \Upsilon$ *be a PFM defined by* $\Psi_\mathbb{F}(\xi_1, \zeta_1) = \langle 1, 0, 0\rangle$, $\Psi_\mathbb{F}(\xi_1, \zeta_2) = \langle 0.1, 0.2, 0.7\rangle$, $\Psi_\mathbb{F}(\xi_1, \zeta_3) = \langle 0.3, 0.2, 0.4\rangle$, $\Psi_\mathbb{F}(\xi_2, \zeta_1) = \langle 0.33, 0.3, 0.33\rangle$, $\Psi_\mathbb{F}(\xi_2, \zeta_2) = \langle 1, 0, 0\rangle$, $\Psi_\mathbb{F}(\xi_2, \zeta_3) = \langle 0.4, 0.2, 0.4\rangle$. *For* $\mathbb{K}_1 = \{\langle\xi, 0.4, 0.4, 0.2\rangle \mid \xi \in \Xi\}$, $\mathbb{K}_2 = \{\langle\xi, 0.32, 0.3, 0\rangle \mid \xi \in \Xi\}$ *and* $Ч_1 = \{\langle\zeta, 0.32, 0.3, 0.33\rangle \mid \zeta \in \Upsilon\}$ *define picture fuzzy topologies* τ*:*$\left(I^3\right)^\Xi \to I^3$, σ*:* $\left(I^3\right)^\Upsilon \to I^3$, *and picture fuzzy ideal* ℓ^P*:* $\left(I^3\right)^\Upsilon \to I^3$ *as follows:*

$$\tau(\mathbb{K}) = \begin{cases} \langle 1,0,0\rangle & \text{if } \mathbb{K} \in \{\flat, \sharp\}, \\ \langle 0.55, 0.1, 0.11\rangle & \text{if } \mathbb{K} = \mathbb{K}_1, \\ \langle 0,1,0\rangle & \text{otherwise}, \end{cases}, \sigma(Ч) = \begin{cases} \langle 1,0,0\rangle & \text{if } Ч \in \{\flat, \sharp\}, \\ \langle 0.32, 0.3, 0.33\rangle & \text{if } Ч = Ч_1, \\ \langle 0,1,0\rangle & \text{otherwise}, \end{cases}$$

$$\ell^P(Ч) = \begin{cases} \langle 1,0,0\rangle & \text{if } Ч = \flat, \\ \langle 0.44, 0.2, 0.3\rangle & \text{if } \flat \subseteq Ч \subseteq \{\langle \zeta, 0.2, 0.2, 0.4\rangle \,|\zeta \in \Upsilon\}, \\ \langle 0,1,0\rangle & \text{otherwise}. \end{cases}$$

Then, $\mathbb{F} : (\Xi, \tau) \nrightarrow (\Upsilon, \sigma, \ell^P)$ is PF^uS (resp. PF^lS) ℓ^P-continuous but is not PF^uS (resp. PF^lS)-continuous because

$$\begin{aligned} \mathbb{K}_2 &= \mathbb{F}^u(Ч_1) \\ &\subseteq int_\tau\left(\mathbb{F}^u(int_\sigma(cl^*(Ч_1, \langle 0.32, 0.3, 0.33\rangle), \langle 0.32, 0.3, 0.33\rangle)), \langle 0.32, 0.3, 0.33\rangle\right) = \sharp, \\ \mathbb{K}_2 &= \mathbb{F}^l(Ч_1) \\ &\subseteq int_\tau\left(\mathbb{F}^l(int_\sigma(cl^*(Ч_1, \langle 0.32, 0.3, 0.33\rangle), \langle 0.32, 0.3, 0.33\rangle)\right), \langle 0.32, 0.3, 0.33\rangle) = \sharp. \end{aligned}$$

but

$$\begin{aligned} \mathbb{K}_2 &= \mathbb{F}^u(Ч_1) \nsubseteq int_\tau\left(\mathbb{F}^u(Ч_1), \langle 0.32, 0.3, 0.33\rangle\right) = \flat, \\ \mathbb{K}_2 &= \mathbb{F}^l(Ч_1) \nsubseteq int_\tau\left(\mathbb{F}^l(Ч_1), \langle 0.32, 0.3, 0.33\rangle\right) = \flat. \end{aligned}$$

Example 3.4. *Let* $\Xi = \{\xi_1, \xi_2\}$, $\Upsilon = \{\zeta_1, \zeta_2, \zeta_3\}$ *and* $\mathbb{F} : \Xi \nrightarrow \Upsilon$ *be a* PFM *defined by* $\Psi_{\mathbb{F}}(\xi_1, \zeta_1) = \langle 1, 0, 0\rangle$, $\Psi_{\mathbb{F}}(\xi_1, \zeta_2) = \langle 0.2, 0.6, 0.2\rangle$, $\Psi_{\mathbb{F}}(\xi_1, \zeta_3) = \langle 0.25, 0.3, 0.4\rangle$, $\Psi_{\mathbb{F}}(\xi_2, \zeta_1) = \langle 0.32, 0.31, 0.15\rangle$,
$\Psi_{\mathbb{F}}(\xi_2, \zeta_2) = \langle 0.2, 0.2, 0.3\rangle$, $\Psi_{\mathbb{F}}(\xi_2, \zeta_3) = \langle 1, 0, 0\rangle$. *For* $\mathbb{K}_1 = \{\langle \xi, 0.1, 0.35, 0.31\rangle \mid \xi \in \Xi\}$,
$\mathbb{K}_2 = \{\langle \xi, 0.1, 0.45, 0.31\rangle \mid \xi \in \Xi\}$, $Ч_1 = \{\langle \zeta, 0.05, 0.36, 0.51\rangle \mid \zeta \in \Upsilon\}$, $Ч_2 = \{\langle \zeta, 0.1, 0.35, 0.51\rangle \mid \zeta \in \Upsilon\}$,
$Ч_3 = \{\langle \zeta, 0.05, 0.36, 0\rangle \mid \zeta \in \Upsilon\}$ *and* $Ч_4 = \{\langle \zeta, 0.1, 0.35, 0\rangle \mid \zeta \in \Upsilon\}$, *and define picture fuzzy topologies*
τ:$(I^3)^\Xi \to I^3$, σ: $(I^3)^\Upsilon \to I^3$, *and picture fuzzy ideal* ℓ^P: $(I^3)^\Upsilon \to I^3$ *as follows.*

$$\tau(\mathbb{K})=\begin{cases}\langle 1,0,0\rangle & \text{if } \mathbb{K}\in\{\flat,\sharp\},\\ \langle 0.6,0.1,0.3\rangle & \text{if } \mathbb{K}\in\{\mathbb{K}_1,\mathbb{K}_2\},\\ \langle 0,1,0\rangle & \text{otherwise,}\end{cases},\ \sigma(\text{Ч})=\begin{cases}\langle 1,0,0\rangle & \text{if } \text{Ч}\in\{\flat,\sharp\},\\ \langle 0.35,0.5,0.15\rangle & \text{if } \text{Ч}=\text{Ч}_1,\\ \langle 0.55,0.2,0.25\rangle & \text{if } \text{Ч}=\text{Ч}_2,\\ \langle 0,1,0\rangle & \text{otherwise,}\end{cases}$$

$$\ell^P(\text{Ч})=\begin{cases}\langle 1,0,0\rangle & \text{if } \text{Ч}=\flat,\\ \langle 0.55,0.15,0.3\rangle & \text{if } \text{Ч}\in\{\text{Ч}_3,\text{Ч}_4\},\\ \langle 0,1,0\rangle & \text{otherwise.}\end{cases}$$

Then, $\mathbb{F}:(\Xi,\tau)\nleftrightarrow(\Upsilon,\sigma,\ell^P)$ is PF^uA (resp. PF^lA)-continuous but is not PF^uA (resp. PF^lA) ℓ^P-continuous because

$\{\langle\xi,0.05,0.36,0\rangle\,|\xi\in\Xi\}=\mathbb{F}^u(\text{Ч}_1)$
$\subseteq int_\tau\left(\mathbb{F}^u(int_\sigma(cl_\sigma(\text{Ч}_1,\langle 0.35,0.5,0.15\rangle),\langle 0.35,0.5,0.15\rangle)),\langle 0.35,0.5,0.15\rangle\right)$
$=\{\langle\xi,0.1,0.35,0\rangle\,|\xi\in\Xi\}$,
$\{\langle\xi,0.1,0.35,0\rangle\,|\xi\in\Xi\}=\mathbb{F}^u(\text{Ч}_2)$
$\subseteq int_\tau\left(\mathbb{F}^u(int_\sigma(cl_\sigma(\text{Ч}_2,\langle 0.35,0.5,0.15\rangle),\langle 0.35,0.5,0.15\rangle)),\langle 0.35,0.5,0.15\rangle\right)$
$=\{\langle\xi,0.1,0.35,0\rangle\,|\xi\in\Xi\}$,
$\{\langle\xi,0.05,0.36,0\rangle\,|\xi\in\Xi\}=\mathbb{F}^l(\text{Ч}_1)$
$\subseteq int_\tau\left(\mathbb{F}^l(int_\sigma(cl_\sigma(\text{Ч}_1,\langle 0.35,0.5,0.15\rangle),\langle 0.35,0.5,0.15\rangle)),\langle 0.35,0.5,0.15\rangle\right)$
$=\{\langle\xi,0.1,0.35,0\rangle\,|\xi\in\Xi\}$,
$\{\langle\xi,0.1,0.35,0\rangle\,|\xi\in\Xi\}=\mathbb{F}^l(\text{Ч}_2)$
$\subseteq int_\tau\left(\mathbb{F}^l(int_\sigma(cl_\sigma(\text{Ч}_2,\langle 0.35,0.5,0.15\rangle),\langle 0.35,0.5,0.15\rangle)),\langle 0.35,0.5,0.15\rangle\right)$
$=\{\langle\xi,0.1,0.35,0\rangle\,|\xi\in\Xi\}$,
but
$\{\langle\xi,0.05,0.36,0\rangle\,|\xi\in\Xi\}=\mathbb{F}^u(\text{Ч}_1)$
$\nsubseteq int_\tau\left(\mathbb{F}^u(int_\sigma(cl^*(\text{Ч}_1,\langle 0.35,0.5,0.15\rangle),\langle 0.35,0.5,0.15\rangle)),\langle 0.35,0.5,0.15\rangle\right)=\flat$,
$\{\langle\xi,0.05,0.36,0\rangle\,|\xi\in\Xi\}=\mathbb{F}^l(\text{Ч}_1)$
$\nsubseteq int_\tau\left(\mathbb{F}^l(int_\sigma(cl^*(\text{Ч}_1,\langle 0.35,0.5,0.15\rangle),\langle 0.35,0.5,0.15\rangle)),\langle 0.35,0.5,0.15\rangle\right)=\flat$.

Theorem 3.7. *For a PFM* $\mathbb{F}:(\Xi,\tau)\nleftrightarrow(\Upsilon,\sigma,\ell^P)$, $\text{Ч}\in\left(I^3\right)^\Upsilon$, $\varsigma\in I_0,\varkappa\in I_1$ *and* $\vartheta\in I_1$, *the following statements are equivalent:*

(1) $\mathbb{F}$ *is* PF^lA ℓ^P*-continuous.*
(2) $\tau\left(\mathbb{F}^l(\text{Ч})\right)\geq\langle\varsigma,\varkappa,\vartheta\rangle$, *if* $\text{Ч}=int_\sigma(cl^*(\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle),\langle\varsigma,\varkappa,\vartheta\rangle)$.
(3) $\tau\left(\mathbb{F}^l(int_\sigma(cl^*(\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle),\langle\varsigma,\varkappa,\vartheta\rangle)))\right)\geq\langle\varsigma,\varkappa,\vartheta\rangle$ *if* $\sigma(\text{Ч})\geq\langle\varsigma,\varkappa,\vartheta\rangle$.

Proof. (1) $\Longrightarrow$ (2) If $\text{Ч}=int_\sigma(cl^*(\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle),\langle\varsigma,\varkappa,\vartheta\rangle)$, then $\sigma(\text{Ч})\geq\langle\varsigma,\varkappa,\vartheta\rangle$. By Theorem 3.5(2),
$\mathbb{F}^l(\text{Ч})\subseteq int_\tau\left(\mathbb{F}^l(int_\sigma(cl^*(\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle),\langle\varsigma,\varkappa,\vartheta\rangle)),\langle\varsigma,\varkappa,\vartheta\rangle\right)=int_\tau\left(\mathbb{F}^l(\text{Ч}),\langle\varsigma,\varkappa,\vartheta\rangle\right)$.

Thus,
$\tau\left(\mathbb{F}^{l}(\text{Ч})\right) \geq \langle\varsigma,\varkappa,\vartheta\rangle$.

(2) $\Leftrightarrow$ (3) Obvious.

(3) $\Longrightarrow$ (1) Let $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in D(\mathbb{F})$, $\text{Ч} \in \left(I^{3}\right)^{\Upsilon}$, $\sigma(\text{Ч}) \geq \langle\varsigma,\varkappa,\vartheta\rangle$ and $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in \mathbb{F}^{l}(\text{Ч})$. Then, by (3) and
$\text{Ч} \subseteq int_{\sigma}(cl^{*}(\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle),\langle\varsigma,\varkappa,\vartheta\rangle)$, $\tau\left(\mathbb{F}^{l}(int_{\sigma}(cl^{*}(\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle),\langle\varsigma,\varkappa,\vartheta\rangle))\right) \geq \langle\varsigma,\varkappa,\vartheta\rangle$, and
$\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in \mathbb{F}^{l}(\text{Ч}) \subseteq \mathbb{F}^{l}(int_{\sigma}(cl^{*}(\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle),\langle\varsigma,\varkappa,\vartheta\rangle))$. Thus, $\mathbb{F}$ is $PF^{l}A$ ℓ^{P}-continuous.

The following theorems are similarly proved as the proof of Theorem 3.7.

Theorem 3.8. *For a PFM* $\mathbb{F}:(\Xi,\tau) \nrightarrow (\Upsilon,\sigma,\ell^{P})$, $\text{Ч} \in \left(I^{3}\right)^{\Upsilon}$, $\varsigma \in I_{0}, \varkappa \in I_{1}$ *and* $\vartheta \in I_{1}$, *the following statements are equivalent:*

(1) $\mathbb{F}$ *is* $PF^{l}A$ ℓ^{P}*-continuous.*

(2) $\tau(\exists\mathbb{F}^{u}(\text{Ч})) \geq \langle\varsigma,\varkappa,\vartheta\rangle$, *if* $\text{Ч} = cl_{\sigma}(int^{*}(\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle),\langle\varsigma,\varkappa,\vartheta\rangle)$.

(3) $\tau(\exists\mathbb{F}^{u}(cl_{\sigma}(int^{*}(\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle),\langle\varsigma,\varkappa,\vartheta\rangle))) \geq \langle\varsigma,\varkappa,\vartheta\rangle$ *if* $\sigma(\exists\ \text{Ч}) \geq \langle\varsigma,\varkappa,\vartheta\rangle$.

Theorem 3.9. *For a normalized PFM* $\mathbb{F}:(\Xi,\tau) \nrightarrow (\Upsilon,\sigma,\ell^{P})$, $\text{Ч} \in \left(I^{3}\right)^{\Upsilon}$, $\varsigma \in I_{0}, \varkappa \in I_{1}$ *and* $\vartheta \in I_{1}$, *the following statements are equivalent:*

(1) $\mathbb{F}$ *is* $PF^{u}A$ ℓ^{P}*-continuous.*

(2) $\tau(\mathbb{F}^{u}(\text{Ч})) \geq \langle\varsigma,\varkappa,\vartheta\rangle$, *if* $\text{Ч} = int_{\sigma}(cl^{*}(\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle),\langle\varsigma,\varkappa,\vartheta\rangle)$.

(3) $\tau(\mathbb{F}^{u}(int_{\sigma}(cl^{*}(\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle),\langle\varsigma,\varkappa,\vartheta\rangle))) \geq \langle\varsigma,\varkappa,\vartheta\rangle$ *if* $\sigma(\text{Ч}) \geq \langle\varsigma,\varkappa,\vartheta\rangle$.

Theorem 3.10. *For a normalized PFM* $\mathbb{F}:(\Xi,\tau) \nrightarrow (\Upsilon,\sigma,\ell^{P})$, $\text{Ч} \in \left(I^{3}\right)^{\Upsilon}$, $\varsigma \in I_{0}, \varkappa \in I_{1}$ *and* $\vartheta \in I_{1}$, *the following statements are equivalent:*

(1) $\mathbb{F}$ *is* $PF^{u}A$ ℓ^{P}*-continuous.*

(2) $\tau\left(\exists\mathbb{F}^{l}(\text{Ч})\right) \geq \langle\varsigma,\varkappa,\vartheta\rangle$ *if* $\text{Ч} = cl_{\sigma}(int^{*}(\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle),\langle\varsigma,\varkappa,\vartheta\rangle)$.

(3) $\tau\left(\exists\mathbb{F}^{l}(cl_{\sigma}(int^{*}(\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle),\langle\varsigma,\varkappa,\vartheta\rangle))\right) \geq \langle\varsigma,\varkappa,\vartheta\rangle$ *if* $\sigma(\exists\ \text{Ч}) \geq \langle\varsigma,\varkappa,\vartheta\rangle$.

Theorem 3.11. *Let* $\mathbb{F}:(\Xi,\tau) \nrightarrow (\Upsilon,\sigma,\ell^{P})$ *be a PFM. Then,* $\mathbb{F}$ *is* $PF^{l}A$ ℓ^{P}*-continuous iff* $cl_{\tau}(\mathbb{F}^{u}(\text{Ч}),\langle\varsigma,\varkappa,\vartheta\rangle) \subseteq \mathbb{F}^{u}(cl_{\sigma}(\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle))$ *for any* $\text{Ч} \in \left(I^{3}\right)^{\Upsilon}$ *with* $\text{Ч} \subseteq cl_{\sigma}(int^{*}(\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle),\langle\varsigma,\varkappa,\vartheta\rangle), \varsigma \in I_{0}, \varkappa \in I_{1}$ *and* $\vartheta \in I_{1}$.

Proof. ($\Rightarrow$) Let $\mathbb{F}$ be a $PF^{l}A$ ℓ^{P}-continuous. Then, for any $\text{Ч} \in \left(I^{3}\right)^{\Upsilon}$ with
$\text{Ч} \subseteq cl_{\sigma}(int^{*}(\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle),\langle\varsigma,\varkappa,\vartheta\rangle) = \text{Д}$(say), where $\text{Д}= cl_{\sigma}(int^{*}(\text{Д},\langle\varsigma,\varkappa,\vartheta\rangle),\langle\varsigma,\varkappa,\vartheta\rangle)$.
By Theorem 3.7, $\tau(\exists\ \mathbb{F}^{u}(\text{Д})) \geq \langle\varsigma,\varkappa,\vartheta\rangle$, and thus
$cl_{\tau}(\mathbb{F}^{u}(\text{Ч}),\langle\varsigma,\varkappa,\vartheta\rangle) \subseteq cl_{\tau}(\mathbb{F}^{u}(\text{Д}),\langle\varsigma,\varkappa,\vartheta\rangle) = \mathbb{F}^{u}(cl_{\sigma}(int^{*}(\text{Д},\langle\varsigma,\varkappa,\vartheta\rangle),\langle\varsigma,\varkappa,\vartheta\rangle))$
$\subseteq \mathbb{F}^{u}(cl_{\sigma}(\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle))$.

($\Leftarrow$) Let $\text{Ч} \in \left(I^{3}\right)^{\Upsilon}$ with $\text{Ч} = cl_{\sigma}(int^{*}(\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle),\langle\varsigma,\varkappa,\vartheta\rangle)$. Then,
$\text{Ч} \subseteq cl_{\sigma}(int^{*}(\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle),\langle\varsigma,\varkappa,\vartheta\rangle)$, and
$cl_{\tau}(\mathbb{F}^{u}(\text{Ч}),\langle\varsigma,\varkappa,\vartheta\rangle) \subseteq \mathbb{F}^{u}(cl_{\sigma}(\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle)) = \mathbb{F}^{u}(\text{Ч})$. Therefore, we obtain
$\tau(\exists\ \mathbb{F}^{u}(\text{Ч})) \geq \langle\varsigma,\varkappa,\vartheta\rangle$. Thus, by Theorem 3.7, $\mathbb{F}$ is $PF^{l}A$ ℓ^{P}-continuous.

The following theorem is similarly proved as the proof of Theorem 3.11.

Theorem 3.12. *Let* $\mathbb{F} : (\Xi, \tau) \looparrowright (\Upsilon, \sigma, \ell^P)$ *be a normalized PFM. Then,* $\mathbb{F}$ *is* PF^uA ℓ^P*-continuous iff*
$cl_\tau\left(\mathbb{F}^l(Ч), \langle\varsigma, \varkappa, \vartheta\rangle\right) \subseteq \mathbb{F}^l(cl_\sigma(Ч, \langle\varsigma, \varkappa, \vartheta\rangle))$ *for any* $Ч \in \left(I^3\right)^\Upsilon$ *with*
$Ч \subseteq cl_\sigma(int^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle)$, $\varsigma \in I_0, \varkappa \in I_1$ *and* $\vartheta \in I_1$.

Definition 3.7. Let $\mathbb{F} : (\Xi, \tau) \looparrowright (\Upsilon, \sigma, \ell^P)$ be a PFM, $\varsigma \in I_0, \varkappa \in I_1$ and $\vartheta \in I_1$. Then, $\mathbb{F}$ is called:

(1) PF^uW ℓ^P-continuous at a fuzzy point $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in D(\mathbb{F})$ iff $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in \mathbb{F}^u(Ч)$ for each $Ч \in \left(I^3\right)^\Upsilon$, $\sigma(Ч) \geq \langle\varsigma, \varkappa, \vartheta\rangle$ there exists $\mathbb{K} \in \left(I^3\right)^\Xi$, $\tau(\mathbb{K}) \geq \langle\varsigma, \varkappa, \vartheta\rangle$ and $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in \mathbb{K}$ such that $\mathbb{K} \cap D(\mathbb{F}) \subseteq \mathbb{F}^u(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle))$.

(2) PF^lW ℓ^P-continuous at a fuzzy point $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in D(\mathbb{F})$ iff $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in \mathbb{F}^l(Ч)$ for each $Ч \in \left(I^3\right)^\Upsilon$, $\sigma(Ч) \geq \langle\varsigma, \varkappa, \vartheta\rangle$ there exists $\mathbb{K} \in \left(I^3\right)^\Xi$, $\tau(\mathbb{K}) \geq \langle\varsigma, \varkappa, \vartheta\rangle$ and $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in \mathbb{K}$ such that $\mathbb{K} \subseteq \mathbb{F}^l(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle))$.

(3) PF^uAW ℓ^P-continuous at a fuzzy point $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in D(\mathbb{F})$ iff $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in \mathbb{F}^u(Ч)$ for each $Ч \in \left(I^3\right)^\Upsilon$, $\sigma(Ч) \geq \langle\varsigma, \varkappa, \vartheta\rangle$ there exists $\mathbb{K} \in \left(I^3\right)^\Xi, \tau(\mathbb{K}) \geq \langle\varsigma, \varkappa, \vartheta\rangle$ and $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in \mathbb{K}$ such that
$\mathbb{K} \cap D(\mathbb{F}) \subseteq cl_\tau(\mathbb{F}^u(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle)$.

(4) PF^lAW ℓ^P-continuous at a fuzzy point $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in D(\mathbb{F})$ iff $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in \mathbb{F}^l(Ч)$ for each $Ч \in \left(I^3\right)^\Upsilon$, $\sigma(Ч) \geq \langle\varsigma, \varkappa, \vartheta\rangle$ there exists $\mathbb{K} \in \left(I^3\right)^\Xi, \tau(\mathbb{K}) \geq \langle\varsigma, \varkappa, \vartheta\rangle$ and $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in \mathbb{K}$ such that
$\mathbb{K} \subseteq cl_\tau(\mathbb{F}^l(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle)$.

(5) PF^uW ℓ^P-continuous (resp. PF^lW ℓ^P-continuous) iff it is PF^uW ℓ^P-continuous (resp. PF^lW ℓ^P-continuous) at every fuzzy point $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in D(\mathbb{F})$.

(6) PF^uAW ℓ^P-continuous (resp. PF^lAW ℓ^P-continuous) iff it is PF^uAW ℓ^P-continuous
(resp. PF^lAW ℓ^P-continuous) at every fuzzy point $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in D(\mathbb{F})$.

Remark 3.3. (1) If $\mathbb{F}$ is normalized PFM, then $\mathbb{F}$ is PF^uW ℓ^P-continuous at a fuzzy point $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in D(\mathbb{F})$ iff
$\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in \mathbb{F}^u(Ч)$ for each $Ч \in \left(I^3\right)^\Upsilon$, $\sigma(Ч) \geq \langle\varsigma, \varkappa, \vartheta\rangle$ there exists $\mathbb{K} \in \left(I^3\right)^\Xi, \tau(\mathbb{K}) \geq \langle\varsigma, \varkappa, \vartheta\rangle$ and $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in \mathbb{K}$ such that $\mathbb{K} \subseteq \mathbb{F}^u(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle))$.

(2) If $\mathbb{F}$ is normalized PFM, then $\mathbb{F}$ is PF^uAW ℓ^P-continuous at a fuzzy point $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in D(\mathbb{F})$ iff $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in \mathbb{F}^u(Ч)$ for each $Ч \in \left(I^3\right)^\Upsilon$, $\sigma(Ч) \geq \langle\varsigma, \varkappa, \vartheta\rangle$ there exists $\mathbb{K} \in \left(I^3\right)^\Xi, \tau(\mathbb{K}) \geq \langle\varsigma, \varkappa, \vartheta\rangle$ and $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in \mathbb{K}$ such that $\mathbb{K} \subseteq cl_\tau(\mathbb{F}^u(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle)$.

(3) PF^uA (resp. PF^lA) ℓ^P-continuity $\Rightarrow$ PF^uW (resp. PF^lW) ℓ^P-continuity $\Rightarrow$ PF^uW (resp. PF^lW)-continuity.

(4) PF^uW (resp. PF^lW) ℓ^{P0}-continuity $\Leftrightarrow$ PF^uW (resp. PF^lW)-continuity.

(5) PF^uW (resp. PF^lW) ℓ^P-continuity $\Rightarrow$ PF^uAW (resp. PF^lAW) ℓ^P-continuity $\Rightarrow$ PF^uAW (resp. PF^lAW)-continuity.

(6) PF^uAW (resp. PF^lAW) ℓ^{P0}-continuity $\Leftrightarrow$ PF^uAW (resp. PF^lAW)-continuity.

Theorem 3.13. *A PFM* $\mathbb{F} : (\Xi, \tau) \nrightarrow (\Upsilon, \sigma, \ell^P)$ *is* PF^lW ℓ^P*-continuous iff* $\mathbb{F}^l(Ч) \subseteq int_\tau(\mathbb{F}^l(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle)$ *for each* $Ч \in (I^3)^\Upsilon$ *with* $\sigma(Ч) \geq \langle\varsigma, \varkappa, \vartheta\rangle$, $\varsigma \in I_0, \varkappa \in I_1$ *and* $\vartheta \in I_1$.

Proof. ($\Rightarrow$) Let $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in D(\mathbb{F})$, $Ч \in (I^3)^\Upsilon$ with $\sigma(Ч) \geq \langle\varsigma, \varkappa, \vartheta\rangle$ and $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in \mathbb{F}^l(Ч)$. Then, there exists $\mathbb{K} \in (I^3)^\Xi$, $\tau(\mathbb{K}) \geq \langle\varsigma, \varkappa, \vartheta\rangle$ and $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in \mathbb{K}$ such that $\mathbb{K} \subseteq \mathbb{F}^l(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle))$. Thus, $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in \mathbb{K} \subseteq int_\tau(\mathbb{F}^l(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle)$, and hence $\mathbb{F}^l(Ч) \subseteq int_\tau(\mathbb{F}^l(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle)$.

($\Leftarrow$) Let $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in D(\mathbb{F})$, $Ч \in (I^3)^\Upsilon$ with $\sigma(Ч) \geq \langle\varsigma, \varkappa, \vartheta\rangle$ and $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in \mathbb{F}^l(Ч)$. Then, $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in \mathbb{F}^l(Ч) \subseteq int_\tau(\mathbb{F}^l(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle)$. Thus,
$\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in \mathbb{K} \subseteq int_\tau(\mathbb{F}^l(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle) \subseteq \mathbb{F}^l(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle))$. Hence, $\mathbb{F}$ is PF^uW ℓ^P-continuous.

The following theorem is similarly proved as the proof of Theorem 3.13.

Theorem 3.14. *A normalized PFM* $\mathbb{F} : (\Xi, \tau) \nrightarrow (\Upsilon, \sigma, \ell^P)$ *is* PF^uW ℓ^P*-continuous iff* $\mathbb{F}^u(Ч) \subseteq int_\tau(\mathbb{F}^u(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle)$ *for each* $Ч \in (I^3)^\Upsilon$ *with* $\sigma(Ч) \geq \langle\varsigma, \varkappa, \vartheta\rangle$, $\varsigma \in I_0, \varkappa \in I_1$ *and* $\vartheta \in I_1$.

The following examples shows that generally PF^uW ℓ^P-continuous and PF^lW ℓ^P-continuous
(resp. PF^uW continuous and PF^lW continuous) multifunction need not be either PF^uA ℓ^P-continuous (resp. PF^uW ℓ^P-continuous) multifunction or PF^lA ℓ^P-continuous (resp. PF^lW ℓ^P-continuous) multifunction.

Example 3.5. *From Example 3.4,* $\mathbb{F} : (\Xi, \tau) \nrightarrow (\Upsilon, \sigma, \ell^P)$ *is* PF^uW *(resp.* PF^lW*)-continuous but is not* PF^uW *(resp.* PF^lW *)* ℓ^P*-continuous because*

$\{\langle\xi, 0.05, 0.36, 0\rangle | \xi \in \Xi\} = \mathbb{F}^u(Ч_1)$
$\subseteq int_\tau(\mathbb{F}^u(cl_\sigma(Ч_1, \langle 0.35, 0.5, 0.15\rangle)), \langle 0.35, 0.5, 0.15\rangle) = \{\langle\xi, 0.1, 0.35, 0\rangle | \xi \in \Xi\}$,
$\{\langle\xi, 0.1, 0.35, 0\rangle | \xi \in \Xi\} = \mathbb{F}^u(Ч_2)$
$\subseteq int_\tau(\mathbb{F}^u(cl_\sigma(Ч_2, \langle 0.35, 0.5, 0.15\rangle)), \langle 0.35, 0.5, 0.15\rangle) = \{\langle\xi, 0.1, 0.35, 0\rangle | \xi \in \Xi\}$,
$\{\langle\xi, 0.05, 0.36, 0\rangle | \xi \in \Xi\} = \mathbb{F}^l(Ч_1)$
$\subseteq int_\tau(\mathbb{F}^l(cl_\sigma(Ч_1, \langle 0.35, 0.5, 0.15\rangle)), \langle 0.35, 0.5, 0.15\rangle) = \{\langle\xi, 0.1, 0.35, 0\rangle | \xi \in \Xi\}$,
$\langle 0.1, 0.35, 0\rangle = \mathbb{F}^l(Ч_2)$
$\subseteq int_\tau(\mathbb{F}^l(cl_\sigma(Ч_2, \langle 0.35, 0.5, 0.15\rangle)), \langle 0.35, 0.5, 0.15\rangle) = \{\langle\xi, 0.1, 0.35, 0\rangle | \xi \in \Xi\}$.
but
$\{\langle\xi, 0.05, 0.36, 0\rangle | \xi \in \Xi\} = \mathbb{F}^u(Ч_1) \nsubseteq int_\tau(\mathbb{F}^u(cl^*(Ч_1, \langle 0.35, 0.5, 0.15\rangle)), \langle 0.35, 0.5, 0.15\rangle)$
$= \flat$,
$\{\langle\xi, 0.05, 0.36, 0\rangle | \xi \in \Xi\} = \mathbb{F}^l(Ч_1) \nsubseteq int_\tau(\mathbb{F}^l(cl^*(Ч_1, \langle 0.35, 0.5, 0.15\rangle)), \langle 0.35, 0.5, 0.15\rangle)$
$= \flat$.

Example 3.6. *From the Example 3.4, for* $\mathbb{K}_1 = \{\langle\xi, 0.6, 0.3, 0.1\rangle \mid \xi \in \Xi\}$, $Ч_1 = \{\langle\zeta, 0.3, 0.6, 0.1\rangle \mid \zeta \in \Upsilon\}$, *define picture fuzzy topologies* τ*:*$(I^3)^\Xi \to I^3$, σ*:* $(I^3)^\Upsilon \to I^3$, *and picture fuzzy ideal* ℓ^P*:* $(I^3)^\Upsilon \to I^3$ *as follows:*

$$\tau(\mathbb{K})=\begin{cases}\langle 1,0,0\rangle & \text{if } \mathbb{K}\in\{\flat,\sharp\},\\ \langle 0.6,0.2,0.2\rangle & \text{if } \mathbb{K}=\mathbb{K}_1,\\ \langle 0,1,0\rangle & \text{otherwise,}\end{cases}\quad,\quad \sigma(\text{Ч})=\begin{cases}\langle 1,0,0\rangle & \text{if Ч}\in\{\flat,\sharp\},\\ \langle 0.5,0.3,0.1\rangle & \text{if Ч}=\text{Ч}_1,\\ \langle 0,1,0\rangle & \text{otherwise,}\end{cases}$$

$$\ell^P(\text{Ч})=\begin{cases}\langle 1,0,0\rangle & \text{if Ч}=\flat,\\ \langle 0.7,0.15,0.15\rangle & \text{if } \{\langle\zeta,0.4,0.5,0.1\rangle\,|\zeta\in\Upsilon\}\subseteq\text{Ч}\subseteq\sharp,\\ \langle 0,1,0\rangle & \text{otherwise.}\end{cases}$$

$\mathbb{F}:(\Xi,\tau)\nrightarrow(\Upsilon,\sigma,\ell^P)$ is PF^uW (resp. PF^lW) ℓ^P-continuous but is not PF^uA (resp. PF^lA) ℓ^P-continuous because
$\{\langle\xi,0.3,0.6,0\rangle\,|\xi\in\Xi\}=\mathbb{F}^u(\text{Ч}_1)$
$\subseteq int_\tau(\mathbb{F}^u(cl^*(\text{Ч}_1,\langle 0.5,0.3,0.1\rangle)),\langle 0.5,0.3,0.1\rangle)=\{\langle\xi,0.3,0.6,0\rangle\,|\xi\in\Xi\}$,
$\{\langle\xi,0.3,0.6,0\rangle\,|\xi\in\Xi\}=\mathbb{F}^l(\text{Ч}_1)$
$\subseteq int_\tau\left(\mathbb{F}^l(cl^*(\text{Ч}_1,\langle 0.5,0.3,0.1\rangle)\right),\langle 0.5,0.3,0.1\rangle)=\{\langle\xi,0.3,0.6,0\rangle\,|\xi\in\Xi\}$,
but
$\{\langle\xi,0.3,0.6,0\rangle\,|\xi\in\Xi\}=\mathbb{F}^u(\text{Ч}_1)$
$\nsubseteq int_\tau(\mathbb{F}^u(int_\sigma(cl^*(\text{Ч}_1,\langle 0.5,0.3,0.1\rangle),\langle 0.5,0.3,0.1\rangle)),\langle 0.5,0.3,0.1\rangle)=\flat$,
$\{\langle\xi,0.3,0.6,0\rangle\,|\xi\in\Xi\}=\mathbb{F}^l(\text{Ч}_1)$
$\nsubseteq int_\tau\left(\mathbb{F}^l(int_\sigma(cl^*(\text{Ч}_1,\langle 0.5,0.3,0.1\rangle),\langle 0.5,0.3,0.1\rangle))\right),\langle 0.5,0.3,0.1\rangle)=\flat$.

Theorem 3.15. *A PFM* $\mathbb{F}:(\Xi,\tau)\nrightarrow(\Upsilon,\sigma,\ell^P)$ *is* PF^lW ℓ^P*-continuous iff* $cl_\tau(\mathbb{F}^u(int^*(\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle)),\langle\varsigma,\varkappa,\vartheta\rangle)\subseteq\mathbb{F}^u(\text{Ч})$ *for each* $\text{Ч}\in\left(I^3\right)^\Upsilon$ *with* $\sigma(\exists\ \text{Ч})\geq\langle\varsigma,\varkappa,\vartheta\rangle$, $\varsigma\in I_0,\varkappa\in I_1$ *and* $\vartheta\in I_1$.

Proof. ($\Rightarrow$) Let $\text{Ч}\in\left(I^3\right)^\Upsilon$ with $\sigma(\exists\ \text{Ч})\geq\langle\varsigma,\varkappa,\vartheta\rangle$. Then, by Theorem 3.13,

$$\begin{aligned}\exists\ \mathbb{F}^u(\text{Ч}) &= \mathbb{F}^l(\exists\ \text{Ч})\subseteq int_\tau(\mathbb{F}^l(cl^*(\exists\ \text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle)),\langle\varsigma,\varkappa,\vartheta\rangle)\\ &= \exists\ cl_\tau(\mathbb{F}^u(int^*(\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle)),\langle\varsigma,\varkappa,\vartheta\rangle).\end{aligned}$$

Thus, $cl_\tau(\mathbb{F}^u(int^*(\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle)),\langle\varsigma,\varkappa,\vartheta\rangle)\subseteq\mathbb{F}^u(\text{Ч})$.

($\Leftarrow$) Let $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle}\in D(\mathbb{F})$, $\text{Ч}\in\left(I^3\right)^\Upsilon$ with $\sigma(\text{Ч})\geq\langle\varsigma,\varkappa,\vartheta\rangle$ and $\xi_t\in\mathbb{F}^l(\text{Ч})$. Then,

$$\begin{aligned}\exists int_\tau(\mathbb{F}^l(cl^*(\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle)),\langle\varsigma,\varkappa,\vartheta\rangle) &= cl_\tau(\mathbb{F}^u(int^*(\exists\ \text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle)),\langle\varsigma,\varkappa,\vartheta\rangle)\\ &\subseteq \mathbb{F}^u(\exists\ \text{Ч})=\exists\ \mathbb{F}^l(\text{Ч}),\end{aligned}$$

and hence, $\mathbb{F}^l(\text{Ч})\subseteq int_\tau(\mathbb{F}^l(cl^*(\text{Ч},\langle\varsigma,\varkappa,\vartheta\rangle)),\langle\varsigma,\varkappa,\vartheta\rangle)$. Thus, it is PF^lW ℓ^P-continuous.

The following theorem is similarly proved as the proof of Theorem 3.15.

Theorem 3.16. *A normalized PFM* $\mathbb{F} : (\Xi, \tau) \nrightarrow (\Upsilon, \sigma, \ell^P)$ *is* PF^uW ℓ^P*-continuous iff* $cl_\tau(\mathbb{F}^l(int^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle) \subseteq \mathbb{F}^l(Ч)$ *for each* $Ч \in (I^3)^\Upsilon$ *with* $\sigma(\dashv Ч) \geq \langle\varsigma, \varkappa, \vartheta\rangle$, $\varsigma \in I_0, \varkappa \in I_1$ *and* $\vartheta \in I_1$.

Theorem 3.17. *If* $\mathbb{F} : (\Xi, \tau) \nrightarrow (\Upsilon, \sigma, \ell^P)$ *is normalized* PF^uW ℓ^P*-continuous and* $\mathbb{F}(\mathbb{K}) \subseteq int_\sigma(cl^*(\mathbb{F}(\mathbb{K}), \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle)$ *for each* $\mathbb{K} \in (I^3)^\Xi$*. Then,* $\mathbb{F}$ *is* PF^uA ℓ^P*-continuous.*

Proof. Let $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in D(\mathbb{F})$, $Ч \in (I^3)^\Upsilon, \sigma(Ч) \geq \langle\varsigma, \varkappa, \vartheta\rangle$ and $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in \mathbb{F}^u(Ч)$. Then, there exists $\mathbb{K} \in (I^3)^\Xi$ with $\tau(\mathbb{K}) \geq \langle\varsigma, \varkappa, \vartheta\rangle$ and $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in \mathbb{K}$ such that $\mathbb{K} \subseteq \mathbb{F}^u(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle))$, then
$\mathbb{F}(\mathbb{K}) \subseteq \mathbb{F}(\mathbb{F}^u(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle))) \subseteq cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle)$. Since $\mathbb{F}(\mathbb{K}) \subseteq int_\sigma(cl^*(\mathbb{F}(\mathbb{K}), \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle) \subseteq int_\sigma(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle)$, hence
$\mathbb{K} \subseteq \mathbb{F}^u(\mathbb{F}(\mathbb{K})) \subseteq \mathbb{F}^u(int_\sigma(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle))$. Then, $\mathbb{F}$ is PF^uA ℓ^P-continuous.

Theorem 3.18. *Let* $\mathbb{F} : (\Xi, \tau) \nrightarrow (\Upsilon, \sigma, \ell^P)$ *be a* PF^lW ℓ^P*-continuous. Then,* $\mathbb{F}^l(Ч) \subseteq int_\tau(\mathbb{F}^l(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle)$ *for any* $Ч \in (I^3)^\Upsilon$ *with* $Ч \subseteq int_\sigma(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle)$, $\varsigma \in I_0, \varkappa \in I_1$ *and* $\vartheta \in I_1$.

Proof. Let $\mathbb{F}$ be a PF^lW ℓ^P-continuous and $Ч \in (I^3)^\Upsilon$ with $Ч \subseteq int_\sigma(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle)$. Then, if $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in \mathbb{F}^l(Ч) \subseteq \mathbb{F}^l(int_\sigma(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle))$, there exists $\mathbb{K} \in (I^3)^\Xi$, $\tau(\mathbb{K}) \geq \langle\varsigma, \varkappa, \vartheta\rangle$ and $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in \mathbb{K}$ such that

$$\mathbb{K} \subseteq (\mathbb{F}^l(cl^*(int_\sigma(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle)) \subseteq \mathbb{F}^l(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle)).$$

Thus, $\mathbb{K} \subseteq int_\tau(\mathbb{F}^l(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle)$, and
$\mathbb{F}^l(Ч) \subseteq int_\tau(\mathbb{F}^l(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle)$.

The following theorem is similarly proved as the proof of Theorem 3.18.

Theorem 3.19. *Let* $\mathbb{F} : (\Xi, \tau) \nrightarrow (\Upsilon, \sigma, \ell^P)$ *be a normalized* PF^uW ℓ^P*-continuous. Then,* $\mathbb{F}^u(Ч) \subseteq int_\tau(\mathbb{F}^u(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle)$ *for any* $Ч \in (I^3)^\Upsilon$ *with* $Ч \subseteq int_\sigma(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle)$, $\varsigma \in I_0, \varkappa \in I_1$ *and* $\vartheta \in I_1$.

Theorem 3.20. *For a PFM* $\mathbb{F} : (\Xi, \tau) \nrightarrow (\Upsilon, \sigma, \ell^P)$, $Ч \in (I^3)^\Upsilon$, $\varsigma \in I_0, \varkappa \in I_1$ *and* $\vartheta \in I_1$*,the following statements are equivalent:*

(1) $\mathbb{F}$ *is* PF^lAW ℓ^P*-continuous.*
(2) $\mathbb{F}^l(Ч) \subseteq int_\tau(cl_\tau(\mathbb{F}^l(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle)$*,if* $\sigma(Ч) \geq \langle\varsigma, \varkappa, \vartheta\rangle$.
(3) $cl_\tau(int_\tau(\mathbb{F}^u(int^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle) \subseteq \mathbb{F}^u(Ч)$*, if* $\sigma(\dashv Ч) \geq \langle\varsigma, \varkappa, \vartheta\rangle$.

Proof. (1) $\Longrightarrow$ (2) Let $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in D(\mathbb{F})$, $Ч \in (I^3)^\Upsilon, \sigma(Ч) \geq \langle\varsigma, \varkappa, \vartheta\rangle$ and $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in \mathbb{F}^l(Ч)$. Then, there exists $\mathbb{K} \in (I^3)^\Xi, \tau(\mathbb{K}) \geq \langle\varsigma, \varkappa, \vartheta\rangle$ and $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in \mathbb{K}$ such that
$\mathbb{K} \subseteq cl_\tau(\mathbb{F}^l(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle)$. Thus,

$\xi_t \in \mathbb{K} \subseteq int_\tau(cl_\tau(\mathbb{F}^l(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle)$, and hence
$\mathbb{F}^l(Ч) \subseteq int_\tau(cl_\tau(\mathbb{F}^l(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle)$.

(2) $\Longrightarrow$ (3) Let $Ч \in (I^3)^\Upsilon$ with $\sigma(\daleth Ч) \geq \langle\varsigma, \varkappa, \vartheta\rangle$. Then, by (2)

$$\begin{aligned} \daleth\, \mathbb{F}^u(Ч) &= \mathbb{F}^l(\daleth Ч) \subseteq int_\tau(cl_\tau(\mathbb{F}^l(cl^*(\daleth Ч, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle) \\ &= \daleth cl_\tau(int_\tau(\mathbb{F}^u(int^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle\,, \end{aligned}$$

thus, $cl_\tau(int_\tau(\mathbb{F}^u(int^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle \subseteq \mathbb{F}^u(Ч)$.

(3) $\Longrightarrow$ (1) Let $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in D(\mathbb{F})$, $Ч \in (I^3)^\Upsilon$, $\sigma(Ч) \geq \langle\varsigma, \varkappa, \vartheta\rangle$ and $\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in \mathbb{F}^l(Ч)$. Then, by (3), we have

$$\begin{aligned} &\daleth int_\tau(cl_\tau(\mathbb{F}^l(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle) \\ = \; &cl_\tau(int_\tau(\mathbb{F}^u(int^*(\daleth Ч, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle \subseteq \mathbb{F}^u(\daleth Ч) = \daleth\, \mathbb{F}^l(Ч), \end{aligned}$$

and hence $\mathbb{F}^l(Ч) \subseteq int_\tau(cl_\tau(\mathbb{F}^l(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle)$. Therefore,
$\xi_{\langle\varsigma,\varkappa,\vartheta\rangle} \in int_\tau(cl_\tau(\mathbb{F}^l(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle)$
$\subseteq cl_\tau(\mathbb{F}^l(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle)$. Thus, $\mathbb{F}$ is $PF^lAW\ell^P$ -continuous.

The following theorem is similarly proved as the proof of Theorem 3.20.

Theorem 3.21. *For a normalized PFM $\mathbb{F} : (\Xi, \tau) \multimap (\Upsilon, \sigma, \ell^P)$, $Ч \in (I^3)^\Upsilon$, $\varsigma \in I_0, \varkappa \in I_1$ and $\vartheta \in I_1$, the following statements are equivalent:*

(1) *$\mathbb{F}$ is $PF^uAW\ \ell^P$-continuous.*

(2) *$\mathbb{F}^u(Ч) \subseteq int_\tau(cl_\tau(\mathbb{F}^u(cl^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle)$, if $\sigma(Ч) \geq \langle\varsigma, \varkappa, \vartheta\rangle$.*

(3) *$cl_\tau\left(int_\tau(\mathbb{F}^l(int^*(Ч, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle\right) \subseteq \mathbb{F}^l(Ч)$, if $\sigma(\daleth Ч) \geq \langle\varsigma, \varkappa, \vartheta\rangle$.*

The following example shows that generally PF^uAW continuous and PF^lAW continuous (resp. $PF^uAW\ \ell^P$-continuous and $PF^lAW\ \ell^P$-continuous) need not be either $PF^uAW\ \ell^P$-continuous (resp. $PF^uW\ \ell^P$-continuous) or $PF^lAW\ \ell^P$-continuous (resp. $PF^lW\ \ell^P$-continuous).

Example 3.7. *Let $\Xi = \{\xi_1, \xi_2\}$, $\Upsilon = \{\zeta_1, \zeta_2, \zeta_3\}$ and $\mathbb{F} : \Xi \multimap \Upsilon$ be a PFM defined by $\Psi_{\mathbb{F}}(\xi_1, \zeta_1) = \langle 0.1, 0.3, 0.6\rangle$, $\Psi_{\mathbb{F}}(\xi_1, \zeta_2) = \langle 1, 0, 0\rangle$, $\Psi_{\mathbb{F}}(\xi_1, \zeta_3) = \langle 0.23, 0.12, 0.4\rangle$, $\Psi_{\mathbb{F}}(\xi_2, \zeta_1) = \langle 0.31, 0.23, 0.43\rangle$,*
$\Psi_{\mathbb{F}}(\xi_2, \zeta_2) = \langle 0.45, 0.1, 0.4\rangle$, $\Psi_{\mathbb{F}}(\xi_2, \zeta_3) = \langle 1, 0, 0\rangle$. For $\mathbb{K}_1 = \{\langle \xi, 0.3, 0.2, 0.1\rangle \mid \xi \in \Xi\}$ and $Ч_1 = \{\langle \zeta, 0.2, 0.5, 0.3\rangle \mid \zeta \in \Upsilon\}$ define picture fuzzy topologies $\tau : (I^3)^\Xi \to I^3$, $\sigma : (I^3)^\Upsilon \to I^3$, and picture fuzzy ideal $\ell^P : (I^3)^\Upsilon \to I^3$ as follows:

$$\tau(\mathbb{K})=\begin{cases}\langle 1,0,0\rangle & \text{if } \mathbb{K}\in\{\flat,\sharp\},\\ \langle 0.35,0.3,0.3\rangle & \text{if } \mathbb{K}=\mathbb{K}_1,\\ \langle 0,1,0\rangle & \text{otherwise,}\end{cases}\quad ,\ \sigma(Ч)=\begin{cases}\langle 1,0,0\rangle & \text{if } Ч\in\{\flat,\sharp\},\\ \langle 0.31,0.31,0.18\rangle & \text{if } Ч=Ч_1,\\ \langle 0,1,0\rangle & \text{otherwise,}\end{cases}$$

$$\ell^P(Ч)=\begin{cases}\langle 1,0,0\rangle & \text{if } Ч=\flat,\\ \langle 0.36,0.31,0.2\rangle & \text{if } \flat\subseteq Ч\subseteq\{\langle\zeta,0.2,0.5,0.21\rangle\,|\zeta\in\Upsilon\},\\ \langle 0,1,0\rangle & \text{otherwise.}\end{cases}$$

Then, (1) $\mathbb{F}:(\Xi,\tau)\nrightarrow(\Upsilon,\sigma,\ell^P)$ is PF^uAW (resp. PF^lAW)-continuous but is not PF^uAW (resp. PF^lAW) ℓ^P-continuous because
$\{\langle\xi,0.2,0.5,0\rangle\,|\xi\in\Xi\}=\mathbb{F}^u(Ч_1)$
$\subseteq int_\tau(cl_\tau(\mathbb{F}^u(cl_\sigma(Ч_1,\langle 0.31,0.31,0.18\rangle)),\langle 0.31,0.31,0.18\rangle),\langle 0.31,0.31,0.18\rangle)=\sharp$,
$\{\langle\xi,0.2,0.5,0\rangle\,|\xi\in\Xi\}=\mathbb{F}^l(Ч_1)$
$\subseteq int_\tau(cl_\tau(\mathbb{F}^l(cl_\sigma(Ч_1,\langle 0.31,0.31,0.18\rangle)),\langle 0.31,0.31,0.18\rangle),\langle 0.31,0.31,0.18\rangle)=\sharp$,
but
$\{\langle\xi,0.2,0.5,0\rangle\,|\xi\in\Xi\}=\mathbb{F}^u(Ч_1)$
$\nsubseteq int_\tau(cl_\tau(\mathbb{F}^u(cl^*(Ч_1,\langle 0.31,0.31,0.18\rangle)),\langle 0.31,0.31,0.18\rangle),\langle 0.31,0.31,0.18\rangle)=\flat$,
$\{\langle\xi,0.2,0.5,0\rangle\,|\xi\in\Xi\}=\mathbb{F}^l(Ч_1)$
$\nsubseteq int_\tau(cl_\tau(\mathbb{F}^l(cl^*(Ч_1,\langle 0.31,0.31,0.18\rangle)),\langle 0.31,0.31,0.18\rangle),\langle 0.31,0.31,0.18\rangle)=\flat$,

(2) For $\mathbb{K}_1=\{\langle\xi,0.3,0.2,0.1\rangle\mid\xi\in\Xi\}$, then $\mathbb{F}:(\Xi,\tau)\nrightarrow(\Upsilon,\sigma,\ell^P)$ is PF^uAW (resp. PF^lAW) ℓ^P-continuous but is not PF^uW (resp. PF^lW) ℓ^P-continuous because
$\{\langle\xi,0.2,0.5,0\rangle\,|\xi\in\Xi\}=\mathbb{F}^u(Ч_1)$
$\subseteq int_\tau(cl_\tau(\mathbb{F}^u(cl^*(Ч_1,\langle 0.31,0.31,0.18\rangle)),\langle 0.31,0.31,0.18\rangle),\langle 0.31,0.31,0.18\rangle)=\sharp$,
$\{\langle\xi,0.2,0.5,0\rangle\,|\xi\in\Xi\}=\mathbb{F}^l(Ч_1)$
$\subseteq int_\tau(cl_\tau(\mathbb{F}^l(cl^*(Ч_1,\langle 0.31,0.31,0.18\rangle)),\langle 0.31,0.31,0.18\rangle),\langle 0.31,0.31,0.18\rangle)=\sharp$,
but
$\{\langle\xi,0.2,0.5,0\rangle\,|\xi\in\Xi\}=\mathbb{F}^u(Ч_1)\nsubseteq int_\tau(\mathbb{F}^u(cl^*(Ч_1,\langle 0.31,0.31,0.18\rangle)),\langle 0.31,0.31,0.18\rangle)=\flat$,
$\{\langle\xi,0.2,0.5,0\rangle\,|\xi\in\Xi\}=\mathbb{F}^l(Ч_1)\nsubseteq int_\tau(\mathbb{F}^l(cl^*(Ч_1,\langle 0.31,0.31,0.18\rangle)),\langle 0.31,0.31,0.18\rangle)=\flat$,

Theorem 3.22. *Let $\mathbb{F}:(\Xi,\tau)\nrightarrow(\Upsilon,\sigma,\ell^P)$ be a normalized PFM, $\mathbb{F}$ be PF^uAW ℓ^P-continuous and PF^lA ℓ^P-continuous. Then, $\mathbb{F}$ is PF^uW ℓ^P-continuous.*

Proof. Let $Ч\in(I^3)^\Upsilon$ with $\sigma(Ч)\geq\langle\varsigma,\varkappa,\vartheta\rangle$ and $\mathbb{F}$ be PF^uAW ℓ^P-continuous. Then, by Theorem 3.21(2),

$$\mathbb{F}^u(Ч)\subseteq int_\tau(cl_\tau(\mathbb{F}^u(cl^*(Ч,\langle\varsigma,\varkappa,\vartheta\rangle)),\langle\varsigma,\varkappa,\vartheta\rangle),\langle\varsigma,\varkappa,\vartheta\rangle).$$

Since $cl_{\sigma}(Ч, \langle\varsigma, \varkappa, \vartheta\rangle) = cl_{\sigma}(int^{*}(cl_{\sigma}(Ч, \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle), \langle\varsigma, \varkappa, \vartheta\rangle)$, it follows from Theorem 3.8(2) that
$\tau\left(\exists\mathbb{F}^{u}\left(cl_{\sigma}(Ч, \langle\varsigma, \varkappa, \vartheta\rangle)\right)\right) \geq \langle\varsigma, \varkappa, \vartheta\rangle$, then $\tau\left(\exists\mathbb{F}^{u}\left(cl^{*}(Ч, \langle\varsigma, \varkappa, \vartheta\rangle)\right)\right) \geq \langle\varsigma, \varkappa, \vartheta\rangle$, and $\mathbb{F}^{u}(Ч) \subseteq int_{\tau}(\mathbb{F}^{u}(cl^{*}(Ч, \langle\varsigma, \varkappa, \vartheta\rangle)), \langle\varsigma, \varkappa, \vartheta\rangle)$. Thus, by Theorem 3.4, $\mathbb{F}$ is $PF^{u}W$ ℓ^{P}-continuous.

The following theorem is similarly proved as the proof of Theorem 3.22.

Theorem 3.23. *Let* $\mathbb{F} : (\Xi, \tau) \looparrowright (\Upsilon, \sigma, \ell^{P})$ *be a normalized PFM,* $\mathbb{F}$ *be* $PF^{l}AW$ ℓ^{P}*-continuous and* $PF^{u}A$ ℓ^{P}*-continuous. Then,* $\mathbb{F}$ *is* $PF^{l}W$ ℓ^{P}*-continuous.*

4. Conclusion

In this paper, we introduced a definition of picture fuzzy implication operation in PFSs. Two operations based on the product form of the essential degrees of positivism, negativism and neutralism are investigated. Two other operations in (PFSs) are established based on the sum and the product of these essential degrees. Depending on the previous four picture fuzzy operations, we defined four PFMTSs over PFSs. Although the variety of objects defined in this paper, all structures defined here did not satisfy some of the Kuratowski closure conditions or the Kuratowski interior conditions. These constructed structures are called "feeble" (PFFMTSs) standing for not all required conditions for a topological (closure or interior) operator are satisfied. Still, based on the simple operations $\Box$ and $\Diamond$, we got accurate definitions of "closure" and "interior" operators for PFMTSs. So, $\Box$ and $\Diamond$ are the standard two modal operators over PFSs introducing two PFMTSs. In future research work, we will extend the work to some other fuzzy environment and derive some more properties for different topological spaces. Further, we will expand our work to consider the picture topological space to be temporal as well as in the corresponding results.

Conflicts of interest:

The authors declare that they have not any conflicts of interest.

Data Availability Statement:

The data sets used and/or analysed during the current study are available from the corresponding author upon reasonable request.

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