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Investigation of Existence and Ulam's type Stability for Coupled Fractal Fractional Differential Equations

Amjad Ali[1], Farha Bibi[1], Zeeshan Ali[2,*], Kamal Shah[3], Bahaaeldin Abdalla[3], Thabet Abdeljawad[3]

[1] *Department of Mathematics and Statistics, University of Swat, KPK, Pakistan*
[2] *Department of Information Management, National Yunlin University of Science and Technology Douliu Taiwan, R. O.C*
[2] *Department of Mathematics and Sciences, Prince Sultan University, P.O. Box 66833, 11586 Riyadh, Saudi Arabia*

Abstract. In this study, we investigate the coupled system of fractal fractional differential equations (FFDEs) from an existence and stability point of view. The area related to coupled FFDEs is significant because its permitting us to analyze and predict the relationships between several variables throughout the real-world phenomena. Coupled systems have numerous applications in different fields, such as modeling of brain activities and spreads of disease in biology and medicine, modeling of mechanical systems, electrical circuits in engineering, and modeling of population dynamics, predator-prey models in ecology and financial mathematical in economics. Furthermore, the afore mentioned area is significant for environmental science in pollution control, climate change modeling, artificial intelligence, and control systems in technology. We analysis a coupled system and make more informed choices in a variety of fields through coupled system of FFDEs. Keeping these informative aspects of coupled systems in mind, this article aims to explore the qualitative analysis such as existence, uniqueness, and stability for the solution of underlying coupled systems of FFDEs. The tools of functional analysis (FA) and fixed point theory (FPT) have been applied to deduce our required results. We have used the Banach contraction principle (BCP) and the Krasnoselskii fixed point theorem (KFPT) to demonstrate the conditions for existence and uniqueness of a solutions (EUS). Additionally, the results related to stability have demonstrated by using Ulam's concept. Subsequently, a suitable example is given to illustrates the validity of this work.

2020 Mathematics Subject Classifications: 26A33, 34A08, 35A07

Key Words and Phrases: Coupled System, Fractal fractional Operator, Existence, Stability, Fixed Point Theorems

*Corresponding author.
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Email addresses: amjadalijc18@gmail.com (A. Ali), farhabibi987@gmail.com (F.Bibi), zeeshan@yuntech.edu.tw (Z. Ali),kshah@psu.edu.sa (K. Shah), babdallah@psu.edu.sa (B.Abdalla), tabdeljawad@psu.edu.sa (T. Abdeljawad)

1. Introduction

The concept of fractional calculus (FC) is an emerging area of recent research due to its significant implementations in different field of life sciences, physical sciences, and social sciences [1]. The idea of fractional order derivative (FODs) was initiated from the well-known correspondence between Leibniz's and L'Hospital in the seventeenth hundred century [2]. The FODs have some advantages over the classical derivatives, these are more flexible, non-local in nature, having memory effects, great degree of freedom and more reliable [3]. FC is the capability to describe the complex dynamics of all those real-world scenarios, which can not be formulated by conventional calculus [4]. Furthermore, FC is substantial and has a multifaceted of applications throughout numerous fields, such as finance, bio-engineering, ecology, physics and graphics. Of course one of the significant aspects of FC is that it can describe and investigate the real-world situation in the form of fractional differential equations (FDEs), or system of FDEs [5]. Researchers are interested to investigate FDEs to provides a powerful framework for formulating those dynamical phenomena which is not describe by classical order DEs accurately [1]. Due to aforesaid applications, mathematicians have been taken keen interest to investigate FDEs. Consequently, researchers have well explored the theory of FDEs from different aspects, such as stability analysis, numerical approximation, and existence of solution, and published plenty of books, articles, and manographs [6]. Meanwhile, the researchers introduced different types of fractional operators (FOs), such as the conformable fractional derivatives (CFDs), the Riemann-Liouville fractional derivatives (RLFDs), the Caputo fractional derivative (CFD), the Grunwald-Letnikov fractional derivative (GLFD) [7], non singular fractional derivatives (NSFDs) like Atangana-Baleanu fractional derivative (ABFD), Caputo-Fabrizio fractional derivative (CFFD), etc [8]. The theory of fractals derivative is old as the FC. In 2017, researchers have extended the concepts of fractional derivatives to fractal and introduced fractals fractional derivatives (FFDs) [9].

The concept of this new type of FFDs was introduced by Atangana [10]. This mentioned class of FFDs consist of two types of orders, the first is usual fractional order, while the second part is known as fractal dimension. An important aspect of the consider derivatives is that they can investigate both FODs and FFDs at same stage simultaneously in the same problem [11]. The FFDs have advanced over the traditional FODs to understanding their geometrical meaning as well as relation with them [12]. The FFDs have very useful consequences in situation, where discontinuity, power-law, and crossover occur, or decaying memory fail to capture the complex behaviors [13]. The concerned FFDs has a accomplished form of FODs which combined both fractal and fractional to more compact form is known as fractal fractional calculus [14]. This newly emerging area of research has attracted the attention of researchers to investigate various features [15].

The coupled system of FDEs have been studied by researchers from various aspects and furnish theory up-to large extend in the sense of ordinary FODs, such as existence, numerical solution and stability analysis [16]. To best to our knowledge the consider type of coupled system of FFDEs are very rarely investigated by researchers. In this context, the aforementioned research directions needs further attention of researchers to explore this theory. It is worthy to highlight the key features of underlying coupled system of FFDEs, that in some circumstances, the actual dynamics behaviors of phenomena cannot be expressed with ordinary tools of FDEs [17]. While, the proposed tools of coupled system of FFDEs are remarkable achievement for investigating such problems. Some well-known examples of coupled systems are host-parasite, plant- pollinator of biological process, grass land-fire, Lorenz model [18], atmosphere-ocean current and tides-coastal erosion are examples of physical systems [19]. According to our study the coupled system of FDEs in sense of fractal derivatives are very rarely studied by researchers and need more attention. Therefore, we investigate the coupled system of FDEs in sense of fractal operator.

Researchers from different fields are interested to investigate numerous aspects as this theory has the potential to address a variety of issues and is a fast growing field of research [20]. There are many methods to develop the EUS of FFDEs, such as FPT and topological degree theory [21]. Another subsequent aspect of fractal fractional calculus is investigating stability analysis of initial value problems of FFDEs. Stability analysis is a technique, used to find the ability of the system to get back to its equilibrium position after introducing some changes. There are different kinds of stability analysis in the present literature of fractal FC [22]. Ulam, raised a question that under what condition does there exists an additive mapping near an approximately additive mapping? In reply to this question Hyres answered Ulam's question for the "additive mapping in complete norm spaces" [23]. After it was considered a type of stability and called it Ulam -Hyers (UH) stability. The said concept was further extended to generalized (GUH), and Ulam-Hyers-Rassias (UHR) and generalized (GUHR) analysis. This concept has many applications in different mathematical fields, including DEs, operator theory, physics, engineering, and computer science. The connection between stability analysis and Hyers-Ulam stability are interconnected concepts of understanding the strength of systems and equations by applying stability analysis to functional equations [24]. This integrated approach can lead to a deeper understanding of complex dynamics and their behavior. The proposed stability has the main concern of researchers, due to extensive applications in various fields. Some the major contributions in proposed field we discuss here. For instance, Xiadjie *et al.*, [25] studied the following non-linear problem in sense of the Caputo's derivative with order ξ as:

$$\begin{aligned}&D^{\xi}\phi(t)=\theta_1(t,\phi(t)),\ \ t\in\Omega=[0,T],\\&\phi(0)+\psi(\phi)=\phi_0,\end{aligned}$$

where $f:\Omega\times R\to R$ is continuous function. The investigation of coupled system for non-linear FDEs has the main focuss of researchers from existence point of view. In this regards, the author's [26] studied the uniqueness of solution for coupled system of FDEs with integral boundary conditions. In the same way, Su [27] established the conditions for EUS for the following coupled system with two-point boundary conditions (BCs) in sense of singular kernel operator

$$\begin{cases} D^{\eta}\phi(t) = \theta_1(t, \psi(t), D^{p}\psi(t)), \quad t \in \Omega = [0, 1], \\ D^{\xi}\phi(t) = \theta_2(t, \phi(t), D^{q}\phi(t)), \\ \phi(0) = \phi(1) = \psi(0) = \psi(1) = 0, \end{cases}$$

where $\eta, \xi \in (1, 2]$, such that $p - \eta$ *and* $q - \xi > 0$, $\theta_1, \theta_2 \in C([0, 1] \times R^2, R)$. On the same fashion, Wang et al., [28] have been presented sufficient conditions for the EUS to the following non-linear CS with three point BCs

$$\begin{cases} D^{\eta}\phi(t) = \theta_1(t, \psi(t)), \quad t \in \Omega = [0, 1], \\ D^{\xi}\phi(t) = \theta_2(t, \phi(t)), \\ \phi(0) = \psi(0) = 0, \ \ \phi(1) = q\phi(p), \ \ \psi 1 b\psi(p), \end{cases}$$

where $\eta, \xi \in (1, 2]$ and $0 \le a, b \le 1, \ 0 < p \le 1$, $\theta_1, \theta_2 \in C([0, 1] \times R, R)$.

After the comprehensive study of present literature, it is clear that FDEs and their coupled systems are well investigated by researchers in term of singular kernel fractional operators. We felt that coupled system of FFDEs are very rarely investigated by researchers. For the better understanding of consider coupled system FFDEs, we carry out investigations and analysis for model via the tools of mathematical analysis. Therefore, main focuses of our research work is to establish the conditions of EUS and stability analysis for underlying coupled system of FFDEs. We use the results of the functional analysis along with FPT [29, 30] to obtain required results. The considered problem is described as follows:

$$\begin{cases} {}^{FD}D^{\eta} r(t) = \xi t^{\xi - 1} f\left(t, r(t), p(t)\right), \quad t \in I = [0, T], \\ {}^{FD}D^{\eta} p(t) = \xi t^{\xi - 1} g(t, r(t), p(t)), \\ r(0) = r_0 + \phi(r), \\ p(0) = p_0 + \psi(p), \end{cases} \tag{1}$$

where ${}^{FD}D^{\eta}$ is the Caputo fractional derivative and $r_0, p_0 \in R, \eta, \xi \in (0, 1]$, $f, g : I \times R^2 \to R$, $\phi, \psi : I = [0, T] \times R \to R$ are continuous functions. The qualitative analysis of the proposed coupled system of FFDEs offer deeper understanding into the complex dynamics systems exhibiting self-similarity and non-locality. The consider class have many applications in various discipline that can model complex phenomena. The coupled system of FFDEs can accurately describe the behavior of complex system in physics, biology, finance and predictive capabilities and understanding the qualitative behavior of solution. The result of this research will also provide new insight into the behavior of coupled system of FFDEs and will have the ability to inspire new applications and research directions.

2. Background Information

In this work, we will use the following notation: the norm define by $\|u\| = \sup\{|u(t)| : t \in [0, T]\}$ for the Banach space denoted by $\mathfrak{U}, \mathfrak{V} = C([0, T], R)$, where S represents a family of all bounded subsets of $B(\mathfrak{U}\times\mathfrak{V})$, and the product $\mathfrak{U}\times\mathfrak{V}$ is a Banach space with norm $\|(u,v)\| = \|u\| + \|v\|$.

Definition 1. *[2] The Riemann-Liouville derivative is defined by*

$$ {}^{FD}D^{\eta}y(t) = \begin{cases} \dfrac{1}{\Gamma(m-\eta)}\left(\dfrac{d}{dt}\right)^{m}\displaystyle\int_0^t (t-\delta)^{m-\eta-1}y(\delta)d\delta, \ m-1<\eta<m, \\ \dfrac{1}{\Gamma(1-\eta)}\dfrac{d}{dt}\displaystyle\int_0^t (t-\delta)^{-\eta}y(\delta)d\delta, \ 0<\eta<1, \\ \dfrac{d^m y}{dt^m}, \ \eta = m. \end{cases} $$

In the same way, the Caputo fractional derivative is defined by

$$ {}^{FD}D^{\eta}y(t) = \begin{cases} \dfrac{1}{\Gamma(m-\eta)}\displaystyle\int_0^t (t-\delta)^{m-\eta-1}\left(\dfrac{d}{d\delta}\right)^{m}y(\delta)d\delta, \ m-1<\eta<m, \\ \dfrac{1}{\Gamma(1-\eta)}\displaystyle\int_0^t (t-\delta)^{-\eta}\dfrac{d}{d\delta}y(\delta)d\delta, \ 0<\eta<1, \\ \dfrac{d^m y}{dt^m}, \ \eta = m. \end{cases} $$

The corresponding fractional integral is defined by

$$ {}^{FI}I^{\eta}y(t) = \frac{1}{\Gamma(\eta)}\int_0^t (t-\delta)^{\eta-1}y(\delta)d\delta, $$

provided that integral on right exists.

Definition 2. *[10] The FFD in sense of Riemann-Liouville operator for function y which is differentiable both in fractional and in fractals sense over $[0,T]$ given by*

$$ {}^{RFF}D^{\eta,\xi}y(t) = \begin{cases} \dfrac{1}{\Gamma(m-\eta)}\left(\dfrac{d}{dt^{\xi}}\right)^{m}\displaystyle\int_0^t (t-\delta)^{m-\eta-1}y(\delta)d\delta, \ m-1<\eta<m, \\ \dfrac{1}{\Gamma(1-\eta)}\dfrac{d}{dt^{\xi}}\displaystyle\int_0^t (t-\delta)^{-\eta}y(\delta)d\delta, \ 0<\eta,\xi<1, \\ \dfrac{1}{\Gamma(m-\eta)}\left(\dfrac{d}{dt}\right)^{m}\displaystyle\int_0^t (t-\delta)^{m-\eta-1}y(\delta)d\delta, \ \xi=1, m-1<\eta<m, \\ \dfrac{d^m y}{dt^m}, \ \xi=1, \ \eta = m. \end{cases} $$

Definition 3. *[10] Let y is continuous function, then the fractals fractional integral is defined as follows:*

$$ {}^{RFF}I^{\eta,\xi}f(t) = \frac{\xi}{\Gamma(\eta)}\int_0^t (t-\delta)^{\eta-1}\delta^{\xi-1}f(\delta)d\delta, $$

where ξ is fractal dimension and η is fractional order.

Lemma 1. *[2] The solution of FDEs*

$$^{FD}D^{\eta}r(t) - \xi t^{\xi-1}e(t) = 0, \;\; r(0) = r_0,$$

is given by

$$r(t) = r_0 +^{FI} I^{\eta}\xi t^{\xi-1}e(t).$$

Definition 4. ***(Fixed Point:)****[29] Let $\mathfrak{U} \times \mathfrak{V}$ be a norm linear space and mapping $H : \mathfrak{U} \times \mathfrak{V} \to \mathfrak{U} \times \mathfrak{V}$, then $(r,p) \in \mathfrak{U} \times \mathfrak{V}$ is be fixed point, if $H(r,p) = (r,p)$*

Definition 5. *[29]* ***(Equi-continuous:)*** *Let a bounded subset B of $\mathfrak{U} \times \mathfrak{V}$ and (r_n, p_n) have a sequence in B for several (r_n, p_n) in B and for all possible $\epsilon > 0$*

$$|\zeta(r_n, p_n)(t) - \zeta(r_n, p_n)(\tau)| \leq \epsilon$$

Theorem 1. *[29] Let $\mathfrak{U} \times \mathfrak{V}$, be a Banach space and mapping $H : \mathfrak{U} \times \mathfrak{V} \to \mathfrak{U} \times \mathfrak{V}$, is contraction, with $H(r,p) = (r,p)$, then H has a unique fixed point in $\mathfrak{U} \times \mathfrak{V}$.*

Theorem 2. ***(Krasnoselskii Theorem:)****[30] Let B be a Banach space and H and G be two operators, such that*

- *H is contraction.*
- *G is compact operator.*
- *Then $p = (H + G)p$ has a fixed point.*

3. Existence of solution

This section of research work is devoted to EUS for the proposed coupled system of FFDEs (1).

Lemma 2. *Let $t \in I$ and $\eta, \xi \in (0,1]$, then the solution of coupled system of FFDEs*

$$\begin{cases} ^{FD}D^{\eta}r(t) = \xi t^{\xi-1} f\left(t, r(t), p(t)\right), \\ ^{FD}D^{\eta}p(t) = \xi t^{\xi-1} g(t, r(t), p(t)), \\ r(0) = r_0 + \phi(r), \\ p(0) = p_0 + \psi(p), \end{cases} \tag{2}$$

is provided by

$$\begin{cases} r(t) = r_0 + \phi(r) + \dfrac{\xi}{\Gamma(\eta)} \displaystyle\int_0^t (t-\delta)^{\eta-1}\delta^{\xi-1} f(\delta, r(\delta), p(\delta))d\delta, \\ p(t) = p_0 + \psi(p) + \dfrac{\xi}{\Gamma(\eta)} \displaystyle\int_0^t (t-\delta)^{\eta-1}\delta^{\xi-1} g(\delta, r(\delta), p(\delta))d\delta. \end{cases} \tag{3}$$

Proof. Let us consider the coupled system of FFDEs (2) in the form of linear equations as:

$$\begin{cases} {}^{FD}D^{\eta}r(t) = \xi t^{\xi-1} f\left(t, r(t), p(t)\right) = \theta_1(t), \\ {}^{FD}D^{\eta}p(t) = \xi t^{\xi-1} g(t, r(t), p(t)) = \theta_2(1), \\ r(0) = r_0 + \phi(r), \\ p(0) = p_0 + \psi(p), \end{cases} \tag{4}$$

where $\theta_1(t),\ \theta_2(t) : I \to R$ are continuous functions.

Applying ${}^{FI}I^{\eta}$ on coupled system of FFDEs (4) and in view of Lemma 1, we obtain

$$\begin{cases} r(t) = h_0 +^{FI} I^{p} \xi t^{\xi-1} \theta_1(t), \\ p(t) = h_0^* +^{FI} I^{\eta} \xi t^{\xi-1} \theta_2(t). \end{cases} \tag{5}$$

Using the condition $r(0) = r_0 + \phi(r)$ and $p(0) = p_0 + \psi(t, e(t))$, we obtain equation (5) in the form of

$$\begin{cases} r(t) = r_0 + \phi(r) + \dfrac{\xi}{\Gamma(\eta)} \displaystyle\int_0^t (t-\delta)^{\eta-1} \delta^{\xi-1} \theta_1(\delta) d\delta. \\ p(t) = p_0 + \psi(p) + \dfrac{\xi}{\Gamma(\eta)} \displaystyle\int_0^t (t-\delta)^{\eta-1} \delta^{\xi-1} \theta_2(\delta) d\delta. \end{cases} \tag{6}$$

In-sight of system of equations (4), the system of equations (6) is equivalent to the following:

$$\begin{aligned} r(t) &= r_0 + \phi(r) + \frac{\xi}{\Gamma(\eta)} \int_0^t (t-\delta)^{\eta-1} \delta^{\xi-1} f(\delta, r(\delta), p(\delta)) d\delta. \\ p(t) &= p_0 + \psi(p) + \frac{\xi}{\Gamma(\eta)} \int_0^t (t-\delta)^{\eta-1} \delta^{\xi-1} g(\delta, r(\delta), p(\delta)) d\delta. \end{aligned} \tag{7}$$

The above system (7), represents the integral representation for our proposed problem (2).

Let us define the following operators, $H : \mathfrak{U} \times \mathfrak{V} \to \mathfrak{U} \times \mathfrak{V}$, such that

$$H(r, p)(t) = (H_1 r(t), H_2 p(t)), \tag{8}$$

. where

$$\begin{aligned} H_1 r(t) &= r_0 + \phi(r), \\ H_2 p(t) &= p_0 + \psi(p). \end{aligned}$$

Also $G^{**} : \mathfrak{U} \times \mathfrak{V} \to \mathfrak{U} \times \mathfrak{V}$, such that

$$G^{**}(r, p)(t) = \left(G_1^*(r, p)(t), G_2^*(r, p)(t)\right), \tag{9}$$

where

$$\begin{aligned} G_1^*(r, p)(t) &= \frac{\xi}{\Gamma(\eta)} \int_0^t (t-\delta)^{\eta-1} \delta^{\xi-1} f(\delta, r(\delta), p(\delta)) d\delta, \\ G_2^*(r, p)(t) &= \frac{\xi}{\Gamma(\eta)} \int_0^t (t-\delta)^{\eta-1} \delta^{\xi-1} g(\delta, r(\delta), p(\delta)) d\delta. \end{aligned}$$

Furthermore, let us define $K = H + G^{**}$, then the operator equation corresponding to system of equation (6) as:

$$(r,p) = K(r,p)(t) = H(r,p)(t) + G^{**}(r,p)(t),$$

which is a solution of the system in operator form.

Now we present some hypothesis which are helpful in building our main existence result.

(M_1) There exist some constant L_ϕ and $L_\psi \in (0,1]$ such that

$$|\phi(r) - \phi(\overline{r})| \leq L_\phi |r - \overline{r}|,$$
$$|\psi(p) - \psi(\overline{p})| \leq L_\psi |p - \overline{p}|.$$

(M_2) There exist some constant $c_1, c_2, d_1, d_2 > 0$ and $c_3, d_3 \geqslant 0$, with

$$|f(t, r(t), p(t))| \leq c_1|r| + c_2|p| + c_3,$$
$$|g(t, r(t), p(t)| \leq d_1|r| + d_2|p| + d_3.$$

(M_3) There exist L_f and L_g such that

$$||f(t,r,p)(t) - f(t, \overline{r}(t), \overline{r}(t)|| \leq L_f \left(||r - \overline{r}|| + ||p - \overline{p}||\right),$$
$$||g(t,r,p)(t) - g(t, \overline{r}(t), \overline{p}(t)|| \leq L_g \left(||r - \overline{r}|| + ||p - \overline{p}||\right).$$

Theorem 3. *Under the assumption (M_1) and (M_2) the coupled system of FFDE (2) has at lest one solution.*

Proof. To obtained results, we needs the following:

Step1: We need to show that H is contraction mapping, so for any $(r,p), (\overline{r}, \overline{p})$.

$$\begin{aligned} ||H(r,p)(t) - H(\overline{r},\overline{p})(t)|| &\leq ||\phi(r) - \phi(\overline{r})|| + ||\psi(p) - \psi(\overline{p}|| \\ &\leq L_\phi |r - \overline{r}| + L_\psi |p - \overline{p}|, \end{aligned}$$

which implies that

$$||H(r,p)(t) - H(\overline{r},\overline{p})(t)|| \leq L||(r,p) - (\overline{r},\overline{p})||, \tag{10}$$

where $L = \max(L_\phi, L_\psi)$. Hence H is contraction.

Step2: Next, we will to prove G^{**} is bounded , for this let $S = \{||(r,p)|| \leq r; (r,p)\}$ be closed and bounded set, then

$$\begin{aligned} \|G^{**}(r,p)(t)\| &= \left\| \frac{\xi}{\Gamma(\eta)} \int_0^t (t-\delta)^{\eta-1}\delta^{\xi-1} f(\delta, r(\delta), p(\delta))d\delta + \frac{\xi}{\Gamma(\eta)} \int_0^t (t-\delta)^{\eta-1}\delta^{\xi-1} g(\delta, r(\delta), p(\delta))d\delta \right\| \\ &\leq \left\| \frac{\xi}{\Gamma(\eta)} \int_0^t (t-\delta)^{\eta-1}\delta^{\xi-1} f(\delta, r(\delta), p(\delta))d\delta \right\| + \left\| \frac{\xi}{\Gamma(\eta)} \int_0^t (t-\delta)^{\eta-1}\delta^{\xi-1} g(\delta, r(\delta), p(\delta))d\delta \right\| \\ &\leq T^{\eta+\xi-1}\mathbf{B}(\eta,\xi)(c_1|r| + c_2|p| + c_3) + T^{\eta+\xi-1}\mathbf{B}(\eta,\xi)(d_1|r| + d_2|p| + d_3), \\ &= T^{\eta+\xi-1}\mathbf{B}(\eta,\xi)\left[(c_1+d_1)|r| + (c_2+d_2)|p| + (c_3+d_3)\right] \leq \infty. \end{aligned}$$

Thus, G^{**} is bounded. For the equi-continuity of G, let $\tau < t \in I$, and

$$\begin{aligned}\|G_1^*(r,p)(t) - G_1^*(r,p)(\tau)\| &= \left\| \frac{\xi}{\Gamma(\eta)} \int_0^t (t-\delta)^{\eta-1}\delta^{\xi-1} f(\delta, r(\delta), p(\delta)) d\delta \right. \\ &\quad \left. - \frac{\xi}{\Gamma(\eta)} \int_0^\tau (\tau-\delta)^{\eta-1}\delta^{\xi-1} f(\delta, r(\delta), p(\delta)) d\delta \right\| \\ &\leq \|f(\delta, r(\delta), p(\delta))\| \left(\int_0^t (t-\delta)^{\eta-1}\delta^{\xi-1} d\delta - \int_0^\tau (\tau-\delta)^{\eta-1}\delta^{\xi-1} d\delta \right) \\ &\leq \|f(\delta, r(\delta), p(\delta))\| \mathbf{B}(\eta,\xi)(t^{\eta+\xi-1} - \tau^{\eta+\xi-1}) \to 0 \;\text{ as }\; t \to \tau.\end{aligned}$$

It follow that

$$\|G_1^*(r,p)(t) - G_1^*(r,p)(\tau)\| \to 0 \;\text{ as }\; t \to \tau.$$

Similarly, one can obtained the following result for:

$$\|G_2^*(r,p)(t) - G_2^*(r,p)(\tau)\| \to 0 \;\text{ as }\; t \to \tau.$$

Therefore G_1^* and G_1^* are continuous and hence G^{**} is continuous and also bounded. Thanks to Krasnoselskii's FPT the system (2) has at lest one solution.

Theorem 4. *Under the assumptions* (M_1) *and* (M_3) *and such* $\Upsilon = L + WT^{\eta+\xi-1}\mathbf{B}(\eta,\xi) \leq 1$ *holds, then the consider system FFDE have unique solution.*

Proof. For this let $(r,p), (\overline{r}, \overline{p}) \in \mathfrak{U} \times \mathfrak{V}$, then from equation (10), one have

$$||H(r,p)(t) - H(\overline{r},\overline{p})(t)|| \leq L||(r,p) - (\overline{r},\overline{p})||, \tag{11}$$

and

$$\begin{aligned}\|G_1^*(r,p)(t) - G_1^*(\overline{r},\overline{p})(t)\| &= \left\| \frac{\xi}{\Gamma(\eta)} \int_0^t (t-\delta)^{\eta-1}\delta^{\xi-1} f(\delta, r(\delta), p(\delta)) d\delta \right. \\ &\quad \left. - \frac{\xi}{\Gamma(\eta)} \int_0^t (t-\delta)^{\eta-1}\delta^{\xi-1} f(\delta, \overline{r}(\delta), \overline{p}(\delta)) d\delta \right\|, \\ &= \frac{\xi}{\Gamma(\eta)} \int_0^T (t-\delta)^{\eta-1}\delta^{\xi-1} \|f(\delta, r(\delta), p(\delta)) - f(\delta, \overline{r}(\delta), \overline{p}(\delta))\| d\delta, \\ &\leq \frac{\xi}{\Gamma(\eta)} T^{\eta+\xi-1}\mathbf{B}(\eta,\xi) L_f(||r-\overline{r}|| + ||p-\overline{p}||,\end{aligned}$$

which implies that

$$\|G_1^*(r,p)(t) - G_1^*(\overline{r},\overline{p})(t)\| \leq T^{\eta+\xi-1}\mathbf{B}(\eta,\xi) L_f \|(r,p)(t) - (\overline{r},\overline{p})(t)\|, \tag{12}$$

and

$$\|G_2^*(r,p)(t) - G_2^*(\overline{r},\overline{p})(t)\| \leq T^{\eta+\xi-1}\mathbf{B}(\eta,\xi) L_g \|(r,p)(t) - (\overline{r},\overline{p})(t)\|. \tag{13}$$

Hence from (12) and (13), we have

$$\begin{aligned}\|G^{**}(r,p)(t)-G^{**}(\overline{r},\overline{p})(t)\| \leq\ & T^{\eta+\xi-1}\mathbf{B}(\eta,\xi)L_f\|(r,p)(t)-(\overline{r},\overline{p})(t)\| \\ &+T^{\eta+\xi-1}\mathbf{B}(\eta,\xi)L_g\|(r,p)(t)-(\overline{r},\overline{p})(t)\|, \\ \leq\ & T^{\eta+\xi-1}\mathbf{B}(\eta,\xi)(L_f+L_g)|(r,p)(t)-(\overline{r},\overline{p})(t)|.\end{aligned}$$

which implies that

$$\|G^{**}(r,p)(t)-G^{**}(\overline{r},\overline{p})(t)\| \leq T^{\eta+\xi-1}\mathbf{B}(\eta,\xi)W\|(r,p)(t)-(\overline{r},\overline{p})(t)\|, \tag{14}$$

where $W=L_f+L_g$. Therefore, from (11) and (14), we have

$$\begin{aligned}\|K(r,p)(t)-K(\overline{r},\overline{p})(t)\| &\leq L\|(r,p)(t)-(\overline{r},\overline{p})(t)\|+WT^{\eta+\xi-1}\mathbf{B}(\eta,\xi)\|(r,p)(t)-(\overline{r},\overline{p})(t)\| \\ &\leq \Upsilon\|(r,p)(t)-(\overline{r},\overline{p})(t)\|,\end{aligned}$$

where $\Upsilon=L+WT^{\eta+\xi-1}\mathbf{B}(\eta,\xi)$. Therefore, K is contraction, thus the system (2) has a unique solution.

4. Stability analysis

This section, is devoted to establishing stability results for the considered coupled system of FFDEs (2). We contend that UH stability idea is important for practical issues in analysis. The interesting feature of stability is that to search a UH stability of a system, who exact solution does not exist, which is typically challenging or time consuming. According to UH stability, there exist a close approximate solution of system to exact solution. As we know that mostly a mathematical models are non-linear in nature and some time its exact solution does not exist or difficult to be obtained. Therefore, we need to find best approximate solution for such problems.

Definition 6. *To obtained the UH stability of the system (2), suppose that exist* $\epsilon=\max(\epsilon_1,\epsilon_2)>0$*, and the system of inequality given by*

$$\begin{cases}|^{FD}D^{\eta}r(t)-\xi t^{\xi-1}f(t,r(t),p(t))|\leq\epsilon_1,\\ |^{FD}D^{\eta}p(t)-\xi t^{\xi-1}g(t,r(t),p(t))|\leq\epsilon_2.\end{cases} \tag{15}$$

Then our system (2) is UH stable, $\exists C_{(1,2)}>0$*, with every solution* (r,p) *of inequality equation (15) there exist unique solution* $(\overline{r},\overline{p})$ *with*

$$|(r,p)(t)-(\overline{r},\overline{p})(t)|\leq\epsilon C_{(1,2)}. \tag{16}$$

Remark : We say that (r,p) is a solution of the system (15), if there exist $\omega_1,\omega_2\in C(I,R)$ which defend upon (r,p), such that

- $|\omega_1(t)|\leq\epsilon_1$,

- $|\omega_2(t)| \leq \epsilon_2$.

and the perturbed system of coupled system of FFDEs as:

$$\begin{cases} {}^{FD}D^{\eta}r(t) = \xi t^{\xi-1}f(t, r(t), p(t)) + \xi t^{\xi-1}\omega_1(t), \\ {}^{FD}D^{\eta}p(t) = \xi t^{\xi-1}g(t, r(t), p(t)) + \xi t^{\xi-1}\omega_2(t). \end{cases} \tag{17}$$

Lemma 3. *Suppose that (r, p) be the solution of the inequality (15), then the following system of inequality hold.*

$$\left| r(t) - \left((r_0 + \phi(r) + \frac{\xi}{\Gamma(\eta)} \int_0^t (t-\delta)^{\eta-1}\delta^{\xi-1}f(\delta, r_i(\delta), p_i(\delta))d\delta) \right) \right| \leq \delta\epsilon_1,$$
$$\left| p(t) - \left((p_0 + \psi(p) + \frac{\xi}{\Gamma(\eta)} \int_0^t (t-\delta)^{\eta-1}\delta^{\xi-1}g(\delta, x_i(\delta), p_i(\delta))d\delta) \right) \right| \leq \delta\epsilon_2.$$

Proof. In-view of Remark(4), we can express the consider system as:

$$\begin{cases} {}^{FD}D^{\eta}r(t) = \xi t^{\xi-1}f(t, r(t), p(t)) + \xi t^{\xi-1}\omega_1(t), \\ {}^{FD}D^{\eta}p(t) = \xi t^{\xi-1}g(t, r(t), p(t)) + \xi t^{\xi-1}\omega_2(t), \\ r(0) = (r_0 + \phi(r)), \\ p(0) = (p_0 + \psi(p)). \end{cases}$$

In light of Lemma 2, we can obtained the following results:

$$r(t) = (r_0 + \phi(r)) + \frac{\xi}{\Gamma(\eta)} \int_0^t (t-\delta)^{\eta-1}\delta^{\xi-1}f(\delta, r(\delta), p(\delta))d\delta) + \frac{\xi}{\Gamma(\eta)} \int_0^t (t-\delta)^{\eta-1}\delta^{\xi-1}\omega_1(\delta)d\delta,$$
$$p(t) = (p_0 + \psi(p)) + \frac{\xi}{\Gamma(\eta)} \int_0^t (t-\delta)^{\eta-1}\delta^{\xi-1}g(\delta, r(\delta), p(\delta))d\delta) + \frac{\xi}{\Gamma(\eta)} \int_0^t (t-\delta)^{\eta-1}\delta^{\xi-1}\omega_2(\delta)d\delta.$$

Now

$$\begin{aligned} \left| r(t) - \left(r_0 + \phi(r) + \frac{\xi}{\Gamma(\eta)} \int_0^t (t-\delta)^{\eta-1}\delta^{\xi-1}f(\delta, r(\delta), p(\delta))d\delta) \right) \right| &= \left| \frac{\xi}{\Gamma(\eta)} \int_0^t (t-\delta)^{\eta-1}\delta^{\xi-1}\omega_1(\delta)d\delta \right| \\ &\leq \frac{\xi}{\Gamma(\eta)} \int_0^T (t-\delta)^{\eta-1}\delta^{\xi-1}\epsilon_1 d\delta \\ &\leq \mathbf{B}(\eta, \xi)T^{\eta+\xi-1}\epsilon_1 \\ &= \delta\epsilon_1, \end{aligned}$$

where $\delta = \mathbf{B}(\eta, \xi)T^{\eta+\xi-1}$. Similarly, one can obtain the following result for second compartment of proposed model:

$$\left| p(t) - \left(p_0 + \psi(p) + \frac{\xi}{\Gamma(\eta)} \int_0^t (t-\delta)^{\eta-1}\delta^{\xi-1}f(\delta, r(\delta), p(\delta))d\delta \right) \right| \leq \delta\epsilon_2,$$

which completes the proof

Theorem 5. *Under the assumptions* (M_1)*-*(M_3)*, the solution of coupled system (2) is UH and GUH stable, if* $\mathbf{c}_1\mathbf{c}_2 L_f L_g \neq 1$.

Proof. Let (r, p) be the arbitrary solution and (τ, κ) be the unique solution of (18):

$$\begin{cases} {}^{FD}D^{\eta,\xi} r(t) = \xi t^{\xi-1} f(t, r(t), p(t)), \\ {}^{FD}D^{\eta,\xi} p(t) = \xi t^{\xi-1} g(t, r(t), p(t)), \\ r(0) = r_0 + \phi(r), \\ p(0) = p_0 + \psi(p). \end{cases} \tag{18}$$

The solution of system (18) is given by:

$$\tau(t) = (r_0 + \phi(\tau)) + \frac{\xi}{\Gamma(\eta)} \int_0^t (t-\delta)^{\eta-1} \delta^{\xi-1} f(\delta, \tau(\delta), \kappa(\delta)) d\delta,$$
$$\kappa(t) = (p_0 + \psi(\kappa)) + \frac{\xi}{\Gamma(\eta)} \int_0^t (t-\delta)^{\eta-1} \delta^{\xi-1} g(\delta, \tau(\delta), \kappa(\delta)) d\delta.$$

Now

$$\begin{aligned} |r(t) - \tau(t)| &= \Bigg| (r_0 + \phi(r) + \frac{\xi}{\Gamma(\eta)} \int_0^t (t-\delta)^{\eta-1} \delta^{\xi-1} f(\delta, r(\delta), p(\delta)) d\delta) - (r_0 + \phi(\tau) \\ &\quad + \frac{\xi}{\Gamma(\eta)} \int_0^t (t-\delta)^{\eta-1} \delta^{\xi-1} f(\delta, \tau(\delta), \kappa(\delta)) d\delta) \Bigg|, \\ &= \Bigg| r(t) - \left(r_0 + \phi(r) + \frac{\xi}{\Gamma(\eta)} \int_0^t (t-\delta)^{\eta-1} \delta^{\xi-1} f(\delta, r(\delta), p(\delta)) d\delta \right) \\ &\quad + (r_0 + \phi(r) + \frac{\xi}{\Gamma(\eta)} \int_0^t (t-\delta)^{\eta-1} \delta^{\xi-1} f(\delta, r(\delta), p(\delta)) d\delta)| \\ &\quad - \left(r_0 + \phi(\tau) + \frac{\xi}{\Gamma(\eta)} \int_0^t (t-\delta)^{\eta-1} \delta^{\xi-1} f(\delta, \tau(\delta), \kappa(\delta) d\delta \right) \Bigg|, \\ &\leq \delta\epsilon_1 + |\phi(r) - \phi(\tau)| + \beta(\eta, \xi) T^{\eta+\xi-1} |f(\delta, t(\delta), p(\delta)) d\delta) - f(\delta, \tau(\delta), \kappa(\delta) d\delta|, \\ &\leq \delta\epsilon_1 + (L_\phi + \delta L_f) |r(t) - \tau(t)| + \delta L_f |p(t) - \kappa(t)|, \end{aligned}$$

Hence, we have

$$|r(t) - \tau(t)| - \frac{\delta L_f}{1 - (L_\phi + \delta L_f)} |p(t) - \kappa(t)| \leq \frac{\delta\epsilon_1}{1 - (L_\phi + \delta L_f)}. \tag{19}$$

Similarly, one can obtain the following result

$$|p(t) - \kappa(t)| - \frac{\delta L_g}{1 - (L_\psi + \delta L_g)} |r(t) - \tau(t)| \leq \frac{\delta\epsilon_2}{1 - (L_\psi + \delta L_g)}. \tag{20}$$

The matrix representation the above inequalities as:

$$\begin{pmatrix} 1 & -L_f\mathbf{c}_1 \\ -L_g\mathbf{c}_2 & 1 \end{pmatrix} \times \begin{pmatrix} |r-\tau| \\ |p-\kappa| \end{pmatrix} \leq \begin{pmatrix} \mathbf{c}_1\epsilon_1 \\ \mathbf{c}_2\epsilon_2, \end{pmatrix},$$

where $\mathbf{c}_1 = \frac{\delta}{1-(L_\phi+\delta L_f)}$ and $\mathbf{c}_2 = \frac{\delta}{1-(L_\psi+\delta L_g)}$, then solving the above matrix, we have

$$|r-\tau| \leq \frac{\mathbf{c}_1\epsilon_1}{1-(\mathbf{c}_1\mathbf{c}_2L_fL_g)} + \frac{\mathbf{c}_1\mathbf{c}_2L_f\epsilon_2}{1-(\mathbf{c}_1\mathbf{c}_2L_fL_g)}, \tag{21}$$

and

$$|p-\kappa| \leq \frac{\mathbf{c}_2\epsilon_2}{1-(\mathbf{c}_1c_{\mathbf{2}}L_fL_g)} + \frac{\mathbf{c}_1\mathbf{c}_2L_g\epsilon_1}{1-(\mathbf{c}_1\mathbf{c}_2L_fL_g)} \tag{22}$$

Adding (21) and (22), we gets

$$\begin{aligned} |r-\tau| + |p-\kappa| \leq & \frac{\mathbf{c}_1\epsilon_1}{1-(\mathbf{c}_1\mathbf{c}_2L_fL_g)} + \frac{\mathbf{c}_1\mathbf{c}_2L_f\epsilon_2}{1-(\mathbf{c}_1\mathbf{c}_2L_fL_g)} \\ & + \frac{\mathbf{c}_2\epsilon_2}{1-(\mathbf{c}_1\mathbf{c}_2L_fL_g)} + \frac{\mathbf{c}_1\mathbf{c}_2L_g\epsilon_1}{1-(\mathbf{c}_1\mathbf{c}_2L_fL_g)}, \\ \leq & \frac{(\mathbf{c}_1+\mathbf{c}_2+\mathbf{c}_1\mathbf{c}_2(L_f+L_g))\epsilon}{1-(\mathbf{c}_1\mathbf{c}_2L_fL_g)}, \end{aligned}$$

which implies that

$$|(r,p)-(\tau,\kappa)| \leq C_{(1,2)}\epsilon, \tag{23}$$

where

$$C_{(1,2)} = \frac{(\mathbf{c}_1+\mathbf{c}_2+\mathbf{c}_1\mathbf{c}_2(L_f+L_g))}{1-(\mathbf{c}_1\mathbf{c}_2L_fL_g)},$$

Thus, the solution of (2) is UH stable. Further, the solution of coupled system of FFDEs (2) is GUH stable,if $\psi : (0,1) \to (0,\infty)$ such that $\psi(\epsilon) = \epsilon$,be a non decreasing function,then we have

$$|(r,p)-(\tau,\kappa)| \leq C_{(1,2)}\psi(\epsilon), \ \ where \ \ \psi(0) = 0. \tag{24}$$

Thus CS is (2) GUH stable

For any function b,we assume the following inequalities holds

$$I^\eta b_i(t) \leq \lambda_{in}b_i(t), \text{where } \ b_i = \{1,2\}$$

Lemma 4. *The solution (r,p) for the selected coupled system of FFDEs*

$$\begin{cases} {}^{FD}D^\eta r(t) = \xi t^{\xi-1}f(t,r(t),p(t)) + \xi t^{\xi-1}b_1(t), \\ {}^{FD}D^\eta p(t) = \xi t^{\xi-1}g(t,r(t),p(t)) + \xi t^{\xi-1}b_2(t), \end{cases}$$

will obeys the following:

$$\begin{aligned} \left| r(t) - \left((r_0 + \phi(r) + \frac{\xi}{\Gamma(\eta)} \int_0^t (t-\delta)^{\eta-1}\delta^{\xi-1}f(\delta,r_i(\delta),p_i(\delta))d\delta \right) \right| \leq \lambda_{1n}b_1(t)\epsilon_1, \\ \left| p(t) - \left((p_0 + \psi(p) + \frac{\xi}{\Gamma(\eta)} \int_0^t (t-\delta)^{\eta-1}\delta^{\xi-1}g(\delta,r_i(\delta),p_i(\delta))d\delta \right) \right| \leq \lambda_{2n}b_2(t)\epsilon_2. \end{aligned} \tag{25}$$

Proof. Using remark 4, its proof is similar to (3)

Theorem 6. *Under the assumptions* (M_1)-(M_3) *and consider Lemma 4 with the condition* $\mathbf{c}_1\mathbf{c}_2L_fL_g \neq 1$, *then the selected system is both UHR and GUHR stable.*

Proof. Let (r,p) be the arbitrary and (τ,κ) unique solution of the coupled system of FFDEs, then one have

$$\begin{aligned}|r(t)-\tau(t)| &= \Bigg|(r_0+\phi(r)+\frac{\xi}{\Gamma(\eta)}\int_0^t (t-\delta)^{\eta-1}\delta^{\xi-1}f(\delta,r(\delta),p(\delta))d\delta)-(r_0+\phi(\tau)\\ &+\frac{\xi}{\Gamma(\eta)}\int_0^t (t-\delta)^{\eta-1}\delta^{\xi-1}f(\delta,\tau(\delta),\kappa(\delta))d\delta)\Bigg|,\\ &= \Bigg|r(t)-\left(r_0+\phi(r)+\frac{\xi}{\Gamma(\eta)}\int_0^t (t-\delta)^{\eta-1}\delta^{\xi-1}f(\delta,r(\delta),p(\delta))d\delta\right)\\ &+(r_0+\phi(r)+\frac{\xi}{\Gamma(\eta)}\int_0^t (t-\delta)^{\eta-1}\delta^{\xi-1}f(\delta,r(\delta),p(\delta))d\delta)|\\ &-\left(r_0+\phi(\tau)+\frac{\xi}{\Gamma(\eta)}\int_0^t (t-\delta)^{\eta-1}\delta^{\xi-1}f(\delta,\tau(\delta),\kappa(\delta)ds\right)\Bigg|,\\ &\leq \lambda_{in}b(t)\epsilon_1+|(\phi(r-\phi(t,\tau(v))|+\beta(\eta,\xi)T^{\eta+\xi-1}|f(\delta,r(\delta),p(\delta)))-f(\delta,\tau(\delta),\kappa(\delta)|,\\ &\leq \lambda_{in}b_1(t)\epsilon_1+L_\phi|r(t)-\tau(t)|+\delta L_f(||p(t)-\tau(t)|+|p(t)-\kappa(t)|).\end{aligned}$$

From the above, we gets

$$|r(t)-\tau(t)|-\frac{\delta L_f}{1-(L_\phi+\delta L_f)}|p(t)-\kappa(t)| \leq \frac{\lambda_{in}b_1(t)\epsilon_1}{1-(L_\phi+\delta L_f)}. \tag{26}$$

Similarly, one can obtain the following result

$$|p(t)-\kappa(t)|-\frac{\delta L_g}{1-(L_\psi+\delta L_g)}|r(t)-\tau(t)| \leq \frac{\lambda_{in}b_2(t)\epsilon_2}{1-(L_\psi+\delta L_g)}. \tag{27}$$

In matrix form, the inequalities (26) and (27) can be expressed as

$$\begin{pmatrix}1 & -L_f\mathbf{c}_1\\ -L_g\mathbf{c}_2 & 1\end{pmatrix}\times\begin{pmatrix}|r-\tau|\\ |p-\kappa|\end{pmatrix}\leq\begin{pmatrix}\mathbf{c}_1\lambda_{in}b_1(t)\epsilon_1\\ \mathbf{c}_2\lambda_{in}b_2(t)\epsilon_2\end{pmatrix},$$

Let $\max\{\lambda_{in}b_1(t),\lambda_{in}b_2(t)\}=\lambda b(t)$, $\max\{\epsilon_1,\epsilon_2\}=\epsilon$ and solving the above matrix, we get

$$|(r,p)-(\tau,\kappa)| \leq D_{(1,2)}\epsilon\lambda b(t), \tag{28}$$

where

$$D_{(1,2)}=\frac{(\mathbf{c}_1+\mathbf{c}_2+\mathbf{c}_1\mathbf{c}_2(L_f+L_g))\lambda}{1-(\mathbf{c}_1\mathbf{c}_2L_fL_g)},$$

Hence, the solution of coupled system of FFDEs (2) is UHR stable. Now, by using above inequality with

$$D_{(1,2,\epsilon)}=\frac{(\mathbf{c}_1+\mathbf{c}_2+\mathbf{c}_1\mathbf{c}_2(L_f+L_g))\lambda\epsilon}{1-(\mathbf{c}_1\mathbf{c}_2L_fL_g)},$$

we have

$$|(r,p)-(\tau,\kappa)| \leq D_{(1,2,\epsilon)}\lambda b(t). \tag{29}$$

So the coupled system of FFDEs (2) is GUHR stable.

5. Numerical Application

This section of our work, is committed to numerical application of our finding. We present a numerical example to elaborate the main finding of this work.

Example 1. *Consider the coupled system of FFDEs with $t \in [0,T]$ as follows*

$$\begin{cases} {}^{FD}D^{\eta}r(t) = \xi t^{\xi-1}\left[\dfrac{e^{-\pi t}}{42+t^2} + \dfrac{|r(t)|}{e^t(50+t^3)} + \dfrac{t^2|sinp(t)|}{51}\right], \\ {}^{FD}D^{\eta}p(t) = \xi t^{\xi-1}\left[\dfrac{t^3}{75} + \dfrac{e^{-t}cos|r(t)|}{39+t^5} + \dfrac{|p(t)|}{39+|cosp(t)|}\right], \\ r(0) = \dfrac{3}{73} + \dfrac{\sin(r)}{76}, \;\; p(0) = \dfrac{17}{49} + \dfrac{\sqrt{p}}{45} \end{cases} \tag{30}$$

From (30), we have the following

$$\begin{aligned} f(t) &= \xi t^{\xi-1}\left[\frac{e^{-\pi t}}{42+t^2} + \frac{|r(t)|}{e^t(500+t^3)} + \frac{t^2|sinp(t)|}{500}\right], \\ g(t) &= \xi t^{\xi-1}\left[\frac{t^3}{75} + \frac{e^{-t}cos|(t)|}{400+t^5} + \frac{|p(t)|}{400+|cosp(t)|}\right], \\ \phi(0) &= \frac{\sin(r)}{76}, \\ \psi(0) &= \frac{\sqrt{p}}{45}. \end{aligned}$$

From coupled system (30), we can easily obtained, $L_\phi = \frac{1}{76},\ L_\psi = \frac{1}{45},\ c_1 = c_2 = \frac{1}{500}, c_3 = \frac{1}{42}d_1 = d_2 = \frac{1}{400}, d_3 = \frac{1}{75}$ and $L_f = \frac{1}{500},\ L_g = \frac{1}{400.}$

Also, $L = \max\{L_\phi, L_\psi\} = \frac{1}{45} < 1$. Let particularly use $T = 1, 2$, we have Hence, the given coupled system of FFDE (30) has at least one solution. Now further we have

$$L + WT^{\eta+\xi-1}\beta(\eta,\xi) = \frac{1}{45} + T^{\eta+\xi-1}\left(\frac{1}{500} + \frac{1}{400}\right)\mathbf{B}\left(\eta,\xi\right) < 1. \tag{31}$$

Here, we demonstrate graphically to show that the quantity $\Upsilon = L + WT^{\eta+\xi-1}\mathbf{B}\left(\eta,\xi\right) < 1$ for $\eta, \xi \in (0,1]$ in figure 1

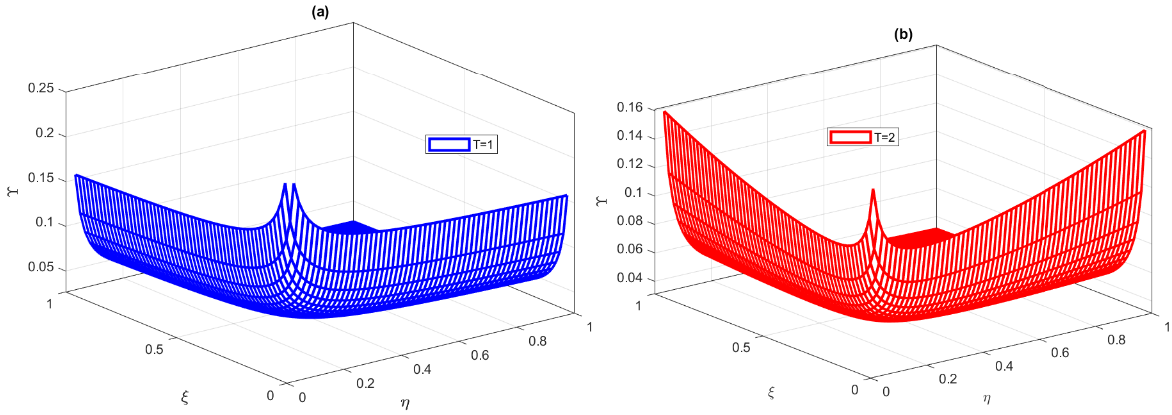

Figure 1: Graphical illustration of quantity Υ in η, ξ space by using $T = 1, 2$.

From figure 1, we see that as the values of ξ, η *increase over* $(0, 1]$*, the value of* Υ *does not exceed 1, which authenticate the result* (31) *for all values* $\eta, \xi \in (0, 1]$*. Hence, the coupled system (30) has a unique solution.*

The problem (30) *is UH and GUH stable, since* $\mathbf{c}_1\mathbf{c}_2 L_f L_g \neq 1$ *for* $\mathbf{c}_1$, $\mathbf{c}_2$*. Similarly, by the same procedure the coupled system* (30) *is UHR and GUHR stable with* $w(t) = s(t) = t$ *for* $t \in (0, 1)$.

6. Conclusion

In this research work, we have investigated a coupled system of FFDEs for existence theory and stability analysis. We have deduced sufficient conditions for the existence, uniqueness of solution to the mentioned problem by using FPT. In addition, appropriate results devoted to UH stability has also been developed. Different kinds of UH stability were deduced for the considered problem. Here, it is remarkable that coupled systems have very rarely investigated for the existence theory of solution and stability analysis. The mentioned type systems have numerous applications in variety of disciplines of science and technology. Also, in the future, the coupled systems of drug therapy, coupled system of mechanical oscillators and coupled system of planetary system can be studied by following our established analysis.

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Competing interest

Does not exist.

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