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On The Distinctive Bi-generations That are Arising Three Frameworks for Maximal and Minimal Bitopologies spaces, Their Relationship to Bitopological Spaces, and Their Respective Applications

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Abstract. Due to the widespread use of various mathematical concepts, operations, relations, findings, numerous writers have established these concepts in minimal spaces. Determining how to create pairwise minimal spaces by utilizing a variety of set operators is what this article is about. Specific kinds of minimal spaces and their classical bitopologies interact with one another to form symmetry. Through the study of sets, we can investigate the characteristics and behaviors of traditional bitopological ideas. Closed spaces are a new class of bitopological spaces that we characterize and assess in this study. We also establish links among this novel category of minimal spaces and other classes of generalized spaces. Furthermore, we introduce and analyze the closed spaces that were originally suggested here, illustrate this innovative idea, make clear the interactions that go along with it, pinpoint the requirements for its successful application, and offer applications and counter-examples. Additional explanations are provided for the pairwise minimal Hausdorff Spaces, pairwise minimal Lindelöf closed spaces, pairwise minimal compact closed spaces, and maximal and minimal bitopologies. With of revenue of these spaces, we look at inverse images having particular bitopological characteristics. The discussion concludes with the identification of related product conclusions for these ideas.

Key Words and Phrases: Maximal and Minimal Bitopologies, Pairwise Minimal Compact Closed Spaces, Pairwise Minimal Lindelöf Closed Spaces, Pairwise Minimal Hausdorff Spaces.

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1. Introduction

Recognizing the significance of topological space in data analysis and some applications, numerous studies have employed a variety of techniques to increase that space, including the idea of minimal spaces. Many research have used a range of methods to expand topological space, including the concept of minimum spaces, because of its importance in data processing and some applications. Numerous authors have established these notions in minimal spaces due to the widespread use of various operations, relations, results, and other aspects in mathematics and related subjects. This article focuses on figuring out how to construct pairwise minimum spaces using various set operators. Symmetry is the result of interactions between certain types of minimum spaces and their classical topologies. A. S. Parhomenko [1] established that compact Hausdorff spaces are minimal Hausdorff in 1939, which is when the idea of minimal topologies was first proposed. Compact Hausdorff spaces are maximally compact in addition to being minimal Hausdorff, as E. Hewitt [2] demonstrated four years later. If non-Hausdorff maximal compact spaces or non-Hausdorff minimal compact spaces exist, R. Vaidyanathaswamy [3] questioned this in 1947.

The existence of these minimal Hausdorff spaces was demonstrated and all minimal Hausdorff spaces were described by A. Ramanathan [4], [5] in the same year. The other portion of Vaidyanathaswamy's query was addressed by Hing Tong [6] in 1948 when he created an illustration of a maximal compact space that wasn't Hausdorff. A. Ramanathan [7] established the maximal compactness of a topological space in 1948. when and only when its compact subsets exactly match the closed sets. In 1963, N. Smythe and C. A. Wilkins [8] created an example of a maximal compact space without isolated points that are strictly weaker than a minimal Hausdorff topology, which was the first significant work on maximal characteristics. Every compact set must be closed for a topological space to be referred to as a compact closed space. Because every compact closed space is a T_1-space and every compact closed space is a T_2-space, the compact closed property can be viewed as a separation axiom between T_1 and T_2. E. Hewitt [2] demonstrated in 1943 that a compact T_2 space is both minimally and maximally compact; for related work, see [7], [8].

If a space is minimally compact closed, is it maximally compact? questioned R. Larson K,[9]. Is there a minimal compact closed topology in every compact closed topology? This is a similar query. This isn't always the case, as W. Fleissner demonstrated. He created a compact closed topology in [10] that is not a minimal compact closed topology. Every Hausdorff compact space is maximally compact and minimally compact, as is well known. However, the author in [8] has demonstrated that there are minimal Hausdorff spaces that are neither maximally compact nor minimally compact, and there are Hausdorff spaces that are neither minimally compact nor maximally compact. Although the notion of a compact closed space is not mentioned, it was demonstrated in the same work that maximal compact spaces are compact closed. Any T_2-space is closed compact; nonetheless, if a space is closed compact, it means that its singletons are closed, which indicates that the space is T_1. This viewpoint suggests that the closed compact quality might be thought of as a sort of separation axiom between T_1 and T_2.

In [9], questioned the assumption that all closed compact spaces that do not accept any strictly coarser closed compact topology must be compact. Such a question, which is also taken into account in [11], naturally fits into inquiries into topologies that are (or are not) minimal or maximal among those enjoying a particular attribute. The growth of General Topology has been greatly influenced by the traditional generalizations of Lindelöf spaces, such as hereditarily and maximally Lindelöf spaces. The class of Lindelöf closed spaces is one particular class of spaces that is relatively new as a notion but has been thoroughly researched in recent years.
In [12] and [13], a topological space is referred to as a Lindelöf closed space if all of its Lindelöf subsets are closed. This idea, which has a tight connection to P-spaces, came forth as a result of research on maximal Lindelöf spaces [14]. Bitopological spaces were first discussed and introduced in [14]. Numerous mathematicians investigated a variety of ideas in bitopological spaces, which has now developed into a significant area of study in general topology. There have been a few generalized topological structures put forth recently. Topological space is crucial for analysis and a wide range of applications; for further information, see One of the key generalizations of the topological space represented by the compact closed and Lindelöf closed spaces.
In this article, we explore the idea of bitopological spaces, maximal and minimal bitopologies, pairwise minimal compact closed spaces, pairwise minimal lindelöf closed spaces, pairwise minimal Hausdorff Spaces, as well as their connections to other bitopological ideas. $O(Z)$ stands for the set of all topologies on Z that have the property O, where O is a topological property, Z is a nonempty set, and O is the property. By including sets, $O(Z)$ is partially sorted. The patial ordering $\leq$ means that such that $(Z, \vartheta_1^{\setminus}, \vartheta_2^{\setminus}) \leq (Z, \vartheta_1, \vartheta_2)$ iff $\vartheta_1^{\setminus} \leq \vartheta_1$ and $\vartheta_2^{\setminus} \leq \vartheta_2$. The paper is organized as follows:Section 2 provides fundamental definitions and theorems for bitopological spaces, including crucial terminologies relevant to our research. Building on these foundations, Section 3 introduces new generations of pairwise compact closed spaces and explores their relationships to other types of spaces. Section 4 expands on this approach by introducing additional properties and novel definitions for pairwise compact closed spaces. In Section 5, we additionally discuss the topological characterizations of pairwise minimal compact closed spaces, supported by a diagram illustrating their interconnections. Section 6 focuses on pairwise Lindel of closed spaces, demonstrating advanced characteristics and oddities in cartesian multiplication under specified conditions. Section 7 expands on this analysis by defining paired minimal Lindelö of closed spaces, together with a diagrammatic description of their structural relationships. Section 8 introduces a novel notion of pairwise minimal Hausdorff spaces and analyzes their properties. Finally, Section 9 finishes the paper by describing applications of minimal spaces in bitopological settings and highlighting intriguing prospects for future research.

2. Preliminaries and Basic Definitions

We provide the fundamental definitions and theorems that we will use to support our primary findings in the following sections. Throughout this publication, we will refer to bitopological spaces as "spaces" to set the scene for our inquiry. We'll start by going through the key terminologies and findings that will be used to this project as a whole.

Definition 2.1. *[10] In $(Z, \vartheta_1, \vartheta_2)$, $Z \subset A$ is bicompact if and only if A is both ϑ_1-compact and ϑ_2-compact.*

Definition 2.2. *[10] A cover $\hat{B}$ of $(Z, \vartheta_1, \vartheta_2)$ is called pairwise open if $\hat{B} \subset \vartheta_1 \cup \vartheta_2$, $\hat{B} \cap \vartheta_i \subset \{A \neq \phi\}$. If every pairwise open cover of $(Z, \vartheta_1, \vartheta_2)$ has a finite subcover, then the space is called pairwise compact.*

Definition 2.3. *[15] A space $(Z, \vartheta_1, \vartheta_2)$ is called pairwise T_1 if for each two distinct points z and n, there are a ϑ_1-open set D and a ϑ_2-open set F such that $z \in D$, $n \notin F$, and $n \in F$, $z \notin D$.*

Definition 2.4. *[15] A space $(Z, \vartheta_1, \vartheta_2)$ is called pairwise Hausdroff (pairwise T_2) if for each two distinct points z and n, there are a ϑ_1-open set Q and a ϑ_2-open set W such that $z \in D$, $n \in F$, and $D \cap F = \phi$.*

Definition 2.5. *[10] A function $\Phi : (Z, \vartheta_1, \vartheta_2) \to (N, \beta_1, \beta_2)$ is called pairwise continuous, if $\Phi_1 : (Z, \vartheta_1) \to (N, \beta_1)$ and $\Phi_2 : (Z, \vartheta_2) \to (N, \beta_2)$ are continuous functions.*

Definition 2.6. *[10]: A function $\Phi : (Z, \vartheta_1, \vartheta_2) \to (N, \beta_1, \beta_2)$ is called pairwise closed, if $\Phi_1 : (Z, \vartheta_1) \to (N, \beta_1)$ and $\Phi_2 : (Z, \vartheta_2) \to (N, \beta_2)$ are closed functions.*

As a result, if $A_1$is closed in ϑ_1, then $\Phi_1(A_1)$ is closed in β_1, and if A_2 is closed in ϑ_2, then $\Phi_2(A_2)$ is closed in β_2.

Definition 2.7. *[15] A bitopological space $(Z, \vartheta_1, \vartheta_2)$ is said to be pairwise locally compact if each $z \in Z$, there exist ϑ_1-open nieghbourhood of Z, whose ϑ_1-closure is pairwise compact or a ϑ_2-open nieghbourhood of Z, whose ϑ_2-closure is pairwise compact.*

Definition 2.8. *[16] A function $\Phi : (Z, \vartheta_1, \vartheta_2) \to (N, \beta_1, \beta_2)$ is called pairwise homomorphism, iff $\Phi_1 : (Z, \vartheta_1) \to (N, \beta_1)$ and $\Phi_2 : (Z, \vartheta_2) \to (N, \beta_2)$ are homomorphism.*

Definition 2.9. *[16] A space (Z, ϑ) is called a P-space if every countable intersection of open sets in ϑ is itself an open set.*

3. New Generations Of Pairwise Compact Closed Spaces

We describe pairwise compact closed spaces in bitopological spaces in this section and demonstrate how they relate to other spaces.

Definition 3.1. *A bitopological space $(Z, \vartheta_1, \vartheta_2)$ is called a pairwise compact closed space if: Every ϑ_1-compact subset of Z is ϑ_2-closed, Every ϑ_2-compact subset of Z is ϑ_1-closed.*

Because each singleton is compact, it is simple to demonstrate that every pairwise Hausdorff space is also a pairwise compact closed space and every pairwise compact closed space is also a pairwise T_1-space. The instances that follow demonstrate that the opposite need not be true.

Example 3.1. *Let ϑ_{cc} be cocountable topology, then $(R, \vartheta_{cc}, \vartheta_{cc})$ is pairwise compact closed space but not pairwise Hausdorff space.*

Example 3.2. *Let ϑ_{cof} be cofinite topology, then $(R, \vartheta_{cof}, \vartheta_{cof})$ is pairwise T_1-spaces, where is not pairwise compact closed space.*

Proposition 3.1. *Let $(Z, \vartheta_1, \vartheta_2)$ be a pairwise locally compact space. If $(Z, \vartheta_1, \vartheta_2)$ is a pairwise compact closed space, then it is a pairwise T_3-space.*

Proof. Consider this $(Z, \vartheta_1, \vartheta_2)$ is pairwise compact closed space. On account of $(Z, \vartheta_1, \vartheta_2)$ is a pairwise locally compact, there exist ϑ_1-open nieghbourhood of $(Z, \vartheta_1, \vartheta_2)$, whose ϑ_1-closure is pairwise compact. Consequently, the set of pairwise compact of nieghbourhood $z \in (Z, \vartheta_1, \vartheta_2)$ shall be a local base of $z \in (Z, \vartheta_1, \vartheta_2)$.Because $(Z, \vartheta_1, \vartheta_2)$ is pairwise compact closed space, the like set is ϑ_2-closed nieghbourhood of $z \in (Z, \vartheta_1, \vartheta_2)$ and retain a local foundation of $z \in (Z, \vartheta_1, \vartheta_2)$. Subsequently $(Z, \vartheta_1, \vartheta_2)$ is pairwise regular and pairwise T_1-space, then $(Z, \vartheta_1, \vartheta_2)$ is pairwise T_3-space, which is T_2-space.

The continuous image of a pairwise compact closed space is not necessarily pairwise compact closed, as illustrated in the following example.

Example 3.3. *Suppose $\Phi : (R, \vartheta_u, \vartheta_u) \to (R, \beta_{ind}, \beta_{ind})$. It is obvious that $(R, \vartheta_u, \vartheta_u)$ is pairwise compact closed space, Nevertheless, $(R, \beta_{ind}, \beta_{ind})$ is not pairwise compact closed space, But whatever $\beta_{ind}-$compact subset is not $\beta_{ind}-$closed.*

Theorem 3.1. *Let $\Phi : (Z, \vartheta_1, \vartheta_2) \overrightarrow{injection} (N, \beta_1, \beta_2)$ be a pairwise continuous function. If (N, β_1, β_2) is a pairwise compact closed space, then $(Z, \vartheta_1, \vartheta_2)$ inherits the pairwise compact closed property.*

Proof. Make U any ϑ_1-compact subset of $(Z, \vartheta_1, \vartheta_2)$, then $\Phi(U)$ is β_1-compact subset in (N, β_1, β_2). Due to the fact that (N, β_1, β_2) is pairwise compact closed space, $\Phi(U)$ is β_2-closed subset of (N, β_1, β_2), and Φ is injection, Consequently $\Phi^{-1}(\Phi(U)) = U$ is ϑ_2-closed subset of $(Z, \vartheta_1, \vartheta_2)$. Comparably, for V is ϑ_2-compact subset of $(Z, \vartheta_1, \vartheta_2)$. It follows that, $(Z, \vartheta_1, \vartheta_2)$ is pairwise compact closed space.

Theorem 3.2. *The property of being a pairwise compact closed space is a bitopological invariant; that is, it is preserved under pairwise homeomorphisms.*

Proof. Assuming that $(Z, \vartheta_1, \vartheta_2)$ is a pairwise compact closed space and $\Phi : (Z, \vartheta_1, \vartheta_2) \to (N, \beta_1, \beta_2)$ be a pairwise homeomorphism and U be any β_1-compact subset of (N, β_1, β_2), then $\Phi^{-1}(U)$ is ϑ_1-compact in $(Z, \vartheta_1, \vartheta_2)$, but $(Z, \vartheta_1, \vartheta_2)$ is pairwise compact closed space, so $\Phi^{-1}(U)$ is ϑ_2-closed in $(Z, \vartheta_1, \vartheta_2)$, then $\Phi^{-1}(\Phi(U)) = U$ is β_2-closed in (N, β_1, β_2). The β_2-compact subset of (N, β_1, β_2) is analogous for V. Thus, pairwise compact closed space (N, β_1, β_2) is defined.

Theorem 3.3. *Being pairwise compact in a closed space is a heritable trait.*

Proof. Assume that $(Z, \vartheta_{1_z}, \vartheta_{2_z})$ is pairwise compact closed space and $(N, \vartheta_{1_n}, \vartheta_{2_n})$ is subspace of $(Z, \vartheta_{1_z}, \vartheta_{2_z})$. Let's assume that U is $\vartheta_{1_n}-$compact subset in $(N, \vartheta_{1_n}, \vartheta_{2_n})$.Due to the fact that $(N, \vartheta_{1_n}, \vartheta_{2_n}) \subseteq (Z, \vartheta_{1_z}, \vartheta_{2_z}), U$ is $\vartheta_{1_z}-$compact subset in $(Z, \vartheta_{1_z}, \vartheta_{2_z})$, but $(Z, \vartheta_{1_z}, \vartheta_{2_z})$ is pairwise compact closed space, so U is $\vartheta_{1_z}-$closed subset in $(Z, \vartheta_{1_z}, \vartheta_{2_z})$, thus U is $\vartheta_{1_n}-$closed subset in $(N, \vartheta_{1_n}, \vartheta_{2_n})$, because $U \cap N = U$. Equivalently for V is $\vartheta_{2_n}-$compact subset in $(N, \vartheta_{1_n}, \vartheta_{2_n})$. As a result $(N, \vartheta_{1_n}, \vartheta_{2_n})$ is pairwise compact closed space.

Theorem 3.4. *Whenever $(Z, \vartheta_1, \vartheta_2)$ be pairwise compact compact closed space and $(N, \vartheta_1, \vartheta_2) \subset (Z, \vartheta_1, \vartheta_2)$, then $(N, \vartheta_1, \vartheta_2)$ is pairwise compact when and only when $(N, \vartheta_1, \vartheta_2)$ is pairwise closed in $(Z, \vartheta_1, \vartheta_2)$.*

Proof. Consider the idea that $(N, \vartheta_1, \vartheta_2)$ is pairwise compact. The pairwise closed space $(N, \vartheta_1, \vartheta_2)$ follows from the fact that $(Z, \vartheta_1, \vartheta_2)$ is a pairwise compact closed space. In the opposite scenario, if $(N, \vartheta_1, \vartheta_2)$ is pairwise closed in $(Z, \vartheta_1, \vartheta_2)$, then $(N, \vartheta_1, \vartheta_2)$ is pairwise compact since $(Z, \vartheta_1, \vartheta_2)$ is pairwise compact.

4. Further properties of Pairwise Compact Closed Spaces

In this section, we provide new definitions as well as additional pairwise compact closed space qualities.

Definition 4.1. *A function $\Phi : (Z, \vartheta_1, \vartheta_2) \to (N, \beta_1, \beta_2)$ is called a pairwise compact-preserving function if the inverse image of every pairwise compact subset of (N, β_1, β_2) is pairwise compact in $(Z, \vartheta_1, \vartheta_2)$.*

Definition 4.2. *A bitopological compact space $(Z, \vartheta_1, \vartheta_2)$ is allegedly pairwise maximal compact topology if $(Z, \vartheta_1, \vartheta_2) \leq (Z, \vartheta_1^{/}, \vartheta_2^{/})$ like that $\vartheta_1 \leq \vartheta_1^{/}$ and $\vartheta_2 \leq \vartheta_2^{/}$ indicates $(Z, \vartheta_1^{/}, \vartheta_2^{/})$ is not pairwise compact.*

Definition 4.3. *A function $\Phi : (Z, \vartheta_1, \vartheta_2) \to (N, \beta_1, \beta_2)$ is allegedly pairwise $K-$function whenever the inverse of any pairwise compact subset of (N, β_1, β_2) is pairwise compact in $(Z, \vartheta_1, \vartheta_2)$ and any pairwise compact subset's image $(Z, \vartheta_1, \vartheta_2)$ is pairwise compact subset in (N, β_1, β_2).*

Theorem 4.1. *Allow $\Phi : (Z, \vartheta_1, \vartheta_2) \xrightarrow{\text{onto closed}} (N, \beta_1, \beta_2)$ be pairwise $K-$function. When $(Z, \vartheta_1, \vartheta_2)$ is pairwise compact closed space, then (N, β_1, β_2) follows.*

Proof. Suppose U be β_1-compact set in (N, β_1, β_2). To demonstrate that U is β_2-closed in (N, β_1, β_2).While Φ is pairwise $K-$function, Consequently $\Phi^{-1}(z)$ is ϑ_1-compact in $(Z, \vartheta_1, \vartheta_2)$, whereas $(Z, \vartheta_1, \vartheta_2)$ is pairwise compact closed space, $\Phi^{-1}(z)$ is ϑ_2-closed in $(Z, \vartheta_1, \vartheta_2)$ otherwise. Due to the fact that Φ is pairwise closed and onto, $\Phi(\Phi^{-1}(U) = U$ is β_2 $-$closed in (N, β_1, β_2). A β_2-compact set in (N, β_1, β_2) is analogous for V. (N, β_1, β_2) is a pairwise compact closed space as a result.

Theorem 4.2. *Allow $\Phi : (Z, \vartheta_1, \vartheta_2) \to (N, \beta_1, \beta_2)$ be pairwise continuous function. When (N, β_1, β_2) is pairwise compact closed space and $(Z, \vartheta_1, \vartheta_2)$ is pairwise compact, subsequently, Φ is pairwise closed function.*

Proof. If U be ϑ_1-closed set in $(Z, \vartheta_1, \vartheta_2)$, yet $(Z, \vartheta_1, \vartheta_2)$ is pairwise compact, then U is ϑ_1-compact in $(Z, \vartheta_1, \vartheta_2)$. Considering that Φ is pairwise continuous function, then $\Phi(U)$ is β_1-compact in (N, β_1, β_2). Due to the fact that (N, β_1, β_2) is pairwise compact closed space, so $\Phi(U)$ is β_1-closed in (N, β_1, β_2). Comparable to$V$ is β_2-closed set in (N, β_1, β_2). As a result, Φ is pairwise closed function.

5. Several Pairwise Minimal Compact Closed Space Theorems

More information about the pairwise minimal compact closed spaces' topological characteristics is included in this section, along with a diagram illustrating how these spaces are connected in general.

Definition 5.1. *It is argued that a bitopological space $(Z, \vartheta_1, \vartheta_2)$ is pairwise minimal compact closed space, $(Z, \vartheta_1, \vartheta_2) \leq (Z, \vartheta_1^{/}, \vartheta_2^{/})$ like that $\vartheta_1^{/} \leq \vartheta_1$ and $\vartheta_2^{/} \leq \vartheta_2$ indicates $(Z, \vartheta_1^{/}, \vartheta_2^{/})$ is not pairwise compact closed compact.*

Theorem 5.1. *A pairwise compact closed spaces is one that is a pairwise minimal compact closed space.*

Proof. Assume that $(Z, \vartheta_1, \vartheta_2)$ is pairwise compact closed spaces and not pairwise minimal compact closed space, then there is $\vartheta_1^{/} \leq \vartheta_1$ and $\vartheta_2^{/} \leq \vartheta_2$ and $(Z, \vartheta_1^{/}, \vartheta_2^{/})$ is pairwise compact closed space. Now that we've established that $\theta_Z : (Z, \vartheta_1, \vartheta_2) \to (Z, \vartheta_1^{/}, \vartheta_2^{/})$ be an identity function, then θ_Z is pairwise continuous and pairwise closed and so pairwise homomorphism. Therefor $\vartheta_1^{/} = \vartheta_1, \ \vartheta_2 = \vartheta_2^{/}$, we obtain contradiction. As a result, $(Z, \vartheta_1, \vartheta_2)$ is pairwise minimal compact closed space.

Example 5.1. *Assuming $Z \neq \phi$ be any finite set and ϑ_{dis} is discrete topology, correspondingly, $(Z, \vartheta_{dis}, \vartheta_{dis})$ is pairwise minimal compact closed space.*

The example that follows demonstrates that pairwise minimal compact closed space is not always represented by its continuous image.

Example 5.2. *Let $\vartheta_{dis}, \ \vartheta_{ind}$ is discrete topology and indiscrete topology respectivly. Consider the following: $\theta_Z : (Z, \vartheta_{dis}, \vartheta_{dis}) \to (Z, \vartheta_{ind}, \vartheta_{ind})$ be an identity function. It is demonstrated Example 3.3 shows that $(Z, \vartheta_{dis}, \vartheta_{dis})$ is pairwise minimal compact closed space, and demonstrated Example 5.1 shows that $(Z, \vartheta_{ind}, \vartheta_{ind})$ is not pairwise minimal compact closed space.*

Theorem 5.2. *Being pairwise minimal compact closed space is a bitopological property.*

Proof. Assume that $(Z, \vartheta_1, \vartheta_2)$ is a pairwise minimal compact closed space and $\Phi : (Z, \vartheta_1, \vartheta_2) \to (N, \beta_1, \beta_2)$ be a pairwise homeomorphism and (N, β_1, β_2) is not pairwise minimal compact closed space, there exists$\beta_1^{/} \leq \beta_1,\ \beta_2^{/} \leq \beta_2$, such that $(N, \beta_1^{/}, \beta_2^{/})$ is pairwise compact closed space. Define $\vartheta_1^{/} = \left\{\Phi^{-1}(B) : B \in \beta_1^{/}\right\}$ is a topology in Z such that $\vartheta_1^{/} \leq \vartheta_1$ and $\tau_2^{/} = \left\{\Phi^{-1}(A) : A \in \beta_2^{/}\right\}$ is a topology in Z such that $\vartheta_2^{/} \leq \vartheta_2$ and so $(Z, \vartheta_1^{/}, \vartheta_2^{/})$ is pairwise compact closed spacewhich is in opposition to $(Z, \vartheta_1, \vartheta_2)$ is pairwise minimal compact closed space. Hence (N, β_1, β_2) is pairwise minimal compact closed space.

Theorem 5.3. *In the event that $(Z_1 \times Z_2,\ \vartheta_1 \times \vartheta_1,\ \vartheta_2\ \times \vartheta_2)$ is pairwise compact closed space. Subsequently, each $(Z_1, \vartheta_1, \vartheta_2),\ (Z_2, \vartheta_1, \vartheta_2)$ is pairwise minimal compact closed space.*

Proof. Because $(Z_1 \times Z_2,\ \vartheta_1 \times \vartheta_1,\ \vartheta_2\ \times \vartheta_2)$ is pairwise compact, then each $(Z_1, \vartheta_1, \vartheta_2),\ (Z_2, \vartheta_1, \vartheta_2)$ is pairwise compact too. In the event that $\{z_2\}$ be a fixed element in $(Z_2, \vartheta_1, \vartheta_2)$, then $(Z_1, \vartheta_1, \vartheta_2) \times \{z_2\}$ is a subspace of $(Z_1 \times Z_2,\ \vartheta_1 \times \vartheta_1,\ \vartheta_2\ \times \vartheta_2)$. In light of this $(Z_1, \vartheta_1, \vartheta_2) \times \{z_2\}$ is pairwise compact compact closed space. But $(Z_1, \vartheta_1, \vartheta_2)$ is pairwise homomorphic to $(Z_1, \vartheta_1, \vartheta_2) \times \{z_2\}$. It follows from this that $(Z_1, \vartheta_1, \vartheta_2)$ is pairwise compact compact closed space. By theorem 5.1, $(Z_1, \vartheta_1, \vartheta_2)$ is pairwise minimal compact closed space. Also, we can demonstrate that $(Z_2, \vartheta_1, \vartheta_2)$ is pairwise minimal compact closed space.

Corollary generalizations of the theorem 5.3 findings are as follows.

Corollary 5.1. *If $Z = \prod_{\rho \in \Lambda} Z_\rho$ is a pairwise compact closed space, then each Z_α is pairwise minimal compact closed space, for each $\rho \in \Lambda$.*

Theorem 5.4. *Suppose that $\Phi : (Z, \vartheta_1, \vartheta_2)\overrightarrow{onto\ closed}\ (N, \beta_1, \beta_2)$ be pairwise continuous function. Whenever $(Z, \vartheta_1, \vartheta_2)$ is pairwise minimal compact closed space, then (N, β_1, β_2) is true.*

Proof. Theorem 4.1 is sufficient to establish that (N, β_1, β_2) is pairwise compact closed space. Suppose U is β_1-compact set in (N, β_1, β_2).To demonstrate that (N, β_1, β_2) is β_2-closed in (N, β_1, β_2). Given that Φ is pairwise closed continuous function, then $\Phi^{-1}(z)$ is ϑ_1-compact in $(Z, \vartheta_1, \vartheta_2)$, and although $(Z, \vartheta_1, \vartheta_2)$ is pairwise minimal compact closed space, so $(Z, \vartheta_1, \vartheta_2)$ is pairwise minimal compact closed space so $\Phi^{-1}(z)$ is ϑ_2 $-$closed in $(Z, \vartheta_1, \vartheta_2)$. However, because Φ is pairwise onto, so $\Phi\ (\Phi^{-1}(U) = U$ is β_2 $-$closed in (N, β_1, β_2). Similar to how$V$ is β_2-compact set in (N, β_1, β_2), so (N, β_1, β_2) is pairwise compact closed space. Given that $(Z, \vartheta_1, \vartheta_2)$ is pairwise compact and Φ is pairwise onto closed function $\Phi(Z) = N$, so (N, β_1, β_2) is pairwise compact. As a result, according to theorem 5.2, (N, β_1, β_2) is pairwise minimal compact closed space.

Theorem 5.5. *If $\Psi : (N, \beta_1, \beta_2) \to (Z, \vartheta_1, \vartheta_2)$ is pairwise hoemorphism, then $(Z, \vartheta_1, \vartheta_2),\ (N, \beta_1^{\backslash}, \beta_2^{\backslash})$ is pairwise compact closed space, while else $(Z, \vartheta_1, \vartheta_2)$ is pairwise compact open space.*

Proof. Assuming $(Z, \vartheta_1, \vartheta_2)$ be pairwise compact closed space, and a function $\Psi : (N, \beta_1, \beta_2) \rightarrow (Z, \vartheta_1, \vartheta_2)$ be a pairwise continuous function therefore $(N, \beta_1^{\setminus}, \beta_2^{\setminus})$ be pairwise compact. In order to demonstrate Ψ is pairwise hoemorphism. Let U be any β_1-compact in (N, β_1, β_2), then U is β_2-closed set. Due to the fact that $(Z, \vartheta_1, \vartheta_2)$ is pairwise compact closed space, then $\Psi(U) = (\Psi^{-1})^{-1}(U)$ is ϑ_2-compact in $(Z, \vartheta_1, \vartheta_2)$ and $\Psi(U)$ is ϑ_1-closed in $(Z, \vartheta_1, \vartheta_2)$, As a result $\Psi^{-1} : (Z, \vartheta_1, \vartheta_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is pairwise continuous function, yet Ψ is pairwise continuous function. Thus, Ψ is pairwise hoemorphism. In order to demonstrate that $(Z, \vartheta_1, \vartheta_2)$ is pairwise compact compact closed space. Allowing the pairwise continuous function $\Psi : (N, \beta_1, \beta_2) \rightarrow (Z, \vartheta_1, \vartheta_2)$ is pairwise hoemorphism. Because every pairwise minimal compact closed space is pairwise compact closed space. It is sufficient to demonstrate that $(Z, \vartheta_1, \vartheta_2)$ is pairwise minimal compact closed space. Take $(Z, \vartheta_1, \vartheta_2) \subset (Z, \vartheta_1^{\setminus}, \vartheta_2^{\setminus})$, it is mean $\vartheta_1 \subset \vartheta_1^{\setminus}$, $\vartheta_2 \subset \vartheta_2^{\setminus}$ and Λ:$(Z, \vartheta_1, \vartheta_2) \rightarrow (Z, \vartheta_1^{\setminus}, \vartheta_2^{\setminus})$ be pairwise continuous function, nevertheless is not pairwise hoemorphism, so ϑ_1, ϑ_2 is not maximal compact. Consequently, $(Z, \vartheta_1, \vartheta_2)$ is pairwise minimal compact closed space. Furthermore, $(Z, \vartheta_1, \vartheta_2)$ is pairwise compact closed space.

Corollary 5.2. *Let's say that a function $\Phi : (Z, \vartheta_1, \vartheta_2) \rightarrow (N, \beta_1, \beta_2)$be pairwise onto continuous function and (N, β_1, β_2) is pairwise compact closed space. As $(Z, \vartheta_1, \vartheta_2)$ is pairwise compact Hausdroff, then (N, β_1, β_2) follows.*

Proof. Given that $(Z, \vartheta_1, \vartheta_2)$ is pairwise compact Hausdroff and Φ is pairwise onto continuous function. Therefore, (N, β_1, β_2) is pairwise compact Hausdroff and pairwise compact closed space.

Theorem 5.6. *Let $\Phi : (Z, \vartheta_1, \vartheta_2)\xrightarrow{\text{one to one}} (N, \beta_1, \beta_2)$ be pairwise $K-$continuous function. If (N, β_1, β_2) is pairwise compact closed space, then $Z = \Phi^{-1}(N)$ is pairwise compact closed space.*

Proof. To demonstrate that U be ϑ_1-compact in $(Z, \vartheta_1, \vartheta_2)$, let's assume that U is ϑ_1-closed in $(Z, \vartheta_1, \vartheta_2)$. U is ϑ_1-compact in $(Z, \vartheta_1, \vartheta_2)$, which means that $\Phi(U)$ is β_1-compact in (N, β_1, β_2). Given that Φ is pairwise $K-$continuous function and (N, β_1, β_2) is pairwise compact closed space, $\Phi(U)$ is β_2-closed in (N, β_1, β_2). However, if Φ is be pairwise continuous function and one to one, $\Phi^{-1}(\Phi(U)) = U$ is ϑ_2-closed in $(Z, \vartheta_1, \vartheta_2)$ as a result, so $Z = \Phi^{-1}(U)$ is pairwise compact closed space. V is ϑ_2-compact in $(Z, \vartheta_1, \vartheta_2)$. The outcome is received.

If we apply the same theorem stages, we will obtain the following corollary.

Corollary 5.3. *Let $\Phi : (Z, \vartheta_1, \vartheta_2)\overrightarrow{\text{one to one}} (N, \beta_1, \beta_2)$ be pairwise compact, pairwise continuous function. If (N, β_1, β_2) is pairwise minimal compact closed space, then $Z = \Phi^{-1}(N)$ is pairwise minimal compact closed space.*

Theorem 5.7. *Every pairwise $K-$function between pairwise minimal compact closed space can be both pairwise continuous and pairwise closed.*

Proof. Assume that $\Phi : (Z, \vartheta_1, \vartheta_2) \to (N, \beta_1, \beta_2)$ are pairwise $K-$function and $(Z, \vartheta_1, \vartheta_2), (N, \beta_1, \beta_2)$ are pairwise minimal compact closed space. Given that Φ is pairwise continuous and that (N, β_1, β_2) is pairwise minimal compact closed space. Consequently, if U is β_1-closed in (N, β_1, β_2), so U is β_2-compact. As a result of the fact that Φ is pairwise $K-$function, $\Phi^{-1}(U)$ is now ϑ_2-compact in $(Z, \vartheta_1, \vartheta_2)$, $yet(Z, \vartheta_1, \vartheta_2)$ is minimal compact closed space. Similarly for$V$ is β_2-closed in (N, β_1, β_2). As a result, Φ is pairwise continuous. Now, Φ is pairwise closed, if R is ϑ_2-compact in $(Z, \vartheta_1, \vartheta_2)$, then $\Phi(R)$ is β_2-compact in (N, β_1, β_2). This means that since Φ is pairwise $K-$function and (N, β_1, β_2) is pairwise minimal compact closed space. As a result, $\Phi(R)$ is β_1-closed in (N, β_1, β_2), Φ is pairwise closed.

Theorem 5.8. *Assume $\Phi : (Z, \vartheta_1, \vartheta_2) \to (N, \beta_1, \beta_2)$ be pairwise $K-$onto function and pairwise closed function. If $(Z, \vartheta_1, \vartheta_2)$ is pairwise minimal compact closed space, then (N, β_1, β_2) is true.*

Proof. As every $(Z, \vartheta_1, \vartheta_2)$ is pairwise minimal compact closed space and Φ is pairwise $K-$onto function, then $N = \Phi(Z)$ and every pairwise minimal compact closed space is pairwise compact, (N, β_1, β_2) is pairwise minimal compact closed space.

If we apply the same theorem stages, we will obtain the following corollary:

Corollary 5.4. *Take $\Phi : (Z, \vartheta_1, \vartheta_2) \to (N, \beta_1, \beta_2)$ is pairwise $K-$onto function and pairwise closed function. In the event where (N, β_1, β_2) is pairwise minimal compact closed space, then $Z = \Phi^{-1}(N)$ holds true.*

6. New Remarks Of Pairwise Lindelöf Closed Spaces

The advanced characteristics of the pairwise minimal Lindelöf closed spaces are highlighted in this part, along with some peculiarities of the cartesian process of multiplication of these spaces in special circumstances.

Definition 6.1. *Let $(Z, \vartheta_1, \vartheta_2)$ be a topological space. We define $(Z, \vartheta_1, \vartheta_2)$ is pairwise Lindelöf closed spaces, when each ϑ_1-Lindelöf subset of $(Z, \vartheta_1, \vartheta_2)$ is ϑ_2-closed in $(Z, \vartheta_1, \vartheta_2)$ and ϑ_2-Lindelöf subspace of $(Z, \vartheta_1, \vartheta_2)$ is ϑ_1-closed in $(Z, \vartheta_1, \vartheta_2)$.*

Remark 6.1. *There are pairwise Lindelöf closed spaces for every pairwise compact closed spaces.*

Example 6.1. *$(R, \vartheta_s, \vartheta_l)$ is not pairwise Lindelöf space. Due to the fact that U is ϑ_s-open subset, then U is ϑ_s-Lindelöf . U is not τ_l-closed though .*

The example below demonstrates that pairwise Lindelöf closed spaces are not always represented by their continuous images.

Example 6.2. *Assume that $\Phi : (R, \vartheta_d, \vartheta_d) \to (R, \vartheta_{ind}, \vartheta_{ind})$ is pairwise continuous function. As a result, whereas $(R, \vartheta_d, \vartheta_d)$ is pairwise Lindelöf closed space, but $(R, \vartheta_{ind}, \vartheta_{ind})$ is not pairwise Lindelöf closed space.*

Theorem 6.1. *While* $\Phi : (Z, \vartheta_1, \vartheta_2) \xrightarrow{injection} (N, \beta_1, \beta_2)$ *be pairwise injection continuous function from* $(Z, \vartheta_1, \vartheta_2)$ *into pairwise Lindelöf closed space* (N, β_1, β_2)*, therefore* $(Z, \vartheta_1, \vartheta_2)$ *is too.*

Proof. Suppose that T is any ϑ_1-Lindelöf subset of $(Z, \vartheta_1, \vartheta_2)$, then $\Phi(T)$ is β_1-lindelöf subset in (N, β_1, β_2). Given that (N, β_1, β_2) is pairwise closed Lindelöf space, then $\Phi(T)$ is β_2-closed of (N, β_1, β_2) and Φ is pairwise injection continuous function, then $\Phi^{-1}(\Phi(T) = T$ is ϑ_2-closed subset of $(Z, \vartheta_1, \vartheta_2)$. Simillarly for J is ϑ_2-Lindelöf subset of $(Z, \vartheta_1, \vartheta_2)$. The $(Z, \vartheta_1, \vartheta_2)$ is ϑ_2-lindelöf subset behaves similarly for J. Consequently, pairwise Lindelöf closed space $(Z, \vartheta_1, \vartheta_2)$ exists.

Theorem 6.2. *Being a pairwise Lindelöf closed space has the bitopological characteristic.*

Proof. Letting $(Z, \vartheta_{z_1}, \vartheta_{z_2})$ be a pairwise Lindelöf closed space and $(N, \vartheta_{n_1}, \vartheta_{n_2})$ be a subspace of $(Z, \vartheta_{z_1}, \vartheta_{z_2})$. Given $\Phi : (Z, \vartheta_{z_1}, \vartheta_{z_2}) \to (N, \vartheta_{n_1}, \vartheta_{n_2})$ be pairwise homeomorphism and U be $\vartheta_{n_1}-$Lindelöf subset of $(N, \vartheta_{n_1}, \vartheta_{n_2})$. Granted that $(Z, \vartheta_{z_1}, \vartheta_{z_2})$ is a pairwise Lindelöf closed space, so $\Phi^{-1}(U)$ is $\vartheta_{z_2}-$closed in $(Z, \vartheta_{z_1}, \vartheta_{z_2})$. As a result, $\Phi^{-1}(\Phi(U) = U$ is $\vartheta_{n_2}-$closed in $(N, \vartheta_{n_1}, \vartheta_{n_2})$.Similar conditions apply V be $\vartheta_{n_2}-$Lindelöf subset of $(N, \vartheta_{n_1}, \vartheta_{n_2})$. Therefore, $(N, \vartheta_{n_1}, \vartheta_{n_2})$ is pairwise Lindelöf closed space.

Theorem 6.3. *Being a pairwise Lindelöf closed space is an inherited quality.*

Proof. Assuming $(Z, \vartheta_{z_1}, \vartheta_{z_2})$ be a pairwise closed Lindelöf space, $(N, \vartheta_{n_1}, \vartheta_{n_2})$ be a subspace of $(Z, \vartheta_{z_1}, \vartheta_{z_2})$ and U be $\vartheta_{n_1}-$Lindelöf subset of $(N, \vartheta_{n_1}, \vartheta_{n_2})$, therefore U is $\vartheta_{z_1}-$Lindelöf subset of $(Z, \vartheta_{z_1}, \vartheta_{z_2})$. However, $(Z, \vartheta_{z_1}, \vartheta_{z_2})$ is a pairwise closed Lindelöf space, so U is $\vartheta_{z_2}-$closed in $(Z, \vartheta_{z_1}, \vartheta_{z_2})$. Nevertheless, $U = U \cap N$ is $\vartheta_{n_2}-$closed in $(N, \vartheta_{n_1}, \vartheta_{n_2})$. The actually imply $(N, \vartheta_{n_1}, \vartheta_{n_2})$ is pairwise Lindelöf closed space.

7. Some Characterisations Of Pairwise Minimal Closed Lindelöf Spaces

More findings about the pairwise minimal Lindelöf closed space's topological properties are presented in this part, along with a diagram illustrating the main connection between these spaces.

Definition 7.1. *A bitopological space* $(Z, \vartheta_1, \vartheta_2)$ *is considered to be pairwise minimal Lindelöf closed space, when and only when* $\vartheta_1^{/} \leq \vartheta_1$*,* $\vartheta_2^{/} \leq \vartheta_2$ *and* $(Z, \vartheta_1^{/}, \vartheta_2^{/})$ *is not pairwise Lindelöf closed space.*

Theorem 7.1. *Any pairwise Lindelöf closed space is a pairwise minimal Lindelöf closed space.*

Proof. While $(Z, \vartheta_1, \vartheta_2)$ be pairwise Lindelöf closed space and is not pairwise minimal Lindelöf closed space, subsequently there are $\vartheta_1^{/}$, $\vartheta_2^{/}$ make $\vartheta_1^{/} \leq \vartheta_1$, $\vartheta_2^{/} \leq \vartheta_2$ and

$(Z, \vartheta_1^/, \vartheta_2^/)$ is pairwise closed Lindelöf space. As $I_x : (Z, \vartheta_1, \vartheta_2) \to (Z, \vartheta_1^/, \vartheta_2^/)$ be pairwise identity, pairwise continuous, pairwise bijection, and pairwise closed function, after which I_x is pairwise homeomorphism, so $\vartheta_1^/ = \vartheta_1,\ \vartheta_2^/ = \vartheta_2$. Contradiction results. Therefore, $(Z, \vartheta_1, \vartheta_2)$ is pairwise minimal closed Lindelöf space.

Example 7.1. *while Z is pairwise countable set, after which $(Z, \vartheta_d, \vartheta_d)$ is pairwise minimal Lindelöf closed space.*

Remark 7.1. *As demonstrated by the following example, pairwise minimal closed lindelöf space is not always the continuous image of pairwise minimal closed lindelöf space.*

Example 7.2. *Assuming Z be countable set and $I_x : (Z, \vartheta_d, \vartheta_d) \to (Z, \vartheta_{ind}, \vartheta_{ind})$ be pairwise identity function on Z, $(Z, \vartheta_d, \vartheta_d)$ is pairwise minimal Lindelöf closed space, however*

$(Z, \vartheta_{ind}, \vartheta_{ind})$ is not pairwise minimal Lindelöf closed space.

Theorem 7.2. *A bitopological property is the state of having pairwise minimal Lindelöf closed space.*

Proof. Let $(Z, \vartheta_{z_1}, \vartheta_{z_2})$ be a pairwise minimal closed Lindelöf space and $\Phi : (Z, \vartheta_{z_1}, \vartheta_{z_2}) \to (N, \vartheta_{n_1}, \vartheta_{n_2})$ be a pairwise homeomorphism and $(N, \vartheta_{n_1}, \vartheta_{n_2})$ is pairwise closed Lindelöf space and is not pairwise minimal Lindelöf closed space. Therefore, there are $\vartheta_{n_1}^/ \leq \vartheta_{n_1},\ \vartheta_{n_2}^/ \leq \vartheta_{n_2}$, and that such $(N, \vartheta_{n_1}^/, \vartheta_{n_2}^/)$ is pairwise Lindelöf closed space. Suppose U is $\vartheta_{n_1}^/-$set in $(N, \vartheta_{n_1}, \vartheta_{n_2})$,such that $\Phi^{-1}(U)$ is $\vartheta_{z_1}-$set in $(Z, \vartheta_{z_1}, \vartheta_{z_2})$, so $\vartheta_{z_1}^/ \leq \vartheta_{z_1}$, but also for$V$ is $\vartheta_{n_2}^/-$set in $(N, \vartheta_{n_1}, \vartheta_{n_2})$, $\Phi^{-1}(V)$ is $\vartheta_{z_2}-$set in $(Z, \vartheta_{z_1}, \vartheta_{z_2})$, so $\vartheta_{z_1}^/ \leq \vartheta_{z_1}$. Consequently, $(Z, \vartheta_{z_1}^/, \vartheta_{z_2}^/)$ is pairwise Lindelöf closed space, which is contradiction with $(Z, \vartheta_{z_1}, \vartheta_{z_2})$ be a pairwise minimal Lindelöf closed space. Therefore, $(N, \vartheta_{n_1}, \vartheta_{n_2})$ is pairwise minimal Lindelöf closed space.

The following corollaries have the same theorem-proof as the following ones.

Corollary 7.1. *Let $(Z, \vartheta_{z_1}, \vartheta_{z_2})$ be a pairwise Lindelöf closed space and $(N, \vartheta_{n_1}, \vartheta_{n_2})$ be pairwise closed subspace of $(Z, \vartheta_{z_1}, \vartheta_{z_2})$, then $(N, \vartheta_{n_1}, \vartheta_{n_2})$ is pairwise Lindelöf closed space.*

Corollary 7.2. *When $(Z, \vartheta_{z_1}, \vartheta_{z_2})$ be a pairwise minimal Lindelöf closed space and $(N, \vartheta_{n_1}, \vartheta_{n_2})$ be pairwise closed subspace of $(Z, \vartheta_{z_1}, \vartheta_{z_2})$, after which $(N, \vartheta_{n_1}, \vartheta_{n_2})$ is pairwise minimal Lindelöf closed space.*

Theorem 7.3. *Assuming that $(Z_1 \times Z_2,\ \vartheta_1 \times \vartheta_1,\ \vartheta_2 \times \vartheta_2)$ is pairwise Lindelöf closed space. Therefore, each $(Z_1, \vartheta_1, \vartheta_2)$, $(Z_2, \vartheta_1, \vartheta_2)$ is pairwise minimal Lindelöf closed space.*

Proof. Lindelöf closed space has the property of being pairwise minimal, which is a bitopological property, if $\{x_2\}$ be a fixed element in $(Z_2, \vartheta_1, \vartheta_2)$, then $(Z_1, \vartheta_1, \vartheta_2) \times \{z_2\}$ is a subspace of $(Z_1 \times Z_2,\ \vartheta_1 \times \vartheta_1,\ \vartheta_2 \times \vartheta_2)$.Therefore $(Z_1, \vartheta_1, \vartheta_2) \times \{z_2\}$ is pairwise Lindelöf closed space. But $(Z_1, \vartheta_1, \vartheta_2) \times \{z_2\}$ is pairwise homomorphic to $(Z_1, \vartheta_1, \vartheta_2)$. By theorem 7.2, $(Z_1, \vartheta_1, \vartheta_2)$ is pairwise Lindelöf closed space. Now since every pairwise Lindelöf closed space is pairwise minimal Lindelöf closed space, therefore $(Z_1, \vartheta_1, \vartheta_2)$ is pairwise minimal Lindelöf closed space.Similar to that, we can demonstrate $(Z_2, \vartheta_1, \vartheta_2)$ is pairwise minimal Lindelöf closed space.

The outcomes of Theorem 7.3 are generalized in the following way.

Corollary 7.3. *If* $Z = \prod_{\alpha \in \Omega} Z_\alpha$ *is a pairwise Lindelöf closed space, then each* Z_α *is pairwise minimal Lindelöf closed space, for each* $\alpha \in \Phi$.

8. A New Defition Of Pairwise Minimal Hausdroff Spaces

We provide a novel definition of pairwise minimal Hausdroff spaces in this section, along with some of its related features.

Definition 8.1. *Can let* $(Z, \vartheta_1, \vartheta_2)$ *be pairwise Hausdroff space. A bitopological space* $(Z, \vartheta_1, \vartheta_2)$ *is allegedly pairwise minimal Hausdroff space, only if and only* $\vartheta_1^{/} \leq \vartheta_1,\ \vartheta_2^{/} \leq \vartheta_2$ *and* $(Z, \vartheta_1^{/}, \vartheta_2^{/})$ *is not pairwise Hausdroff space.*

Theorem 8.1. *Every pairwise locally compact minimal compact closed space is pairwise minimal Hausdroff space.*

Proof. Suppose $(Z, \vartheta_{z_1}, \vartheta_{z_2})$ is a pairwise locally compact minimal compact closed space, so $(Z, \vartheta_{z_1}, \vartheta_{z_2})$ is pairwise locally compact and pairwise compact closed space. Consequently, $(Z, \vartheta_{z_1}, \vartheta_{z_2})$ is pairwise Hausdroff space. Suppose $(Z, \vartheta_{z_1}, \vartheta_{z_2})$ is not pairwise minimal Hausdroff space, so there exists $\vartheta_{z_1}^{/} \leq \vartheta_{z_1},\ \vartheta_{z2}^{/} \leq \vartheta_{z_2}$, like that $(Z, \vartheta_{z_1}^{/},\ \vartheta_{z_2}^{/})$ is pairwise Hausdroff space, implies $(Z, \vartheta_{z_1}^{/},\ \vartheta_{z_2}^{/})$ is pairwise compact closed space. Consequently, a contradiction results. Therefore $(Z, \vartheta_{z_1}, \vartheta_{z_2})$ is pairwise minimal Hausdroff space.

The following corollary's proof resembles that of the aforementioned theorem.

Corollary 8.1. *For each pairwise locally compact minimal Lindelöf closed space is pairwise minimal Hausdroff space.*

Theorem 8.2. *Every pairwise Hausdroff minimal compact closed space is pairwise minimal Hausdroff space.*

Proof. Let $(Z, \vartheta_1, \vartheta_2)$ be pairwise Hausdroff minimal compact closed space, then $(Z, \vartheta_1, \vartheta_2)$ is pairwise Hausdroff compact closed space and $(Z, \vartheta_1, \vartheta_2)$ is not pairwise minimal Hausdroff space, hence, there are $\vartheta_1^{/},\ \vartheta_2^{/}$ such that, $\vartheta_1^{/} \leq \vartheta_1,\ \vartheta_2^{/} \leq \vartheta_2$ and $(Z, \vartheta_1^{/},\ \vartheta_2^{/})$ is pairwise Hausdroff space and so $(Z, \vartheta_1^{/},\ \vartheta_2^{/})$ is pairwise compact closed space. A contradiction results because $(Z, \vartheta_1, \vartheta_2)$ be pairwise minimal compact closed space. Hence $(Z, \vartheta_1, \vartheta_2)$ is pairwise minimal Hausdroff space.

The following corollary's proof is equivalent to that of the aforementioned theorem.

Corollary 8.2. *Each pairwise Hausdroff minimal compact closed space is pairwise minimal compact closed space.*

Theorem 8.3. *Every pairwise regular minimal compact closed space is pairwise minimal Hausdroff space.*

Proof. Let $(Z, \vartheta_1, \vartheta_2)$ be pairwise regular minimal compact closed space, then $(Z, \vartheta_1, \vartheta_2)$ is pairwise regular compact closed space. Since every pairwise compact closed space is pairwise T_1- space. Hence $(Z, \vartheta_1, \vartheta_2)$ is regular and T_1- space so it is T_2- space. Suppose $(Z, \vartheta_1, \vartheta_2)$ is not pairwise minimal Hausdroff space, so there exist $\vartheta_1^/, \ \vartheta_2^/$ such that, $\vartheta_1^/ \leq \vartheta_1, \ \vartheta_2^/ \leq \vartheta_2$ and $(Z, \vartheta_1^/, \ \vartheta_2^/)$ is pairwise compact closed space, which is incongruous; as a result $(Z, \vartheta_1, \vartheta_2)$ is pairwise minimal Hausdroff space. $(Z, \vartheta_1, \vartheta_2), \ (N, \beta_1, \beta_2)$

Theorem 8.4. *Suppose* $(Z \times N, \ \vartheta_1 \times \beta_1, \ \vartheta_2 \ \times \beta_2)$ *is pairwise regular compact closed space, then each* $(Z, \vartheta_1, \vartheta_2), \ (N, \beta_1, \beta_2)$ *is pairwise minimal Hausdroff space.*

Proof. Since $(Z, \vartheta_1, \vartheta_2), \ (N, \beta_1, \beta_2)$ is pairwise compact closed space, so each $(Z, \vartheta_1, \vartheta_2), \ (N, \beta_1, \beta_2)$ is pairwise minimal compact closed space, and therefore $(Z, \vartheta_1, \vartheta_2), \ (N, \beta_1, \beta_2)$ is pairwise regular, then it is pairwise minimal Hausdroff space.

Theorem 8.5. *If* $(Z, \vartheta_1, \vartheta_2), \ (N, \beta_1, \beta_2)$ *are pairwise Hausdroff compact closed space,* $(Z \times N, \ \vartheta_1 \times \beta_1, \ \vartheta_2 \ \times \beta_2)$ *is pairwise regular minimal compact closed space.*

Proof. Since $(Z, \vartheta_1, \vartheta_2), \ (N, \beta_1, \beta_2)$ are pairwise Hausdroff compact closed space, then $(Z \times N, \ \vartheta_1 \times \beta_1, \ \vartheta_2 \ \times \beta_2)$ is pairwise Hausdroff compact closed space, so by theorem 8.3, $(Z \times N, \ \vartheta_1 \times \beta_1, \ \vartheta_2 \ \times \beta_2)$ is pairwise compact closed space, and so is pairwise minimal compact closed space, by theorem 8.4, $(Z \times N, \ \vartheta_1 \times \beta_1, \ \vartheta_2 \ \times \beta_2)$ is pairwise regular minimal compact closed space.

Corollary 8.3. *Every pairwise Lindelöf closed space is pairwise compact closed space.*

Theorem 8.6. *For pairwise closed compact $P-$space* $(Z, \vartheta_1, \vartheta_2)$, *the following is equivalent:*

Let $(Z, \vartheta_1, \vartheta_2)$ is pairwise minimal Hausdroff space, if and only if $(Z, \vartheta_1, \vartheta_2)$ is pairwise Hausdroff minimal Lindelöf closed space.

Proof. $\Rightarrow$Let $(Z, \vartheta_1, \vartheta_2)$ is pairwise minimal Hausdroff space, $(Z, \vartheta_1, \vartheta_2)$ is pairwise

Hausdroff space, so $(Z, \vartheta_1, \vartheta_2)$ is pairwise compact closed space, so pairwise Lindelöf closed space. Hence $(Z, \vartheta_1, \vartheta_2)$ is pairwise minimal Lindelöf closed space.

$\Leftarrow$ Let $(Z, \vartheta_1, \vartheta_2)$ is pairwise Hausdroff minimal Lindelöf closed space, then $(Z, \vartheta_1, \vartheta_2)$ is pairwise Hausdroff Lindelöf closed space, so $(Z, \vartheta_1, \vartheta_2)$ is pairwise compact closed space, and so $(Z, \vartheta_1, \vartheta_2)$ is pairwise minimal Lindelöf closed space. Hence by theorem 8.6 $(Z, \vartheta_1, \vartheta_2)$ is pairwise minimal Hausdroff space

Theorem 8.7. *For pairwise closed compact $P-$space $(Z, \vartheta_1, \vartheta_2)$, the following is equivalent:*

Let $(Z, \vartheta_1, \vartheta_2)$ is pairwise minimal compact space, if and only if $(Z, \vartheta_1, \vartheta_2)$ is pairwise minimal Lindelöf closed space.

Proof. $\Rightarrow$ Let $(Z, \vartheta_1, \vartheta_2)$ is pairwise minimal compact space, then $(Z, \vartheta_1, \vartheta_2)$ is pairwise compact space, so $(Z, \vartheta_1, \vartheta_2)$ is pairwise T_2. Since $(Z, \vartheta_1, \vartheta_2)$ is $P-$space and pairwise compact closed space, then $(Z, \vartheta_1, \vartheta_2)$ is pairwise Lindelöf closed space. Hence $(Z, \vartheta_1, \vartheta_2)$ is pairwise minimal Lindelöf closed space.

$\Leftarrow$ Let $(Z, \vartheta_1, \vartheta_2)$ is pairwise minimal Lindelöf closed space, then $(Z, \vartheta_1, \vartheta_2)$ is pairwise Lindelöf closed space, so $(Z, \vartheta_1, \vartheta_2)$ is pairwise compact closed space. Hence $(Z, \vartheta_1, \vartheta_2)$ is pairwise minimal compact space.

9. Types of minimal spaces in bitopological spaces; application

Minimal space in bitopological spaces provide intriguing opportunities for future and predictive applications. These space provide a foundation for simplifying complicated systems by focusing on critical components and relationships, making it easier to study and anticipate actions in multivariable environments. Traditional approaches frequently struggle with qualitative features of systems, such as those seen in social or educational contexts where quantification is problematic. By reducing the system to its most basic structure, we may better describe and comprehend the underlying dynamics, allowing for more accurate predictions and insights. Potentially improve medical decision-making processes. By using simple structures, we were able to speed the analysis of complicated medical data, focusing on the reduction and fundamental decision qualities. This would enable a more realistic comparison of decision-making outcomes among patients with comparable and dissimilar symptoms. Furthermore, combining minimal spaces with a variable precision rough set model may improve the accuracy and reliability of medical diagnosis and treatment regimens.In machine learning, AI and big data; using minimal compact and Lindelöf spaces in bitopological systems would improve data processing by identifying essential subsets and optimizing sampling across dual topologies. This would minimize dataset size, increase computational efficiency, and preserve accuracy, making predictive models more scalable and adaptable to complex systems with multiple topologies. When applied to urban planning and smart cities, it has the potential to optimize city layouts and infrastructure by taking into account several variables at once. Minimal compact areas allow for effective distribution of facilities while balancing physical land use and social connectivity, resulting in more adaptive urban environments. Meanwhile, Lindelöf spaces could improve the architecture of interconnected networks by identifying minimal hubs that service various connections, allowing for scalable infrastructure that can meet changing demands without requiring proportional resource increases. Overall, this strategy would result in smarter, more resilient city designs.The unifying thread running across all applications is efficiency. Minimal compact and Lindelöf spaces give frameworks for getting optimal results with little resources while improving predictability and scalability in complicated systems.

10. Conclusions

The relationships among pairwise minimal compact closed spaces, pairwise minimal Lindelöf closed spaces, and pairwise minimal Hausdorff spaces in bitopological spaces were examined in this study. According to the compact, Lindelöf, and Hausdorff spaces notion that is here proposed, the study determined the prerequisites for harmonizing the closed sets. We looked at the relationship between these two ideas and described them using several sets. This study's secondary goal was to draw attention to some intricate closed-set features and some peculiarities of the cartesian process of multiplying these functions in specific circumstances. Furthermore, key aspects of these concepts as well as a few instructive situations were carefully investigated. We identified their fundamental characteristics in general and made clear the requirements for establishing similar linkages between them. We went through their main traits and demonstrated how they work together. Additionally, the report highlighted the characteristics of these functions and offered numerous examples of them. The exploration of the various potential futures for these functions will begin with these spaces. Further variations of these regions may be explored in future research [17], [18], [19],[20], [21], and [22] respectively.

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