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On Gould-Hopper-based bivariate Fubini polynomials

Raimah G. Bago[1,*], Normalah S. Abdulcarim[1]

[1] *Department of Mathematics, College of Natural Sciences and Mathematics, Mindanao State University Main Campus, 9700 Marawi City, Philippines*

Abstract. In this paper, we incorporate the bivariate Fubini polynomials with Gould-Hopper polynomials to introduce new polynomials called Gould-Hopper-based bivariate Fubini polynomials by modifying the classical generating function of the bivariate Fubini polynomials. Also, properties such as addition formula, explicit formula, implicit formula, recurrence formula and symmetric identities are obtained. Moreover, some relations between the Gould-Hopper-based bivariate Fubini polynomials and some of the other special polynomials and numbers, such as the 2-variable Gould-Hopper polynomials and the Stirling numbers of the second kind were investigated.

2020 Mathematics Subject Classifications: 05A15, 26A24, 26A36

Key Words and Phrases: Gould-Hopper polynomials, Fubini numbers and polynomials, bivariate Fubini polynomials, generating functions, derivatives, integration

1. Introduction

In 1975, Tanny introduced the classical Fubini polynomials $F_n(y)$ which are defined in [1] by:

$$F_n(y) = \sum_{k=1}^{n} k!S(n,k)y^k, \tag{1}$$

where $S(n,k)$ is the Stirling numbers of the second kind. These polynomials can be generated by

$$\frac{1}{1-y(e^t-1)} = \sum_{n=0}^{\infty} F_n(y)\frac{t^n}{n!}. \tag{2}$$

Note that when setting $y = 1$, gives $F_n(1) = F_n$, the classical Fubini number. One calls these numbers the Fubini numbers, ordered Bell numbers, or geometric numbers. In

*Corresponding author.
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Email addresses: bago.rg36@s.msumain.edu.ph (R. G. Bago),
normalah.abdulcarim@msumain.edu.ph (N. S. Abdulcarim)

2017, Kargin defined the bivariate Fubini polynomials $F_n(x,y)$ (see[2]), by the following generating function

$$\sum_{n=0}^{\infty} F_n(x,y)\frac{t^n}{n!} = \frac{e^{xt}}{1-y(e^t-1)}. \tag{3}$$

Another well-researched special polynomial are the Hermite polynomials introduced by French mathematician Charles Hermite in the mid-19th century, which are deemed to be the most beneficial orthogonal special functions during the classical period. The Hermite polynomials $H_n(x)$ defined in [3], are given by

$$e^{-t^2+2tx} = \sum_{n\geq 0} H_n(x)\frac{t^n}{n!}. \tag{4}$$

Later on, it was then generalized by Appell and de Fériet [4] as follows:

$$H_n(x,y) = n!\sum_{r=0}^{\infty} \frac{y^r x^{n-2r}}{r!(n-2r)!}, \tag{5}$$

which are now well-known as the 2-variable Hermite Kampe de Fériet polynomials. These polynomials exponentially defined by the following generating function:

$$\sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!} = e^{xt+yt^2}. \tag{6}$$

Then in 2019, another type of Hermite polynomials involving Fubini polynomials was introduced by Khan et al [5], which was the Hermite-Fubini polynomials. They defined the 3-variable Hermite-Fubini polynomials by means of the following generating function

$$\frac{e^{xt+yt^2}}{1-z(e^t-1)} = \sum_{n=0}^{\infty} {}_HF_n(x,y;z)\frac{t^n}{n!}.$$

For $y=0$ in the above equation, they obtain the bivariate Fubini polynomials. They also investigate some properties of these polynomials and then establish summation formulas and derive symmetric identities.

Recently, researchers have studied and investigated another type of special polynomial called Gould-Hopper polynomials. These polynomials sometimes also called higher-order Hermite or Kampe de Fériet polynomials. In 1962, Gould and Hopper [6] defined the Gould-Hopper polynomials $H_n^{(j)}(x,y)$ by means of the following generating function

$$e^{xt+yt^j} = \sum_{n=0}^{\infty} H_n^{(j)}(x,y)\frac{t^n}{n!}. \tag{7}$$

These polynomials are represented by the series:

$$H_n^{(j)}(x,y) = n!\sum_{k=0}^{\left[\frac{n}{j}\right]} \frac{x^{n-jk}y^k}{(n-jk)!k!}. \tag{8}$$

When setting $j = 2$, $H_n^{(2)}(x,y) = H_n(x,y)$, where $H_n(x,y)$ is the 2-variable Hermite Kampe̍ de Fériet polynomials.

In this paper, the authors introduced the Gould-Hopper-based bivariate Fubini polynomials which we define in parallel to the definition of 3-variable Hermite-Fubini polynomials described in [5]. We use j instead of 2 making it a generalization of Hermite-Fubini polynomials and investigate some of its properties.

2. Preliminaries

Definition 1. [3] The n **falling factorial** of order k, denoted by $(n)_k$, is defined as

$$\begin{aligned}(n)_k &= \prod_{i=1}^{k}(n-i+1)\\ &= \frac{n!}{(n-k)!}\\ &= n(n-1)...(n-k+1), \quad \text{if} \quad k \geq 1; \quad (n)_0 = 1.\end{aligned}$$

Theorem 1. [7](**Newton's Binomial Theorem**) *For all real numbers r,*

$$(1+x)^r = \sum_{i=0}^{\infty}\binom{r}{i}x^i,$$

where

$$\binom{r}{i} = \begin{cases} 1 & ,\text{if} \quad i = 0,\\ \frac{r(r-1)...(r-i+1)}{i!} & ,\text{if} \quad i > 0.\end{cases}$$

Remark 1. The following series arises in the Binomial theorem where r is any positive real number,

$$(x+y)^{-r} = \sum_{i=0}^{\infty}(-1)^i\binom{r+i-1}{i}x^{-r-i}y^i, \quad (|y| < |x|).$$

Example 1. Consider the expression $\left[1 - y(e^t-1)\right]^{-1}$. By applying Remark 1, we have

$$\begin{aligned}\left[1 - y(e^t-1)\right]^{-1} &= \sum_{i=0}^{\infty}(-1)^i\binom{1+i-1}{i}(1)^{-1-i}[-y(e^t-1)]^i\\ &= \sum_{i=0}^{\infty}y^i(e^t-1)^i.\end{aligned}$$

Definition 2. [8] **Geometric series** are series of the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-r} \tag{9}$$

in which a and r are fixed real numbers and $a \neq 0$. The series can also be written as $\sum_{n=0}^{\infty} ar^n$. The **ratio** r can be positive or negative.

Example 2. For positive integers a and b. Consider the geometric series

$$\sum_{n=0}^{a-1} e^{btn}. \tag{10}$$

Note that

$$\begin{aligned} e^{bt}\sum_{n=0}^{a-1} e^{btn} - \sum_{n=0}^{a-1} e^{btn} =& (e^{bt})\left[(1) + (1)(e^{bt}) + (1)(e^{bt})^2 + \cdots + (1)(e^{bt})^{a-1}\right] \\ &- \left[(1) + (1)(e^{bt}) + (1)(e^{bt})^2 + \cdots + (1)(e^{bt})^{a-1}\right] \\ e^{bt}\sum_{n=0}^{a-1} e^{btn} - \sum_{n=0}^{a-1} e^{btn} =& \left[e^{bt} + (e^{bt})^2 + (e^{bt})^3 + \cdots + (e^{bt})^a\right] \\ &- \left[1 + e^{bt} + (e^{bt})^2 + \cdots + (e^{bt})^{a-1}\right] \\ \sum_{n=0}^{a-1} e^{btn}\left(e^{bt} - 1\right) =& (e^{bt})^a - 1. \end{aligned}$$

So,

$$\sum_{n=0}^{a-1} e^{btn} = \frac{(e^{bt})^a - 1}{e^{bt} - 1}. \tag{11}$$

Theorem 2. [9] *The following formula holds*

$$\sum_{m,n=0}^{\infty} f(m+n)\frac{x^m}{m!}\frac{y^n}{n!} = \sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!}. \tag{12}$$

Theorem 3. [10] *Let* $A(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}$ *and* $B(t) = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!}$ *be the exponential generating function for the sequence* (a_n) *and* (b_n)*, respectively. Then* $A(t)B(t)$ *is given by*

$$\left(\sum_{n=0}^{\infty} a_n \frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} b_n \frac{t^n}{n!}\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}\frac{t^n}{n!}. \tag{13}$$

Theorem 4. [3] *The Stirling numbers of the second kind* $S(n,k)$ *have a vertical generating function given by:*

$$\frac{1}{k!}(e^t - 1)^k = \sum_{k=0}^{n} S(n,k)\frac{t^n}{n!}. \tag{14}$$

Definition 3. [1] **The classical Fubini polynomials or geometric polynomials** $F_n(y)$ are defined by

$$F_n(y) = \sum_{k=0}^{n} S(n,k)k!y^k, \tag{15}$$

where $S(n,k)$ are the Stirling numbers of the second kind.

Setting $y = 1$, we obtain the n^{th} Fubini number F_n, defined by

$$F_n = \sum_{k=0}^{n} S(n,k)k!. \tag{16}$$

Theorem 5. [1] *The Fubini polynomials satisfy the following generating function*

$$\sum_{n=0}^{\infty} F_n(y)\frac{t^n}{n!} = \frac{1}{1 - y(e^t - 1)}. \tag{17}$$

Specializing to the case $y = 1$ yields

$$\sum_{n=0}^{\infty} F_n\frac{t^n}{n!} = \frac{1}{2 - e^t}. \tag{18}$$

Definition 4. [2] The **bivariate Fubini polynomials** $F_n(x; y)$ are defined by the following generating function

$$\frac{e^{xt}}{1 - y(e^t - 1)} = \sum_{n=0}^{\infty} F_n(x; y)\frac{t^n}{n!}. \tag{19}$$

Theorem 6. [2] *The bivariate Fubini polynomials satisfy the following*

$$yF_n(x + 1, y) = (1 + y)F_n(x, y) - x^n. \tag{20}$$

Definition 5. [11] The **Gould-Hopper polynomials** $H_n^{(j)}(x, y)$ are defined by the following generating function

$$e^{xt+yt^j} = \sum_{n=0}^{\infty} H_n^{(j)}(x, y)\frac{t^n}{n!}. \tag{21}$$

Theorem 7. [11] *The Gould-Hopper polynomials $H_n^{(j)}(x, y)$ satisfy the following generating function*

$$H_n^{(j)}(x, y) = n!\sum_{r=0}^{\left[\frac{n}{j}\right]} \frac{y^r x^{n-jr}}{r!(n - jr)!}. \tag{22}$$

Setting $j = 2$ gives $H_n^{(2)}(x, y) = H_n(x, y)$, where $H_n(x, y)$ is the 2-variable Hermite Kampe̒ de Fériet polynomials.

3. Gould-Hopper-based bivariate Fubini polynomials

Definition 6. Let $j \in \mathbb{N}, j \geq 2$. The **Gould-Hopper-based bivariate Fubini polynomials**, denoted by ${}_HF_n^{(j)}(x, y; z)$, are defined by the following generating function

$$\sum_{n=0}^{\infty} {}_HF_n^{(j)}(x, y; z)\frac{t^n}{n!} = \frac{e^{xt+zt^j}}{1 - y(e^t - 1)}, \tag{23}$$

where $|t| < 2\pi$, if $\frac{y}{y+1} = 1$, and $|t| < \left|\ln\left(\frac{y}{y+1}\right)\right|$, if $\frac{y}{y+1} \neq 1$.

Observe that when $j = 2$,

$$\sum_{n=0}^{\infty} {}_HF_n^{(2)}(x, y; z)\frac{t^n}{n!} = \frac{e^{xt+zt^2}}{1 - y(e^t - 1)} = \sum_{n=0}^{\infty} {}_HF_n(x, y; z)\frac{t^n}{n!}, \tag{24}$$

which are the **3-variable Hermite-Fubini polynomials**. Furthermore, when $z = 0$,

$$\sum_{n=0}^{\infty} {}_HF_n^{(j)}(x, y; 0)\frac{t^n}{n!} = \frac{e^{xt}}{1 - y(e^t - 1)} = \sum_{n=0}^{\infty} F_n(x, y)\frac{t^n}{n!}, \tag{25}$$

which are the **bivariate Fubini polynomials**. When $x = z = 0$,

$$\sum_{n=0}^{\infty} {}_HF_n^{(j)}(0, y; 0)\frac{t^n}{n!} = \frac{1}{1 - y(e^t - 1)} = \sum_{n=0}^{\infty} F_n(y)\frac{t^n}{n!}, \tag{26}$$

which are the **classical Fubini polynomials**. And when $x = z = 0, y = 1$,

$${}_HF_n^{(j)}(0, 1; 0) = \frac{1}{2 - e^t} = \sum_{n=0}^{\infty} F_n\frac{t^n}{n!},$$

which is the n**th Fubini number**.

Theorem 8. *For* $n \geq 0$,

$${}_HF_n^{(j)}(x + u, y; z + v) = \sum_{r=0}^{n} \binom{n}{r} {}_HF_{n-r}^{(j)}(x, y; z)H_r^{(j)}(u, v). \tag{27}$$

Proof. By Definition 6,

$$\begin{aligned}\sum_{n=0}^{\infty} {}_HF_n^{(j)}(x + u, y; z + v)\frac{t^n}{n!} &= \frac{e^{(x+u)t+(z+v)t^j}}{1 - y(e^t - 1)} \\ &= \frac{e^{xt+zt^j}}{1 - y(e^t - 1)} \cdot e^{ut+vt^j}.\end{aligned}$$

Then again by applying Definition 6 and Definition 5 to the right-hand side of the above equation, we have

$$\sum_{n=0}^{\infty} {}_HF_n^{(j)}(x+u,y;z+v)\frac{t^n}{n!} = \sum_{n=0}^{\infty} {}_HF_n^{(j)}(x,y;z)\frac{t^n}{n!} \cdot \sum_{r=0}^{\infty} H_r^{(j)}(u,v)\frac{t^r}{r!}. \tag{28}$$

Moreover, applying Theorem 3 to the right-hand side of equation (28) we get

$$\sum_{n=0}^{\infty} {}_HF_n^{(j)}(x+u,y;z+v)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}\binom{n}{r} {}_HF_{n-r}^{(j)}(x,y;z)H_r^{(j)}(u,v)\right)\frac{t^n}{n!}. \tag{29}$$

Comparing the coefficients of $\frac{t^n}{n!}$ yield (27). □

Remark 2. *Setting $j=2$ in Theorem 8, the following formula involving Hermite-Fubini polynomials holds:*

$${}_HF_n(x+u,y;z+v) = \sum_{r=0}^{n}\binom{n}{r} {}_HF_{n-r}(x,y;z)H_r(u,v).$$

Setting $z=0$ in Theorem 8, a relation between the Gould-Hopper-based bivariate Fubini polynomials, bivariate Fubini polynomials and Gould-Hopper polynomials were established in the following corollary.

Corollary 1. *For $n \geq 0$, the following equation holds:*

$${}_HF_n^{(j)}(x+u,y;v) = \sum_{r=0}^{n}\binom{n}{r} F_{n-r}(x,y)H_r^{(j)}(u,v).$$

Setting $j=2$ and $z=0$ in Theorem 8, a relation between the 3-variable-Hermite Fubini polynomials, bivariate Fubini polynomials and 2-variable Hermite polynomials will be established in the following corollary.

Corollary 2. *For $n \geq 0$, the following equation holds:*

$${}_HF_n(x+u,y;v) = \sum_{r=0}^{n}\binom{n}{r} F_{n-r}(x,y)H_r(u,v).$$

Theorem 9. *For $n \geq 0$,*

$${}_HF_n^{(j)}(x+u,y;z) = \sum_{m=0}^{n}\binom{n}{m} x^{n-m} {}_HF_m^{(j)}(u,y;z).$$

Proof. By Definition 6,

$$\sum_{n=0}^{\infty} {}_HF_n^{(j)}(x+u,y;z)\frac{t^n}{n!} = \frac{e^{ut+zt^j}}{1-y(e^t-1)} \cdot e^{xt}.$$

Then, again applying Definition 6 to the right-hand side of the above equation and expressing e^{xt} in its series form, we have

$$\sum_{n=0}^{\infty} {}_HF_n^{(j)}(x+u,y;z)\frac{t^n}{n!} = \sum_{m=0}^{\infty} {}_HF_m^{(j)}(u,y;z)\frac{t^m}{m!} \cdot \sum_{n=0}^{\infty} x^n \frac{t^n}{n!}. \tag{30}$$

Moreover, applying Theorem 3 to equation (30) we get

$$\sum_{n=0}^{\infty} {}_HF_n^{(j)}(x+u,y;z)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} x^{n-m} {}_HF_m^{(j)}(u,y;z) \right) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ yield to the desired result. □

Setting $j = 2$ in Theorem 9, a formula involving 3-variable-Hermite Fubini polynomials will be established in the following corollary.

Corollary 3. *For $n \geq 0$, the following equation holds:*

$${}_HF_n(x+u,y;z) = \sum_{m=0}^{n} \binom{n}{m} x^{n-m} {}_HF_m(u,y;z).$$

Setting $z = 0$ in Theorem 9, a formula involving bivariate Fubini polynomials will be established in the following corollary.

Corollary 4. *For $n \geq 0$, the following equation holds:*

$$F_n(x+u,y) = \sum_{m=0}^{n} \binom{n}{m} x^{n-m} F_m(u,y).$$

Theorem 10. *For $n \geq 0$, the following formula for Gould-Hopper-based bivariate Fubini polynomials holds:*

$$y {}_HF_n^{(j)}(x+1,y;z) = (1+y) {}_HF_n^{(j)}(x,y;z) - H_n^{(j)}(x,z). \tag{31}$$

Proof. Using Definition 6,

$$\begin{aligned}
\sum_{n=0}^{\infty} y {}_HF_n^{(j)}(x+1,y;z)\frac{t^n}{n!} &= \frac{ye^{(x+1)t+zt^j}}{1-y(e^t-1)} \\
&= \frac{e^{xt+zt^j} - e^{xt+zt^j} + ye^{xt+zt^j} - ye^{xt+zt^j} + ye^{(x+1)t+zt^j}}{1-y(e^t-1)} \\
&= \frac{e^{xt+zt^j} + ye^{xt+zt^j} - e^{xt+zt^j}(1-ye^t+y)}{1-y(e^t-1)} \\
&= \frac{e^{xt+zt^j}}{1-y(e^t-1)} + \frac{ye^{xt+zt^j}}{1-y(e^t-1)} - e^{xt+zt^j}.
\end{aligned}$$

Then, by using Definition 6 and Definition 5 to the right-hand side of the above equation we have

$$\begin{aligned}\sum_{n=0}^{\infty} y_H F_n^{(j)}(x+1,y;z)\frac{t^n}{n!} &= \sum_{n=0}^{\infty} {}_H F_n^{(j)}(x,y;z)\frac{t^n}{n!} + \sum_{n=0}^{\infty} y_H F_n^{(j)}(x,y;z)\frac{t^n}{n!} - \sum_{n=0}^{\infty} H_n^{(j)}(x,z)\frac{t^n}{n!} \\ &= (1+y)\sum_{n=0}^{\infty} {}_H F_n^{(j)}(x,y;z)\frac{t^n}{n!} - \sum_{n=0}^{\infty} H_n^{(j)}(x,z)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left[(1+y)_H F_n^{(j)}(x,y;z) - H_n^{(j)}(x,z)\right]\frac{t^n}{n!}.\end{aligned}$$

Thus, comparing the coefficients of $\frac{t^n}{n!}$ in the above equation we get (31). □

When we set $j=2$ in Theorem 10, a relationship between 3-variable Hermite-Fubini polynomials and 2-variable Hermite polynomials will be established in the following corollary.

Corollary 5. *For $n \geq 0$, the following equation holds:*

$$y_H F_n(x+1,y;z) = (1+y)_H F_n(x,y;z) - H_n(x,z).$$

Remark 3. *Setting $z=0$, Theorem 10 reduces to Theorem 6.*

Theorem 11. *For $n \geq 0$, the following formula for Gould-Hopper-based bivariate Fubini polynomials holds:*

$$H_n^{(j)}(x,z) = {}_H F_n^{(j)}(x,y;z) - y_H F_n^{(j)}(x+1,y;z) + y_H F_n^{(j)}(x,y;z).$$

Proof. Note that

$$\begin{aligned} e^{xt+zt^j} &= \frac{1-y(e^t-1)}{1-y(e^t-1)} \cdot e^{xt+zt^j} \\ &= \frac{e^{xt+zt^j} - ye^t(e^{xt+zt^j}) + y(e^{xt+zt^j})}{1-y(e^t-1)} \\ &= \frac{e^{xt+zt^j}}{1-y(e^t-1)} - y\frac{e^{(x+1)t+zt^j}}{1-y(e^t-1)} + y\frac{e^{xt+zt^j}}{1-y(e^t-1)}.\end{aligned}$$

Then, applying Definition 5 to the left-hand side and Definition 6 to the right-hand side of the above equation we have

$$\begin{aligned}\sum_{n=0}^{\infty} H_n^{(j)}(x,z)\frac{t^n}{n!} &= \sum_{n=0}^{\infty} {}_H F_n^{(j)}(x,y;z)\frac{t^n}{n!} - y\sum_{n=0}^{\infty} {}_H F_n^{(j)}(x+1,y;z)\frac{t^n}{n!} + y\sum_{n=0}^{\infty} {}_H F_n^{(j)}(x,y;z)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left[{}_H F_n^{(j)}(x,y;z) - y_H F_n^{(j)}(x+1,y;z) + y_H F_n^{(j)}(x,y;z)\right]\frac{t^n}{n!}.\end{aligned}$$

Therefore, comparing the coefficients of $\frac{t^n}{n!}$ yields to the desired result. □

Theorem 12. *For $n \geq 0$ and $y_1 \neq y_2$, the following formula for Gould-Hopper-based bivariate Fubini polynomials holds:*

$$\sum_{k=0}^{n}\binom{n}{k}{}_HF_{n-k}^{(j)}(x_1,y_1;z_1)\,{}_HF_k^{(j)}(x_2,y_2;z_2)=\frac{y_2\,{}_HF_n^{(j)}(x_1+x_2,y_2;z_1+z_2)-y_1\,{}_HF_n^{(j)}(x_1+x_2,y_1;z_1+z_2)}{y_2-y_1}.$$

Proof. Using Definition 6

$$\sum_{n=0}^{\infty}{}_HF_n^{(j)}(x_1,y_1;z_1)\frac{t^n}{n!}\sum_{k=0}^{\infty}{}_HF_k^{(j)}(x_2,y_2;z_2)\frac{t^k}{k!}=\frac{e^{x_1t+z_1t^j}}{1-y_1(e^t-1)}\cdot\frac{e^{x_2t+z_2t^j}}{1-y_2(e^t-1)}.$$

Then, applying Theorem 3 to the left-hand side of the above equation we get

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}{}_HF_{n-k}^{(j)}(x_1,y_1;z_1)\,{}_HF_k^{(j)}(x_2,y_2;z_2)\right)\frac{t^n}{n!}=\frac{e^{x_1t+z_1t^j}}{1-y_1(e^t-1)}\cdot\frac{e^{x_2t+z_2t^j}}{1-y_2(e^t-1)}. \tag{32}$$

Note that the right-hand side of equation (32) can be expressed as

$$\begin{aligned}\frac{e^{x_1t+z_1t^j}}{1-y_1(e^t-1)}\cdot\frac{e^{x_2t+z_2t^j}}{1-y_2(e^t-1)}&=\frac{e^{(x_1+x_2)t+(z_1+z_2)t^j}}{(1-y_1(e^t-1))(1-y_2(e^t-1))}\cdot\frac{y_2-y_1}{y_2-y_1}\\&=\frac{y_2e^{(x_1+x_2)t+(z_1+z_2)t^j}-y_1e^{(x_1+x_2)t+(z_1+z_2)t^j}}{(1-y_2(e^t-1))(1-y_1(e^t-1))(y_2-y_1)}\\&=\left(\frac{y_2e^{(x_1+x_2)t+(z_1+z_2)t^j}}{1-y_2(e^t-1)}-\frac{y_1e^{(x_1+x_2)t+(z_1+z_2)t^j}}{1-y_1(e^t-1)}\right)\left(\frac{1}{y_2-y_1}\right).\end{aligned} \tag{33}$$

Thus, applying Definition 6 to the right-hand side of (33) we have

$$\begin{aligned}&\left(\sum_{n=0}^{\infty}y_2\,{}_HF_n^{(j)}(x_1+x_2,y_2;z_1+z_2)\frac{t^n}{n!}-\sum_{n=0}^{\infty}y_1\,{}_HF_n^{(j)}(x_1+x_2,y_1;z_1+z_2)\frac{t^n}{n!}\right)\left(\frac{1}{y_2-y_1}\right)\\&\quad=\sum_{n=0}^{\infty}\left(\frac{y_2\,{}_HF_n^{(j)}(x_1+x_2,y_2;z_1+z_2)-y_1\,{}_HF_n^{(j)}(x_1+x_2,y_1;z_1+z_2)}{y_2-y_1}\right)\frac{t^n}{n!}.\end{aligned}$$

So that,

$$\begin{aligned}&\frac{e^{x_1t+z_1t^j}}{1-y_1(e^t-1)}\cdot\frac{e^{x_2t+z_2t^j}}{1-y_2(e^t-1)}\\&\quad=\sum_{n=0}^{\infty}\left(\frac{y_2\,{}_HF_n^{(j)}(x_1+x_2,y_2;z_1+z_2)-y_1\,{}_HF_n^{(j)}(x_1+x_2,y_1;z_1+z_2)}{y_2-y_1}\right)\frac{t^n}{n!}.\end{aligned} \tag{34}$$

Moreover, equating the left-hand side of (32) and the right-hand side of (34) we get

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}{}_HF_{n-k}^{(j)}(x_1,y_1;z_1)\,{}_HF_k^{(j)}(x_2,y_2;z_2)\right)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\frac{y_2 {}_H F_n^{(j)}(x_1 + x_2, y_2; z_1 + z_2) - y_1 {}_H F_n^{(j)}(x_1 + x_2, y_1; z_1 + z_2)}{y_2 - y_1} \right) \frac{t^n}{n!}.$$

Furthermore, comparing the coefficients of $\frac{t^n}{n!}$ yields the desired result. □

In the next theorems, we derive some explicit formulae for Gould-Hopper-based bivariate Fubini polynomials.

Theorem 13. *For $n \geq 0$, the following formula for Gould-Hopper-based bivariate Fubini polynomials holds:*

$$ {}_H F_n^{(j)}(x, y; z) = \sum_{m=0}^{n} \binom{n}{m} F_{n-m}(y) H_m^{(j)}(x, z). \tag{35}$$

Proof. By Definition 6,

$$\sum_{n=0}^{\infty} {}_H F_n^{(j)}(x, y; z) \frac{t^n}{n!} = \frac{e^{xt+zt^j}}{1 - y(e^t - 1)} = \frac{1}{1 - y(e^t - 1)} e^{xt+zt^j}. \tag{36}$$

Then by applying Theorem 5 and Definition 5 to the right-hand side of the equation (36), we get

$$\sum_{n=0}^{\infty} {}_H F_n^{(j)}(x, y; z) \frac{t^n}{n!} = \sum_{n=0}^{\infty} F_n(y) \frac{t^n}{n!} \sum_{m=0}^{\infty} H_m^{(j)}(x, z) \frac{t^m}{m!}.$$

Moreover, applying Theorem 3 to the above equation we have

$$\sum_{n=0}^{\infty} {}_H F_n^{(j)}(x, y; z) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} F_{n-m}(y) H_m^{(j)}(x, z) \right) \frac{t^n}{n!}.$$

Finally, comparing the coefficients of $\frac{t^n}{n!}$ yields to the desired result. □

Remark 4. *When $j = 2$ in Theorem 13, the following formula involving Hermite-Fubini polynomials holds:*

$$ {}_H F_n(x, y; z) = \sum_{m=0}^{n} \binom{n}{m} F_{n-m}(y) H_m(x, z).$$

Theorem 14. *For $n \geq 0$ and $p, q \in \mathbb{R}$, the following summation formula for Gould-Hopper-based bivariate Fubini polynomials holds:*

$$ {}_H F_n^{(j)}(px, y; qz) = \sum_{k=0}^{n} \sum_{r=0}^{\left[\frac{k}{j}\right]} \frac{(n)_k}{r!(k - jr)!} {}_H F_{n-k}^{(j)}(x, y; z)(p - 1)^{k-jr}(q - 1)^r z^r x^{k-jr}. \tag{37}$$

Proof. By applying Definition 6

$$\begin{aligned}\sum_{n=0}^{\infty} {}_{H}F_n^{(j)}(px,y;qz)\frac{t^n}{n!} &= \frac{e^{pxt+qzt^j}}{1-y(e^t-1)} \\ &= \frac{e^{xt+zt^j}}{1-y(e^t-1)} \cdot e^{(p-1)xt+(q-1)zt^j}.\end{aligned}$$

Then, applying Definition 6 again and Definition 5 to the right-hand side of the above equation we have

$$\sum_{n=0}^{\infty} {}_{H}F_n^{(j)}(px,y;qz)\frac{t^n}{n!} = \sum_{n=0}^{\infty} {}_{H}F_n^{(j)}(x,y;z)\frac{t^n}{n!} \cdot \sum_{n=0}^{\infty} H_n^{(j)}((p-1)x,(q-1)z)\frac{t^n}{n!}. \tag{38}$$

Note that, by applying Theorem 7

$$\begin{aligned}\sum_{n=0}^{\infty} H_n^{(j)}((p-1)x,(q-1)z)\frac{t^n}{n!} &= \sum_{n=0}^{\infty} n! \sum_{r=0}^{\left[\frac{n}{j}\right]} \frac{((q-1)z)^r((p-1)x)^{n-jr}}{r!(n-jr)!}\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{r=0}^{\left[\frac{n}{j}\right]} \frac{n!((q-1)z)^r((p-1)x)^{n-jr}}{r!(n-jr)!}\frac{t^n}{n!}.\end{aligned}$$

Substituting this to equation (38), gives

$$\sum_{n=0}^{\infty} {}_{H}F_n^{(j)}(px,y;qz)\frac{t^n}{n!} = \sum_{n=0}^{\infty} {}_{H}F_n^{(j)}(x,y;z)\frac{t^n}{n!} \cdot \sum_{n=0}^{\infty} \sum_{r=0}^{\left[\frac{n}{j}\right]} \frac{n!((q-1)z)^r((p-1)x)^{n-jr}}{r!(n-jr)!}\frac{t^n}{n!}.$$

Thus, by applying Theorem 3 to the right-hand side of the above equation we get

$$\begin{aligned}\sum_{n=0}^{\infty} {}_{H}F_n^{(j)}(px,y;qz)\frac{t^n}{n!} &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \sum_{r=0}^{\left[\frac{k}{j}\right]} \binom{n}{k} {}_{H}F_{n-k}^{(j)}(x,y;z) \frac{k!((q-1)z)^r((p-1)x)^{k-jr}}{r!(k-jr)!} \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \sum_{r=0}^{\left[\frac{k}{j}\right]} \frac{n!}{k!(n-k)!} {}_{H}F_{n-k}^{(j)}(x,y;z) \frac{k!((q-1)z)^r((p-1)x)^{k-jr}}{r!(k-jr)!} \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \sum_{r=0}^{\left[\frac{k}{j}\right]} \frac{n!}{(n-k)!} {}_{H}F_{n-k}^{(j)}(x,y;z) \frac{((q-1)z)^r((p-1)x)^{k-jr}}{r!(k-jr)!} \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \sum_{r=0}^{\left[\frac{k}{j}\right]} (n)_k {}_{H}F_{n-k}^{(j)}(x,y;z) \frac{((q-1)z)^r((p-1)x)^{k-jr}}{r!(k-jr)!} \right) \frac{t^n}{n!}\end{aligned}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{r=0}^{\left[\frac{k}{j}\right]}\frac{(n)_k}{r!(k-jr)!}{}_HF_{n-k}^{(j)}(x,y;z)(p-1)^{k-jr}(q-1)^r z^r x^{k-jr}\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ yields (37). □

Setting $p = 1$ and $q = 1$ in Theorem 14, the following corollary involving Gould-Hopper-based bivariate Fubini polynomials holds.

Corollary 6. *For $n \geq 0$, the following equation holds:*

$${}_HF_n^{(j)}(x,y;z) = \sum_{k=0}^{n}\sum_{r=0}^{\left[\frac{k}{j}\right]}\frac{(n)_k}{r!(k-jr)!}{}_HF_{n-k}^{(j)}(x,y;z)z^r x^{k-jr}.$$

In the subsequent theorems, we give the symmetric identities for Gould-Hopper-based bivariate Fubini polynomials.

Theorem 15. *For integers a, b and $n \geq 0$, the following symmetric identity for Gould-Hopper-based bivariate Fubini polynomials holds:*

$$\sum_{r=0}^{n}\binom{n}{r}b^r a^{n-r}{}_HF_{n-r}^{(j)}(bx,y;b^j z){}_HF_r^{(j)}(ax,y;a^j z)$$
$$= \sum_{r=0}^{n}\binom{n}{r}a^r b^{n-r}{}_HF_{n-r}^{(j)}(ax,y;a^j z){}_HF_r^{(j)}(bx,y;b^j z). \tag{39}$$

Proof. Consider the function

$$A(t) = \frac{(e^{abxt+a^j b^j zt^j})^2}{(1-y(e^{at}-1))(1-y(e^{bt}-1))}$$
$$= \frac{e^{bx(at)+b^j z(at)^j}}{(1-y(e^{at}-1))}\cdot\frac{e^{ax(bt)+a^j z(bt)^j}}{(1-y(e^{bt}-1))}. \tag{40}$$

Then, by applying Definition 6 to equation (40) we have

$$A(t) = \sum_{n=0}^{\infty}{}_HF_n^{(j)}(bx,y;b^j z)\frac{(at)^n}{n!}\sum_{n=0}^{\infty}{}_HF_n^{(j)}(ax,y;a^j z)\frac{(bt)^n}{n!}$$
$$= \sum_{n=0}^{\infty}a^n{}_HF_n^{(j)}(bx,y;b^j z)\frac{t^n}{n!}\sum_{n=0}^{\infty}b^n{}_HF_n^{(j)}(ax,y;a^j z)\frac{t^n}{n!}.$$

Thus, applying the Theorem 3 to the equation above we get

$$A(t) = \sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}\binom{n}{r}b^r a^{n-r}{}_HF_{n-r}^{(j)}(bx,y;b^j z)\cdot{}_HF_r^{(j)}(ax,y;a^j z)\right)\frac{t^n}{n!}. \tag{41}$$

Note that, $A(t)$ can also be expressed as follows

$$A(t) = \frac{e^{ax(bt)+a^j z(bt)^j}}{(1-y(e^{bt}-1))} \cdot \frac{e^{bx(at)+b^j z(at)^j}}{(1-y(e^{at}-1))}. \tag{42}$$

Applying Definition 6 to equation (42), gives

$$A(t) = \sum_{n=0}^{\infty} b^n {}_H F_n^{(j)}(ax, y; a^j z) \frac{t^n}{n!} \sum_{n=0}^{\infty} a^n {}_H F_n^{(j)}(bx, y; b^j z) \frac{t^n}{n!}.$$

We then apply Theorem 3 to the equation above and obtain

$$\begin{aligned} A(t) &= \sum_{n=0}^{\infty} \left(\sum_{r=0}^{n} \binom{n}{r} b^{n-r} {}_H F_{n-r}^{(j)}(ax, y; a^j z) \cdot a^r {}_H F_r^{(j)}(bx, y; b^j z) \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r} {}_H F_{n-r}^{(j)}(ax, y; a^j z) \cdot {}_H F_r^{(j)}(bx, y; b^j z) \right) \frac{t^n}{n!}. \end{aligned} \tag{43}$$

Furthermore, equating equations (41) and (43) yields

$$\begin{aligned} &\sum_{n=0}^{\infty} \left(\sum_{r=0}^{n} \binom{n}{r} b^r a^{n-r} {}_H F_{n-r}^{(j)}(bx, y; b^j z) {}_H F_r^{(j)}(ax, y; a^j z) \right) \frac{t^n}{n!} \\ &\quad = \sum_{n=0}^{\infty} \left(\sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r} {}_H F_{n-r}^{(j)}(ax, y; a^j z) {}_H F_r^{(j)}(bx, y; b^j z) \right) \frac{t^n}{n!}. \end{aligned}$$

Finally, comparing the coefficients of $\frac{t^n}{n!}$ we get (39). □

Theorem 16. *For positive integers a, b and $n \geq 0$, the following symmetric identity for Gould-Hopper-based bivariate Fubini polynomials holds:*

$$\begin{aligned} &\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{a-1} \sum_{l=0}^{b-1} b^k a^{n-k} {}_H F_{n-k}^{(j)}(bx + \frac{b}{a} i + l, y; b^j z) F_k(au, y) \\ &\quad = \sum_{k=0}^{n} \binom{n}{k} \sum_{l=0}^{a-1} \sum_{i=0}^{b-1} a^k b^{n-k} {}_H F_{n-k}^{(j)}(ax + \frac{a}{b} i + l, y; a^j z) F_k(bu, y). \end{aligned} \tag{44}$$

Proof. Consider the following function:

$$\begin{aligned} B(t) &= \frac{e^{ab(x+u)t + a^j b^j z t^j} (e^{abt}-1)^2}{(1-y(e^{at}-1))(1-y(e^{bt}-1))(e^{at}-1)(e^{bt}-1)} \\ &= \frac{e^{abxt + a^j b^j z t^j}}{1-y(e^{at}-1)} \cdot \frac{(e^{bt})^a - 1}{e^{bt}-1} \cdot \frac{(e^{at})^b - 1}{e^{at}-1} \cdot \frac{e^{abut}}{1-y(e^{bt}-1)}. \end{aligned} \tag{45}$$

By using Example 2,

$$\sum_{i=0}^{a-1} e^{bti} = \frac{(e^{bt})^a - 1}{e^{bt} - 1}, \tag{46}$$

and

$$\sum_{l=0}^{b-1} e^{atl} = \frac{(e^{at})^b - 1}{e^{at} - 1}. \tag{47}$$

Substituting equations (46) and (47) to equation (45), we get

$$\begin{aligned} B(t) &= \frac{e^{bx(at)+b^j z(at)^j}}{1 - y(e^{at} - 1)} \cdot \sum_{i=0}^{a-1} e^{bti} \cdot \sum_{l=0}^{b-1} e^{atl} \cdot \frac{e^{au(bt)}}{1 - y(e^{bt} - 1)} \\ &= \frac{e^{bx(at)+b^j z(at)^j}}{1 - y(e^{at} - 1)} \cdot \sum_{i=0}^{a-1} \sum_{l=0}^{b-1} e^{(\frac{b}{a}i+l)at} \cdot \frac{e^{au(bt)}}{1 - y(e^{bt} - 1)} \\ &= \frac{\sum_{i=0}^{a-1} \sum_{l=0}^{b-1} e^{(bx+\frac{b}{a}i+l)at+b^j z(at)^j}}{1 - y(e^{at} - 1)} \cdot \frac{e^{au(bt)}}{1 - y(e^{bt} - 1)}. \end{aligned} \tag{48}$$

Then, by applying Definition 6 and Definition 4 to equation (48) we have

$$B(t) = \sum_{n=0}^{\infty} \sum_{i=0}^{a-1} \sum_{l=0}^{b-1} a^n {}_H F_n^{(j)}(bx + \frac{b}{a}i + l, y; b^j z) \frac{t^n}{n!} \cdot \sum_{n=0}^{\infty} b^n F_n(au, y) \frac{t^n}{n!}. \tag{49}$$

Thus, by applying Theorem 3 to equation (49) we get

$$\begin{aligned} B(t) &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{a-1} \sum_{l=0}^{b-1} a^{n-k} {}_H F_{n-k}^{(j)}(bx + \frac{b}{a}i + l, y; b^j z) \cdot b^k F_k(au, y) \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{a-1} \sum_{l=0}^{b-1} b^k a^{n-k} {}_H F_{n-k}^{(j)}(bx + \frac{b}{a}i + l, y; b^j z) F_k(au, y) \right) \frac{t^n}{n!}. \end{aligned} \tag{50}$$

Moreover, note that $B(t)$ can also be expressed as follows

$$\begin{aligned} B(t) &= \frac{e^{ab(x+u)t+a^j b^j z t^j}(e^{abt} - 1)^2}{(1 - y(e^{at} - 1))(1 - y(e^{bt} - 1))(e^{at} - 1)(e^{bt} - 1)} \\ &= \frac{e^{abxt+a^j b^j z t^j}}{1 - y(e^{bt} - 1)} \cdot \frac{(e^{at})^b - 1}{e^{at} - 1} \cdot \frac{(e^{bt})^a - 1}{e^{bt} - 1} \cdot \frac{e^{abut}}{1 - y(e^{at} - 1)}. \end{aligned} \tag{51}$$

Again, by using Example 2

$$\sum_{i=0}^{b-1} e^{ati} = \frac{(e^{at})^b - 1}{e^{at} - 1}, \tag{52}$$

and

$$\sum_{l=0}^{a-1} e^{btl} = \frac{(e^{bt})^a - 1}{e^{bt} - 1}. \tag{53}$$

Substituting equations (52) and (53) to equation (51), we get

$$\begin{aligned} B(t) &= \frac{e^{ax(bt)+a^j z(bt)^j}}{1 - y(e^{bt}-1)} \cdot \sum_{i=0}^{b-1} e^{ati} \cdot \sum_{l=0}^{a-1} e^{btl} \cdot \frac{e^{bu(at)}}{1-y(e^{at}-1)} \\ &= \frac{\sum_{i=0}^{b-1}\sum_{l=0}^{a-1} e^{(\frac{a}{b}i+l)bt} e^{ax(bt)+a^j z(bt)^j}}{1-y(e^{bt}-1)} \cdot \frac{e^{bu(at)}}{1-y(e^{at}-1)} \\ &= \frac{\sum_{i=0}^{b-1}\sum_{l=0}^{a-1} e^{(ax+\frac{a}{b}i+l)bt+a^j z(bt)^j}}{1-y(e^{bt}-1)} \cdot \frac{e^{bu(at)}}{1-y(e^{at}-1)}. \end{aligned} \tag{54}$$

Applying Definition 6 and Definition 4 to (54), gives

$$B(t) = \sum_{n=0}^{\infty}\sum_{l=0}^{a-1}\sum_{i=0}^{b-1} b^n {}_HF_n^{(j)}(ax + \frac{a}{b}i + l, y; a^j z)\frac{t^n}{n!} \cdot \sum_{n=0}^{\infty} a^n F_n(bu, y)\frac{t^n}{n!}. \tag{55}$$

We then apply the Theorem 3 to equation (55) and obtain

$$B(t) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\sum_{l=0}^{a-1}\sum_{i=0}^{b-1} a^k b^{n-k} {}_HF_{n-k}^{(j)}(ax + \frac{a}{b}i + l, y; a^j z)F_k(bu, y)\right)\frac{t^n}{n!}. \tag{56}$$

Furthermore, equating (50) and (56) yields

$$\begin{aligned} &\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\sum_{l=0}^{b-1} b^k a^{n-k} {}_HF_{n-k}^{(j)}(bx + \frac{b}{a}i + l, y; b^j z)F_k(au, y)\right)\frac{t^n}{n!} \\ &\quad = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\sum_{l=0}^{a-1}\sum_{i=0}^{b-1} a^k b^{n-k} {}_HF_{n-k}^{(j)}(ax + \frac{a}{b}i + l, y; a^j z)F_k(bu, y)\right)\frac{t^n}{n!}. \end{aligned} \tag{57}$$

Finally, comparing the coefficients of $\frac{t^n}{n!}$ we obtain (44). □

The following theorem is the recurrence relation for Gould-Hopper-based bivariate Fubini polynomials in the coming theorem.

Theorem 17. *The Gould-Hopper-based bivariate Fubini polynomials satisfy the following recurrence relation:*

$$\begin{aligned} {}_HF_{n+1}^{(j)}(x, y; z) = \sum_{m=0}^{n}\binom{n}{m} y {}_HF_n^{(j)}(x+1, y; z)F_{n-m}(y) + x {}_HF_n^{(j)}(x, y; z) \\ + jz(n)_{j+1} {}_HF_{n-j+1}^{(j)}(x, y; z). \end{aligned}$$

Proof. Note that

$$\frac{e^{xt+zt^j}}{1-y(e^t-1)} = \sum_{n=0}^{\infty} {}_HF_n^{(j)}(x,y;z)\frac{t^n}{n!}.$$

Differentiating both sides with respect to t, we have

$$\begin{aligned}\frac{d}{dt}\left[\frac{e^{xt+zt^j}}{1-y(e^t-1)}\right] &= \frac{d}{dt}\left[\sum_{n=0}^{\infty} {}_HF_n^{(j)}(x,y;z)\frac{t^n}{n!}\right]\\ &= \sum_{n=0}^{\infty}\frac{d}{dt}\left[{}_HF_n^{(j)}(x,y;z)\frac{t^n}{n!}\right]. \qquad (58)\end{aligned}$$

Now, for $n \geq 1$,

$$\frac{d}{dt}\left[{}_HF_n^{(j)}(x,y;z)\frac{t^n}{n!}\right] = n\,{}_HF_n^{(j)}(x,y;z)\frac{t^{n-1}}{n!}. \qquad (59)$$

By substituting equation (59) to equation (58), we get

$$\begin{aligned}\frac{d}{dt}\left[\frac{e^{xt+zt^j}}{1-y(e^t-1)}\right] &= \sum_{n=1}^{\infty} n\,{}_HF_n^{(j)}(x,y;z)\frac{t^{n-1}}{n!}\\ &= \sum_{n=0}^{\infty} {}_HF_{n+1}^{(j)}(x,y;z)\frac{t^n}{n!}. \qquad (60)\end{aligned}$$

Note that

$$\frac{e^{xt+zt^j}}{1-y(e^t-1)} = e^{xt+zt^j}(1-y(e^t-1))^{-1}.$$

Thus, using derivative of a product, we have

$$\begin{aligned}&\frac{d}{dt}\left[\frac{e^{xt+zt^j}}{1-y(e^t-1)}\right]\\ &= \frac{d}{dt}\left[e^{xt+zt^j}(1-y(e^t-1))^{-1}\right]\\ &= e^{xt+zt^j}\frac{d}{dt}\left[(1-y(e^t-1))^{-1}\right] + (1-y(e^t-1))^{-1}\left(e^{xt+zt^j}(x+jzt^{j-1})\right). \qquad (61)\end{aligned}$$

Observe that

$$\begin{aligned}\frac{d}{dt}\left[(1-y(e^t-1))^{-1}\right] &= -1(1-y(e^t-1))^{-2}(-ye^t)\\ &= (1-y(e^t-1))^{-1}(1-y(e^t-1))^{-1}(ye^t). \qquad (62)\end{aligned}$$

Thus, by substituting equation (62) to equation (61) gives

$$\frac{d}{dt}\left[\frac{e^{xt+zt^j}}{1-y(e^t-1)}\right] = e^{xt+zt^j}(1-y(e^t-1))^{-1}(1-y(e^t-1))^{-1}(ye^t)$$

$$
\begin{aligned}
&+ (1-y(e^t-1))^{-1}\left(e^{xt+zt^j}(x+jzt^{j-1})\right)\\
=&y\left[e^{(x+1)t+zt^j}(1-y(e^t-1))^{-1}\right](1-y(e^t-1))^{-1}\\
&+x\left[e^{xt+zt^j}(1-y(e^t-1))^{-1}\right]\\
&+jzt^{j-1}\left[e^{xt+zt^j}(1-y(e^t-1))^{-1}\right]. \qquad (63)
\end{aligned}
$$

Moreover, applying Definition 6 and Theorem 5 to the right-hand side of equation (63) we have

$$
\begin{aligned}
&\frac{d}{dt}\left[\frac{e^{xt+zt^j}}{1-y(e^t-1)}\right]\\
=&y\sum_{n=0}^{\infty} {}_HF_n^{(j)}(x+1,y;z)\frac{t^n}{n!}\sum_{n=0}^{\infty}F_n(y)\frac{t^n}{n!}+x\sum_{n=0}^{\infty} {}_HF_n^{(j)}(x,y;z)\frac{t^n}{n!}\\
&+jzt^{j-1}\sum_{n=0}^{\infty} {}_HF_n^{(j)}(x,y;z)\frac{t^n}{n!}.
\end{aligned}
$$

Hence, applying Theorem 3 to the right-hand side of the above equation gives

$$
\begin{aligned}
&\frac{d}{dt}\left[\frac{e^{xt+zt^j}}{1-y(e^t-1)}\right]\\
=&y\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m} {}_HF_n^{(j)}(x+1,y;z)F_{n-m}(y)\frac{t^n}{n!}+x\sum_{n=0}^{\infty} {}_HF_n^{(j)}(x,y;z)\frac{t^n}{n!}\\
&+jzt^{j-1}\sum_{n=0}^{\infty} {}_HF_n^{(j)}(x,y;z)\frac{t^n}{n!}. \qquad (64)
\end{aligned}
$$

But note that,

$$
\begin{aligned}
&jzt^{j-1}\sum_{n=0}^{\infty} {}_HF_n^{(j)}(x,y;z)\frac{t^n}{n!}\\
&\quad= jz\sum_{n=0}^{\infty} {}_HF_n^{(j)}(x,y;z)\frac{t^{n+j-1}}{n!}\\
&\quad= jz\sum_{n=0}^{\infty}(n+j-1)\dots(n+1) {}_HF_n^{(j)}(x,y;z)\frac{t^{n+j-1}}{(n+j-1)!}\\
&\quad= jz\sum_{n=0}^{\infty}(n)_{j+1} {}_HF_{n-j+1}^{(j)}(x,y;z)\frac{t^n}{n!}\\
&\quad= jz\sum_{n=0}^{\infty}(n)_{j+1} {}_HF_{n-j+1}^{(j)}(x,y;z)\frac{t^n}{n!}. \qquad (65)
\end{aligned}
$$

Substituting equation (65) to equation (64) yields

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{e^{xt+zt^j}}{1-y(e^t-1)}\right]\\
&=y\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}{}_HF_n^{(j)}(x+1,y;z)F_{n-m}(y)\frac{t^n}{n!}+x\sum_{n=0}^{\infty}{}_HF_n^{(j)}(x,y;z)\frac{t^n}{n!}\\
&\quad+jz(n)_{j+1}\sum_{n=0}^{\infty}{}_HF_{n-j+1}^{(j)}(x,y;z)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}y{}_HF_n^{(j)}(x+1,y;z)F_{n-m}(y)+x{}_HF_n^{(j)}(x,y;z)\\
&\quad+jz(n)_{j+1}{}_HF_{n-j+1}^{(j)}(x,y;z)\frac{t^n}{n!}.
\end{aligned} \tag{66}$$

We then equate equation (60) and equation (66) and obtain

$$\begin{aligned}
\sum_{n=0}^{\infty}{}_HF_{n+1}^{(j)}(x,y;z)\frac{t^n}{n!}=&\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}y{}_HF_n^{(j)}(x+1,y;z)F_{n-m}(y)\\
&+x{}_HF_n^{(j)}(x,y;z)+jz(n)_{j+1}{}_HF_{n-j+1}^{(j)}(x,y;z)\frac{t^n}{n!}.
\end{aligned}$$

Finally, comparing the coefficients of $\frac{t^n}{n!}$ yields to the desired result. □

In the next theorem, we introduced the partial derivative formulae for Gould-Hopper-based bivariate Fubini polynomials.

Theorem 18. *For $n \geq 0$, the following partial derivatives for the Gould-Hopper-based bivariate Fubini polynomials holds:*

$$\text{i.}\qquad \frac{\partial}{\partial x}{}_HF_{n+1}^{(j)}(x,y;z)=(n+1){}_HF_n^{(j)}(x,y;z), \tag{67}$$

$$\text{ii.}\qquad \frac{\partial}{\partial z}{}_HF_{n+j}^{(j)}(x,y;z)=(n+j)_j\,{}_HF_n^{(j)}(x,y;z). \tag{68}$$

Proof. Note that

$$\sum_{n=0}^{\infty}{}_HF_n^{(j)}(x,y;z)\frac{t^n}{n!}=e^{xt+zt^j}(1-y(e^t-1))^{-1}. \tag{69}$$

Differentiating both sides with respect to x, we have

$$\begin{aligned}
\frac{\partial}{\partial x}\left[\sum_{n=0}^{\infty}{}_HF_n^{(j)}(x,y;z)\frac{t^n}{n!}\right]&=\frac{\partial}{\partial x}\left[e^{xt+zt^j}(1-y(e^t-1))^{-1}\right]\\
&=te^{xt+zt^j}(1-y(e^t-1))^{-1}.
\end{aligned}$$

But note that,

$$\frac{\partial}{\partial x}\left[\sum_{n=0}^{\infty} {}_HF_n^{(j)}(x,y;z)\frac{t^n}{n!}\right] = \sum_{n=0}^{\infty}\frac{\partial}{\partial x}{}_HF_n^{(j)}(x,y;z)\frac{t^n}{n!}.$$

Thus,

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial x}{}_HF_n^{(j)}(x,y;z)\frac{t^n}{n!} = te^{xt+zt^j}(1-y(e^t-1))^{-1}. \tag{70}$$

By applying Definition 6 to the right-hand side of equation (70) we get

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial x}{}_HF_n^{(j)}(x,y;z)\frac{t^n}{n!} = \sum_{n=0}^{\infty} {}_HF_n^{(j)}(x,y;z)\frac{t^{n+1}}{n!}.$$

Differentiating both sides of the above equation with respect to t, we have

$$\sum_{n=1}^{\infty}\frac{\partial}{\partial x}{}_HF_n^{(j)}(x,y;z)\frac{t^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty}(n+1){}_HF_n^{(j)}(x,y;z)\frac{t^n}{n!}. \tag{71}$$

Reindexing the left-hand side of equation (71), we have

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial x}{}_HF_{n+1}^{(j)}(x,y;z)\frac{t^n}{n!} = \sum_{n=0}^{\infty}(n+1){}_HF_n^{(j)}(x,y;z)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ yields to (67).
Moreover, we will prove (68) analogously. That is, differentiating both side of equation (69) with respect to z gives us

$$\begin{aligned}\frac{\partial}{\partial z}\left[\sum_{n=0}^{\infty} {}_HF_n^{(j)}(x,y;z)\frac{t^n}{n!}\right] &= \frac{\partial}{\partial z}\left[e^{xt+zt^j}(1-y(e^t-1))^{-1}\right]\\ &= t^je^{xt+zt^j}(1-y(e^t-1))^{-1}.\end{aligned}$$

But note that,

$$\frac{\partial}{\partial z}\left[\sum_{n=0}^{\infty} {}_HF_n^{(j)}(x,y;z)\frac{t^n}{n!}\right] = \sum_{n=0}^{\infty}\frac{\partial}{\partial z}{}_HF_n^{(j)}(x,y;z)\frac{t^n}{n!}.$$

Hence,

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial z}{}_HF_n^{(j)}(x,y;z)\frac{t^n}{n!} = t^je^{xt+zt^j}(1-y(e^t-1))^{-1}. \tag{72}$$

Then, by applying Definition 6 to the right-hand side of equation (72) we get

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial z}{}_HF_n^{(j)}(x,y;z)\frac{t^n}{n!} = \sum_{n=0}^{\infty} {}_HF_n^{(j)}(x,y;z)\frac{t^{n+j}}{n!}.$$

Differentiating both sides of the above equation with respect to t, we have

$$\sum_{n=1}^{\infty} \frac{\partial}{\partial z} {}_H F_n^{(j)}(x,y;z) \frac{t^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} (n+j) {}_H F_n^{(j)}(x,y;z) \frac{t^{n+j-1}}{n!}. \tag{73}$$

Reindexing the left-hand side of equation (73), we have

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial z} {}_H F_{n+1}^{(j)}(x,y;z) \frac{t^{n}}{n!} = \sum_{n=0}^{\infty} (n+j) {}_H F_n^{(j)}(x,y;z) \frac{t^{n+j-1}}{n!}.$$

Again, differentiating both sides of the above equation with respect to t, we have

$$\sum_{n=1}^{\infty} \frac{\partial}{\partial z} {}_H F_{n+1}^{(j)}(x,y;z) \frac{t^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} (n+j)(n+j-1) {}_H F_n^{(j)}(x,y;z) \frac{t^{n+j-2}}{n!}. \tag{74}$$

Reindexing the left-hand side of equation (74), we have

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial z} {}_H F_{n+2}^{(j)}(x,y;z) \frac{t^{n}}{n!} = \sum_{n=0}^{\infty} (n+j)(n+j-1) {}_H F_n^{(j)}(x,y;z) \frac{t^{n+j-2}}{n!}.$$

Continue doing this, until we arrive with

$$\sum_{n=1}^{\infty} \frac{\partial}{\partial z} {}_H F_{n+j-1}^{(j)}(x,y;z) \frac{t^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} (n+j)(n+j-1)\dots(n+1) {}_H F_n^{(j)}(x,y;z) \frac{t^{n}}{n!}. \tag{75}$$

Reindexing the left-hand side of equation (75), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\partial}{\partial z} {}_H F_{n+j}^{(j)}(x,y;z) \frac{t^{n}}{n!} &= \sum_{n=0}^{\infty} (n+j)(n+j-1)\dots(n+1) {}_H F_n^{(j)}(x,y;z) \frac{t^{n}}{n!} \\ &= \sum_{n=0}^{\infty} (n+j)_j \, {}_H F_n^{(j)}(x,y;z) \frac{t^{n}}{n!}. \end{aligned}$$

Finally, comparing the coefficients of $\frac{t^n}{n!}$ yields to (68). □

The subsequent theorem will established the integral formulae for Gould-Hopper-based bivariate Fubini polynomials.

Theorem 19. *For $n \geq 0$, the following formulae for Gould-Hopper-based bivariate Fubini polynomials holds:*

$$i. \qquad \int {}_H F_n^{(j)}(x,y;z)dx = \frac{1}{(n+1)} {}_H F_{n+1}^{(j)}(x,y;z), \tag{76}$$

$$ii. \qquad \int {}_H F_n^{(j)}(x,y;z)dz = \frac{1}{(n+j)_j} {}_H F_{n+j}^{(j)}(x,y;z). \tag{77}$$

Proof. From (67), it follows that

$$\frac{\partial}{\partial x}\frac{1}{(n+1)}{}_HF^{(j)}_{n+1}(x,y;z) = {}_HF^{(j)}_n(x,y;z). \tag{78}$$

Integrating both sides of (78) with respect to x gives

$$\frac{1}{(n+1)}{}_HF^{(j)}_{n+1}(x,y;z) = \int {}_HF^{(j)}_n(x,y;z)dx.$$

Thus, we obtain (76).
Moreover, we will prove (77) analogously. By using (68) we obtain

$$\frac{\partial}{\partial z}\frac{1}{(n+j)_j}{}_HF^{(j)}_{n+j}(x,y;z) = {}_HF^{(j)}_n(x,y;z). \tag{79}$$

Integrating both sides of (79) with respect to z gives

$$\frac{1}{(n+j)_j}{}_HF^{(j)}_{n+j}(x,y;z) = \int {}_HF^{(j)}_n(x,y;z)dz.$$

□

The next theorem will derive the implicit formula for Gould-Hopper-based bivariate Fubini polynomials.

Theorem 20. *For $n \geq 0$, the following implicit summation formula for Gould-Hopper-based bivariate Fubini polynomials holds:*

$${}_HF^{(j)}_{q+l}(w,y;z) = \sum_{p,r=0}^{q,l}\binom{q}{p}\binom{l}{r}(w-x)^{p+r}\,{}_HF^{(j)}_{q+l-p-r}(x,y;z). \tag{80}$$

Proof. Replacing t by $t+u$ in Definition 6, we have

$$\sum_{n=0}^{\infty}{}_HF^{(j)}_n(x,y;z)\frac{(t+u)^n}{n!} = \frac{e^{x(t+u)+z(t+u)^j}}{1-y(e^{t+u}-1)}. \tag{81}$$

Applying Theorem 2 to the left-hand side of equation (81), we get

$$\sum_{n=0}^{\infty}{}_HF^{(j)}_n(x,y;z)\frac{(t+u)^n}{n!} = \sum_{q,l=0}^{\infty}{}_HF^{(j)}_{q+l}(x,y;z)\frac{t^q}{q!}\frac{u^l}{l!}. \tag{82}$$

Then, equating (81) and (82) gives

$$\sum_{q,l=0}^{\infty}{}_HF^{(j)}_{q+l}(x,y;z)\frac{t^q}{q!}\frac{u^l}{l!} = \frac{e^{x(t+u)+z(t+u)^j}}{1-y(e^{t+u}-1)}. \tag{83}$$

Multiplying both sides of equation (83) by $\frac{1}{e^{x(t+u)}}$, we have

$$e^{-x(t+u)} \sum_{q,l=0}^{\infty} {}_HF_{q+l}^{(j)}(x,y;z) \frac{t^q}{q!} \frac{u^l}{l!} = \frac{e^{z(t+u)^j}}{1-y(e^{t+u}-1)}. \tag{84}$$

Replacing x by w in equation (84), we get

$$e^{-w(t+u)} \sum_{q,l=0}^{\infty} {}_HF_{q+l}^{(j)}(w,y;z) \frac{t^q}{q!} \frac{u^l}{l!} = \frac{e^{z(t+u)^j}}{1-y(e^{t+u}-1)}. \tag{85}$$

Hence, equating (84) and (85) and multiplying both sides by $\frac{1}{e^{-w(t+u)}}$ we have

$$\begin{aligned}
e^{-x(t+u)} \sum_{q,l=0}^{\infty} {}_HF_{q+l}^{(j)}(x,y;z) \frac{t^q}{q!} \frac{u^l}{l!} &= e^{-w(t+u)} \sum_{q,l=0}^{\infty} {}_HF_{q+l}^{(j)}(w,y;z) \frac{t^q}{q!} \frac{u^l}{l!} \\
e^{-x(t+u)} e^{w(t+u)} \sum_{q,l=0}^{\infty} {}_HF_{q+l}^{(j)}(x,y;z) \frac{t^q}{q!} \frac{u^l}{l!} &= \sum_{q,l=0}^{\infty} {}_HF_{q+l}^{(j)}(w,y;z) \frac{t^q}{q!} \frac{u^l}{l!} \\
e^{-x(t+u)+w(t+u)} \sum_{q,l=0}^{\infty} {}_HF_{q+l}^{(j)}(x,y;z) \frac{t^q}{q!} \frac{u^l}{l!} &= \sum_{q,l=0}^{\infty} {}_HF_{q+l}^{(j)}(w,y;z) \frac{t^q}{q!} \frac{u^l}{l!} \\
e^{(w-x)(t+u)} \sum_{q,l=0}^{\infty} {}_HF_{q+l}^{(j)}(x,y;z) \frac{t^q}{q!} \frac{u^l}{l!} &= \sum_{q,l=0}^{\infty} {}_HF_{q+l}^{(j)}(w,y;z) \frac{t^q}{q!} \frac{u^l}{l!}. \qquad (86)
\end{aligned}$$

Note that, expressing $e^{(w-x)(t+u)}$ in its exponential form and applying Theorem 2 gives

$$\begin{aligned}
e^{(w-x)(t+u)} &= \sum_{n=0}^{\infty} (w-x)^n \frac{(t+u)^n}{n!} \\
&= \sum_{q,l=0}^{\infty} (w-x)^{q+l} \frac{t^q}{q!} \frac{u^l}{l!}. \qquad (87)
\end{aligned}$$

Now, substituting (87) to (86) we find

$$\sum_{q,l=0}^{\infty} (w-x)^{q+l} \frac{t^q}{q!} \frac{u^l}{l!} \sum_{q,l=0}^{\infty} {}_HF_{q+l}^{(j)}(x,y;z) \frac{t^q}{q!} \frac{u^l}{l!} = \sum_{q,l=0}^{\infty} {}_HF_{q+l}^{(j)}(w,y;z) \frac{t^q}{q!} \frac{u^l}{l!}. \tag{88}$$

Thus, applying Theorem 3 to the left-hand side of equation (88) we get

$$\sum_{q,l=0}^{\infty} \left(\sum_{p,r=0}^{q,l} \binom{q}{p} \binom{l}{r} (w-x)^{p+r} {}_HF_{q+l-p-r}^{(j)}(x,y;z) \right) \frac{t^q}{q!} \frac{u^l}{l!}$$

$$= \sum_{q,l=0}^{\infty} {}_H F_{q+l}^{(j)}(w,y;z) \frac{t^q}{q!} \frac{u^l}{l!}.$$

Comparing the coefficients of $\frac{t^q}{q!} \frac{u^l}{l!}$ in the above equation we get (80). □

Setting $l = 0$ in Theorem 20, it will deduce to the following corollary.

Corollary 7. *For $n \geq 0$, the following formula for Gould-Hopper-based bivariate Fubini polynomials holds:*

$${}_H F_q^{(j)}(w,y;z) = \sum_{p=0}^{q} \binom{q}{p} (w-x)^p {}_H F_{q-p}^{(j)}(x,y;z).$$

In the following theorem, a relationship of Gould-Hopper-based bivariate Fubini polynomials with Gould-Hopper polynomials and Stirling number of the second kind will be introduced.

Theorem 21. *For $n \geq 0$, the following relationship holds:*

$${}_H F_n^{(j)}(x,y;z) = \sum_{m=0}^{n} \binom{n}{m} H_{n-m}^{(j)}(x,z) \sum_{k=0}^{m} k! y^k S(m,k). \tag{89}$$

Proof. By Definition 6, we have

$$\sum_{n=0}^{\infty} {}_H F_n^{(j)}(x,y;z) \frac{t^n}{n!} = e^{xt+zt^j} \frac{1}{1-y(e^t-1)}. \tag{90}$$

Then, applying Definition 5 and Theorem 5 to the right-hand side of equation (90) gives us

$$\sum_{n=0}^{\infty} {}_H F_n^{(j)}(x,y;z) \frac{t^n}{n!} = \sum_{n=0}^{\infty} H_n^{(j)}(x,z) \frac{t^n}{n!} \sum_{m=0}^{\infty} F_m(y) \frac{t^m}{m!}. \tag{91}$$

Hence, applying Definition 3 to the right-hand side of equation (91) we have

$$\begin{aligned} \sum_{n=0}^{\infty} {}_H F_n^{(j)}(x,y;z) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} H_n^{(j)}(x,z) \frac{t^n}{n!} \sum_{m=0}^{\infty} \sum_{k=0}^{m} S(m,k) k! y^k \frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty} H_n^{(j)}(x,z) \frac{t^n}{n!} \sum_{m=0}^{\infty} \sum_{k=0}^{m} k! y^k S(m,k) \frac{t^m}{m!}. \end{aligned} \tag{92}$$

Thus, applying Theorem 3 to the right-hand side of equation (92), we obtain

$$\sum_{n=0}^{\infty} {}_H F_n^{(j)}(x,y;z) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} H_{n-m}^{(j)}(x,z) \sum_{k=0}^{m} k! y^k S(m,k) \right) \frac{t^n}{n!}. \tag{93}$$

Comparing the coefficients of $\frac{t^n}{n!}$ yields (89). □

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