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The Double Sumudu-Sawi Transform

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Abstract. The paper explores integral transforms and their broader generalizations. The primary aim is to develop a new integral transform that combines the Hybrid Sumudu and Sawi transforms, investigating their properties, existence, and inversion theorem. We introduce recent findings on partial differential equations in higher dimensions and broaden the scope of the double convolution theorem to encompass two-dimensional scenarios. Utilizing these novel properties and theorems, we solve particular types of differential equations, showcasing their practical applications in physics and various scientific domains.

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Key Words and Phrases: Sumudu transform, Sawi transform, The Double Sumudu-Sawi transform.

1. Introduction

Integral transforms are powerful mathematical tools that convert functions into different domains, making them easier to analyze and manipulate. Once transformed, a function can be reverted to its original form using the inverse transform. These transformations play a crucial role in engineering, economics, physics, and chemistry, helping simplify complex real-world problems.

As mathematical challenges grow, researchers continue to develop more general classes of differential equations and innovative analytical techniques. One of the most widely used integral transforms is the Laplace transform, introduced in 1780. In recent years, new transforms such as the Sumudu transform in [1] and Sawi transform in [2] have gained attention for their unique properties and diverse applications.

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Beyond single-variable transforms, double transforms have been developed to handle multi-variable differential equations. Among these are the Double Laplace Transform in [3], Double Mellin-ARA Transform in [4], the Double Sumudu Transform [5], the Double Sumudu-Shehu Transform [6], Double Laplace-Sawi Transform [7], Double Sawi Transform in [8], and the Double ARA-Sawi Transform in [9]. These methods provide new ways to solve high-dimensional equations, making them essential tools in modern mathematical analysis.

In this work, we introduce a novel Double Sumudu-Sawi Transform (DS-SWT) designed to extend the scope of differential equation analysis. We explore its fundamental properties, establishing the conditions for its existence and demonstrating its power in convolution theory and derivative operations. By applying this new transform, we uncover innovative approaches to solving partial differential equations and integro-differential equations, paving the way for more efficient mathematical tools.

2. Sumudu and Sawi transforms

This section offers an overview of the individual transforms, emphasizing the key properties of the Sumudu and Sawi transforms, and highlighting their distinct features and applications.

2.1. Sumudu transform

Definition 1. *The Sumudu transform of a continuous function $r(\xi)$ on $(0,\infty)$ is defined as follows*

$$R(\delta) = S(r(\xi)) = \frac{1}{\delta}\int_{0}^{\infty} e^{-\frac{\xi}{\delta}} r(\xi) d\xi, \delta \in \mathbb{C}.$$

The fundamental properties of the Sumudu transform are presented as follows:

Let $R(\delta) = S(r(\xi))$, then for nonzero constants β and γ, we have

$$S(\beta r_1(\xi) + \gamma r_2(\xi)) = \beta S(r_1(\xi)) + \gamma S(r_2(\xi)), \tag{1}$$

where $r_1(\xi)$ and $r_2(\xi)$ are continuous functions on $(0,\infty)$.

$$S(\xi^{\beta}) = \Gamma(\beta+1)\delta^{\beta}, \tag{2}$$

$$S(e^{\beta\xi}) = \frac{1}{1-\delta\beta},\ \beta \in \mathbb{R}, \tag{3}$$

$$S(r'(\xi)) = \frac{R(\delta)}{\delta} - \frac{r(0)}{\delta}, \tag{4}$$

$$S(r''(\xi)) = \frac{R(\delta)}{\delta^2} - \frac{r(0)}{\delta^2} - \frac{r'(0)}{\delta}. \tag{5}$$

2.2. The Sawi transform

Definition 2. *The Sawi transform of a continuous function $f(\chi)$ on $(0,\infty)$ expressed as follows*

$$F(\epsilon) = W(f(\chi)) = \frac{1}{\epsilon^2}\int_0^\infty e^{-\frac{\chi}{\epsilon}} f(\chi)d\chi.$$

Let us examine the fundamental properties that characterize the Sawi transform.

Suppose that $F_1(\epsilon) = W(f_1(\chi))$ and $F_2(\epsilon) = W(f_2(\chi))$, with u and v as nonzero real numbers, the following properties hold

$$W(uf_1(\chi) + vf_2(\chi)) = uW(f_1(\chi)) + vW(f_2(\chi)), \tag{6}$$

$$W(\chi^u) = \Gamma(u+1)\epsilon^{u-1}, \tag{7}$$

$$W(e^{v\chi}) = \frac{1}{\epsilon\,(1 - v\epsilon)}, \tag{8}$$

$$W(f'(\chi)) = \frac{1}{\epsilon}F(\epsilon) - \frac{1}{\epsilon^2}f(0), \tag{9}$$

$$W(f''(\chi)) = \frac{1}{\epsilon^2}F(\epsilon) - \frac{1}{\epsilon^3}f(0) - \frac{1}{\epsilon^2}f'(0). \tag{10}$$

3. The Double Sumudu-Sawi transform

This section introduces the Double Sumudu-Sawi Transformation (DS-SWT). We begin by outlining its fundamental properties, including linearity and inversion. Next, we present a new result related to partial derivatives, as well as a novel outcome regarding the convolution theorem. Additionally, we explain how these results are applied to compute the DS-SWT of several basic functions. The definition of the DS-SWT is:

$$H(\delta,\epsilon) = S_\xi W_\chi(h(\xi,\chi)) = \frac{1}{\delta\epsilon^2}\int_0^\infty\int_0^\infty e^{-\frac{\xi}{\delta}-\frac{\chi}{\epsilon}} h(\xi,\chi)\ d\xi d\chi, \tag{11}$$

where $h(\xi,\chi)$ is a continuous function on $(0,\infty)\times(0,\infty)$.

Clearly, $S_\xi W_\chi(h(\xi,\chi))$ is linear transformation. In fact, for nonzero constants u and v, we have

$S_\xi W_\chi(uh_1(\xi,\chi){+}vh_2(\xi,\chi))$

$$
\begin{aligned}
&= \frac{1}{\delta\epsilon^2}\int_0^\infty\int_0^\infty e^{-\frac{\xi}{\delta}-\frac{\chi}{\epsilon}}(uh_1(\xi,\chi)+vh_2(\xi,\chi))\,d\xi d\chi\\
&= \frac{u}{\delta\epsilon^2}\int_0^\infty\int_0^\infty e^{-\frac{\xi}{\delta}-\frac{\chi}{\epsilon}}h_1(\xi,\chi)\,d\xi d\chi+\frac{v}{\delta\epsilon^2}\int_0^\infty\int_0^\infty e^{-\frac{\xi}{\delta}-\frac{\chi}{\epsilon}}h_2(\xi,\chi)\,d\xi d\chi\\
&= uS_\xi W_\chi(h_1(\xi,\chi))+vS_\xi W_\chi(h_2(\xi,\chi)).
\end{aligned}
$$

If $h(\xi,\chi)$ can be written as $h(\xi,\chi)=g(\xi)f(\chi)$ for some continuous functions g and f, then $S_\xi W_\chi(h(\xi,\chi))=S(g(\xi))W(f(\chi))$. In fact

$$
\begin{aligned}
S_\xi W_\chi(h(\xi,\chi)) &= S_\xi W_\chi(g(\xi)f(\chi))\\
&= \frac{1}{\delta\epsilon^2}\int_0^\infty\int_0^\infty e^{-\frac{\xi}{\delta}-\frac{\chi}{\epsilon}}g(\xi)f(\chi)d\xi d\chi\\
&= \left(\frac{1}{\delta}\int_0^\infty e^{-\frac{\xi}{\delta}}g(\xi)d\xi\right)\left(\frac{1}{\epsilon^2}\int_0^\infty e^{-\frac{\chi}{\epsilon}}f(\chi)d\chi\right)\\
&= S(g(\xi))W(f(\chi)).
\end{aligned}
$$

3.1. DS-SWT for some basic functions

(i)

$$
S_\xi W_\chi(1)=\frac{1}{\delta\epsilon^2}\int_0^\infty\int_0^\infty e^{-\frac{\xi}{\delta}-\frac{\chi}{\epsilon}}d\xi d\chi
$$

$$
=\left(\frac{1}{\delta}\int_0^\infty e^{-\frac{\xi}{\delta}}d\xi\right)\left(\frac{1}{\epsilon^2}\int_0^\infty e^{-\frac{\chi}{\epsilon}}d\chi\right)=1\times\frac{1}{\epsilon}=\frac{1}{\epsilon},\ \mathrm{Re}(\frac{1}{\delta})>0.
$$

(ii)

$$
S_\xi W_\chi(\xi^u\chi^v)=\frac{1}{\delta\epsilon^2}\int_0^\infty\int_0^\infty e^{-\frac{\xi}{\delta}-\frac{\chi}{\epsilon}}\xi^u\chi^v d\xi d\chi
$$

$$
=\left(\frac{1}{\delta}\int_0^\infty \xi^u e^{-\frac{\xi}{\delta}}d\xi\right)\left(\frac{1}{\epsilon^2}\int_0^\infty \chi^v e^{-\frac{\chi}{\epsilon}}d\chi\right)
$$

$$
=\Gamma(v+1)\delta^u\times\Gamma(v+1)\epsilon^{v-1}
$$

$$
=\delta^u\epsilon^{v-1}\Gamma(u+1)\Gamma(v+1),\ \mathrm{Re}(\frac{1}{\delta})>0 \text{ and } \mathrm{Re}(u)>-1.
$$

(iii)

$$S_\xi W_\chi(e^{u\xi+v\chi}) = \frac{1}{\delta\epsilon^2}\int_0^\infty\int_0^\infty e^{-\frac{\xi}{\delta}-\frac{\chi}{\epsilon}} e^{u\xi+v\chi} d\xi d\chi$$

$$= \left(\frac{1}{\delta}\int_0^\infty e^{u\xi-\frac{\xi}{\delta}} d\xi\right)\left(\frac{1}{\epsilon^2}\int_0^\infty e^{v\chi-\frac{\chi}{\epsilon}} d\chi\right) = \frac{1}{1-\delta u} \times \frac{1}{\epsilon\,(1-v\epsilon)}$$

$$= \frac{1}{\epsilon\,(1-\delta u)\,(1-v\epsilon)},\ \operatorname{Re}(\frac{1}{\delta}) > \operatorname{Re}(u).$$

3.2. Existence condition for DS-SWT

Definition 3. *A function $h(\xi,\chi)$ is said to be of exponential orders u and v on $0 \le \xi < \infty$ and $0 \le \chi < \infty$. If there exist $K, X, Y > 0$ such that $|h(\xi,\chi)| \le Ke^{u\xi+v\chi}$, for all $\xi > X$, $\chi > Y$.*

Theorem 1. *Let $h(\xi,\chi)$ be a continuous function on the region $[0,\infty)\times[0,\infty)$ of exponential orders u and v. Then $H(\delta,\epsilon)$ exists for δ, ϵ and γ whenever $\operatorname{Re}\left(\frac{1}{\delta}\right) > u$ and $\operatorname{Re}\left(\frac{1}{\epsilon}\right) > v$.*

Proof. We have

$$\begin{aligned}
|H(\delta,\epsilon)| &= \left|\frac{1}{\delta\epsilon^2}\int_0^\infty\int_0^\infty e^{-\frac{\xi}{\delta}-\frac{\chi}{\epsilon}} h(\xi,\chi)\ d\xi d\chi\right| \le \frac{1}{\delta\epsilon^2}\int_0^\infty\int_0^\infty e^{-\frac{\xi}{\delta}-\frac{\chi}{\epsilon}}\,|h(\xi,\chi)|\, d\xi d\chi \\
&\le K\frac{1}{\delta\epsilon^2}\int_0^\infty\int_0^\infty e^{-\frac{\xi}{\delta}-\frac{\chi}{\epsilon}} e^{u\xi+v\chi} d\xi d\chi = K\left(\int_0^\infty e^{-(\delta-u)\xi} d\xi\right)\left(\frac{1}{\epsilon^2}\int_0^\infty e^{-(\frac{1}{\epsilon}-v)\chi} d\chi\right) \\
&= \frac{K}{\epsilon\,(\delta-u)\,(1-v\epsilon)}.
\end{aligned}$$

where $\operatorname{Re}(\delta) > u$ and $\operatorname{Re}\left(\frac{1}{\epsilon}\right) > v$.

3.3. Derivatives properties

Now, we present some basic properties of the DS-SWT

Let $H(\delta,\epsilon) = S_\xi W_\chi(h(\xi,\chi))$ where $h(\xi,\chi)$ is a continuous function on $(0,\infty)\times(0,\infty)$. Then

(i)

$$S_\xi W_\chi \left(\frac{\partial h(\xi,\chi)}{\partial \xi} \right) = \frac{1}{\delta} H(\delta,\epsilon) - \frac{1}{\delta} W(h(0,\chi)), \tag{12}$$

(ii)

$$S_\xi W_\chi \left(\frac{\partial^2 h(\xi,\chi)}{\partial \xi^2} \right) = \frac{1}{\delta^2} H(\delta,\epsilon) - \frac{1}{\delta^2} W(h(0,\chi)) - \frac{1}{\delta} W(h_\xi(0,\chi)),$$

(iii)

$$S_\xi W_\chi \left(\frac{\partial h(\xi,\chi)}{\partial \chi} \right) = \frac{1}{\epsilon} H(\delta,\epsilon) - \frac{1}{\epsilon^2} S(h(\xi,0)), \tag{13}$$

(iv)

$$S_\xi W_\chi \left(\frac{\partial^2 h(\xi,\chi)}{\partial \chi^2} \right) = \frac{1}{\epsilon^2} H(\delta,\epsilon) - \frac{1}{\epsilon^3} S(h(\xi,0)) - \frac{1}{\epsilon^2} S(h_\chi(\xi,0)), \tag{14}$$

(v)

$$S_\xi W_\chi \left(\frac{\partial^2 h(\xi,\chi)}{\partial \xi \partial \chi} \right) = \frac{1}{\delta\epsilon} H(\delta,\epsilon) - \frac{1}{\delta\epsilon^2} S(h(\xi,0)) - \frac{1}{\delta\epsilon} W(h(0,\chi)) + \frac{1}{\delta\epsilon^2} h(0,0). \tag{15}$$

Proof. (1) $S_\xi W_\chi \left(\frac{\partial h(\xi,\chi)}{\partial \xi} \right) = \frac{1}{\delta\epsilon^2} \int\limits_0^\infty \int\limits_0^\infty e^{-\frac{\xi}{\delta} - \frac{\chi}{\epsilon}} \frac{\partial h(\xi,\chi)}{\partial \xi}\, d\xi d\chi = \frac{1}{\delta\epsilon^2} \int\limits_0^\infty e^{-\frac{\chi}{\epsilon}} \int\limits_0^\infty e^{-\frac{\xi}{\delta}} \frac{\partial h(\xi,\chi)}{\partial \xi} d\xi d\chi.$
By integrating by parts, we get

$S_\xi W_\chi \left(\frac{\partial h(\xi,\chi)}{\partial \xi} \right) = \frac{1}{\delta\epsilon^2} \int\limits_0^\infty e^{-\frac{\chi}{\epsilon}} \left(-h(0,\chi) + \frac{1}{\delta} \int\limits_0^\infty e^{-\frac{\xi}{\delta}} h(\xi,\chi)\, d\xi \right) d\chi$
$= -\frac{1}{\delta\epsilon^2} \int\limits_0^\infty e^{-\frac{\chi}{\epsilon}} h(0,\chi) d\chi + \frac{1}{\delta^2\epsilon^2} \int\limits_0^\infty \int\limits_0^\infty e^{-\frac{\xi}{\delta} - \frac{\chi}{\epsilon}} h(\xi,\chi)\, d\xi d\chi$
$= \frac{1}{\delta} H(\delta,\epsilon) - \frac{1}{\delta} W(h(0,\chi)).$

The proof of Equations 2, 13, 14 and 15 can be obtained in the same manner.

3.4. Convolution Theorem of DS-SWT

Let $F(\xi,\chi)$ represent the Heaviside unit step function, which is defined as follows:

$$F(\xi - u, \chi - v) = \begin{cases} 1, & \xi > u \text{ and } \chi > v \\ 0, & \text{otherwise} \end{cases}$$

Then we have the following lemma

Lemma 1. *Let $h(\xi,\chi)$ be a continuous function on $(0,\infty)\times(0,\infty)$ and $F(\xi,\chi)$ be the Heaviside unit step function. Then $S_\xi W_\chi(h(\xi-u,\chi-v)F(\xi-u,\chi-v)) = e^{-\frac{u}{\delta}-\frac{v}{\epsilon}} S_\xi W_\chi(h(\xi,\chi)$.*

Proof. We have

$$S_\xi W_\chi(h(\xi-u,\chi-v)F(\xi-u,\chi-v)) \tag{16}$$

$$= \frac{1}{\delta\epsilon^2}\int_0^\infty\int_0^\infty e^{-\frac{\xi}{\delta}-\frac{\chi}{\epsilon}} h(\xi-u,\chi-v)F(\xi-u,\chi-v)d\xi d\chi$$

$$= \frac{1}{\delta\epsilon^2}\int_u^\infty\int_v^\infty e^{-\frac{\xi}{\delta}-\frac{\chi}{\epsilon}} h(\xi-u,\chi-v)d\xi d\chi.$$

Now, by making the substitution $z=\xi-u$ and $w=\chi-v$, equation 16 becomes:

$$\begin{aligned} S_\xi W_\chi(h(\xi-u,\chi-v)F(\xi-u,\chi-v)) &= \frac{1}{\delta\epsilon^2}\int_0^\infty\int_0^\infty e^{-\frac{(z+u)}{\delta}-\frac{(w+v)}{\epsilon}} h(z,w)dzdw \\ &= e^{-\frac{u}{\delta}-\frac{v}{\epsilon}} S_\xi W_\chi(h(\xi,\chi)). \end{aligned}$$

Definition 4. *Let $h(\xi,\chi)$ and $k(\xi,\chi)$ be continuous functions. We define the convolution in the DS-SWT as*

$$(h**k)(\xi,\chi) = \int_0^\xi\int_0^\chi h(\xi-u,\chi-v)k(u,v)dudv.$$

In the following theorem, we compute DS-SWT of the convolution of two functions

Theorem 2. *Let $H(\delta,\epsilon) = S_\xi W_\chi(h(\xi,\chi))$ and $K(\delta,\epsilon) = S_\xi W_\chi(k(\xi,\chi))$. Then*

$$S_\xi W_\chi((h**k)(\xi,\chi)) = \delta\epsilon^2 H(\delta,\epsilon)K(\delta,\epsilon).$$

Proof.

$S_\xi W_\chi((h**k)(\xi,\chi))$

$$\begin{aligned} &= \frac{1}{\delta\epsilon^2}\int_0^\infty\int_0^\infty e^{-\frac{\xi}{\delta}-\frac{\chi}{\epsilon}} (h**k)(\xi,\chi)d\xi d\chi \\ &= \frac{1}{\delta\epsilon^2}\int_0^\infty\int_0^\infty e^{-\frac{\xi}{\delta}-\frac{\chi}{\epsilon}} \left(\int_0^\xi\int_0^\chi h(\xi-u,\chi-v)k(u,v)dudv\right) d\xi d\chi. \end{aligned} \tag{17}$$

Using the Heaviside unit step function, We can write equation 17 as

$$S_\xi W_\chi((h{*}{*}h)(\xi,\chi))$$
$$= \frac{1}{\delta\epsilon^2}\int_0^\infty\int_0^\infty e^{-\frac{\xi}{\delta}-\frac{\chi}{\epsilon}}\left(\int_0^\infty\int_0^\infty h(\xi-u,\chi-v)F(\xi-u,\chi-v)k(u,v))dudv\right)d\xi d\chi$$
$$= \int_0^\infty\int_0^\infty k(u,v)\left(\frac{1}{\delta\epsilon^2}\int_0^\infty\int_0^\infty e^{-\frac{\xi}{\delta}-\frac{\chi}{\epsilon}}h(\xi-u,\chi-v)F(\xi-u,\chi-v)d\xi d\chi\right)dudv.$$

So by Lemma 1, We have

$$\begin{aligned} S_\xi W_\chi((h**k)(\xi,\chi)) &= H(\delta,\epsilon)\int_0^\infty\int_0^\infty k(u,v)e^{-\frac{u}{\delta}-\frac{v}{\epsilon}}dudv \\ &= \delta\epsilon^2 H(\delta,\epsilon)K(\delta,\epsilon). \end{aligned}$$

In Table 1, we have the DAHT of some basic functions.

Table 1: Table of DAHT

$h(\xi,\chi)$	$S_\xi W_\chi(h(\xi,\chi))$
1	$\frac{1}{\epsilon}$, $\mathrm{Re}(\frac{1}{\delta})>0$
$\xi^u\chi^v$	$\delta^u\epsilon^{v-1}\Gamma(u+1)\Gamma(v+1)$, $\mathrm{Re}(\frac{1}{\delta})>0$ and $\mathrm{Re}(u)>-1$
$e^{u\xi+v\chi}$	$\frac{1}{\epsilon(1-\delta u)(1-v\epsilon)}$, $\mathrm{Re}(\frac{1}{\delta})>\mathrm{Re}(u)$.
$e^{i(u\xi+v\chi)}$	$\frac{-1}{\epsilon(\delta u+i)(i+v\epsilon)}$, $\mathrm{Im}(u)+\mathrm{Re}(\frac{1}{\delta})>0$
$\sin(u\xi+v\chi)$	$\frac{u\delta+\epsilon v}{\epsilon(1+u^2\delta^2)(1+v^2\epsilon^2)}$, $\lvert\mathrm{Im}(u)\rvert<\mathrm{Re}(\frac{1}{\delta})$
$\cos(u\xi+v\chi)$	$\frac{1-\delta\epsilon uv}{\epsilon(1+u^2\delta^2)(1+v^2\epsilon^2)}$, $\lvert\mathrm{Im}(u)\rvert<\mathrm{Re}(\frac{1}{\delta})$
$\sinh(u\xi+v\chi)$	$\frac{u\delta+\epsilon v}{\epsilon(1-\delta^2u^2)(1-v^2\epsilon^2)}$, $\mathrm{Re}(\frac{1}{\delta})>\mathrm{Re}(u)$ and $\mathrm{Re}(\frac{1}{\delta}+u)>0$
$\cosh(u\xi+v\chi)$	$\frac{1+\delta\epsilon uv}{\epsilon(1-\delta^2u^2)(1-v^2\epsilon^2)}$, $\mathrm{Re}(\frac{1}{\delta})>\mathrm{Re}(u)$ and $\mathrm{Re}(\frac{1}{\delta}+u)>0$
$g(\xi)f(\chi)$	$S(g(\xi))W(f(\chi))$
$h(\xi-u,\chi-v)H(\xi-u,\chi-v)$	$e^{-\frac{u}{\delta}-\frac{v}{\epsilon}}S_\xi W_\chi(h(\xi,\chi)$
$(h**k)(\xi,\chi)$	$\delta\epsilon^2 S_\xi W_\chi(h(\xi,\chi))S_\xi W_\chi(k(\xi,\chi))$
$J_0\left(c\sqrt{\xi\chi}\right)$	$\frac{4}{\epsilon(4+c^2\delta\epsilon)}$, $\mathrm{Re}\left(\frac{1}{\delta}+\frac{c^2\epsilon}{4}\right)>0$

4. Applications

In this section, we use the DS-SWT for solving PDEs and Integro PDEs

4.1. DS-SWT for solving PDEs

Consider the PDE of the form

$$A_1 h_{\xi\xi} + A_2 h_{\xi\chi} + A_3 h_{\chi\chi} + A_4 h_{\xi} + A_5 h_{\chi} + A_6 h(\xi,\chi) = k(\xi,\chi) \tag{18}$$

With ICs

$h(\xi,0) = g_1(\xi)$, $h_\chi(\xi,0) = g_2(\xi)$

and BCs

$h(0,\chi) = f_1(\chi)$, $h_\xi(0,\chi) = f_2(\chi)$

and assuming $h(0,0) = \Phi$

Given that $h(\xi,\chi)$ is the unknown function, $k(\xi,\chi)$ is the source term, and $A_1, A_2, ..., A_6$ and Φ are constants, we aim to apply the DS-SWT to Equation 18.

To achieve this, we first apply the single Sumudu transform to the ICs and the single Sawi transform to the BCs.

$S(g_1(\xi)) = G_1(\xi)$, $S(g_2(\xi)) = G_2(\xi)$, $W(f_1(\chi)) = F_1(\chi)$ and $W(f_2(\chi)) = F_2(\chi)$

By applying the DS-SWT to Equation (18), we have

$$\begin{aligned} &A_1 S_\xi W_\chi (h_{\xi\xi}) + A_2 S_\xi W_\chi (h_{\xi\chi}) + A_3 S_\xi W_\chi (h_{\chi\chi}) + A_4 S_\xi W_\chi (h_\xi) \\ &+A_5 S_\xi W_\chi (h_\chi) + A_6 S_\xi W_\chi (h(\xi,\chi)) = S_\xi W_\chi (k(\xi,\chi)) \end{aligned} \tag{19}$$

By the properties of the derivatives in Equations $(12) - (15)$, we get

$$\begin{aligned} &A_1 \left(\frac{1}{\delta^2} H(\delta,\epsilon) - \frac{1}{\delta^2} F_1(\chi) - \frac{1}{\delta} F_2(\chi) \right) \\ &+A_2 \left(\frac{1}{\delta\epsilon} H(\delta,\epsilon) - \frac{1}{\delta\epsilon^2} G_1(\xi) - \frac{1}{\delta\epsilon} F_1(\chi) + \frac{1}{\delta\epsilon^2}\Phi \right) \\ &+A_3 \left(\frac{1}{\epsilon^2} H(\delta,\epsilon) - \frac{1}{\epsilon^3} G_1(\xi) - \frac{1}{\epsilon^2} G_2(\xi) \right) + A_4 \left(\frac{1}{\delta} H(\delta,\epsilon) - \frac{1}{\delta} F_1(\chi) \right) \\ &+A_5 \left(\frac{1}{\epsilon} H(\delta,\epsilon) H(\delta,\epsilon) - \frac{1}{\epsilon^2} G_1(\xi) \right) + A_6 H(\delta,\epsilon) = K(\delta,\epsilon) \end{aligned} \tag{20}$$

Simplify Equation 20 as follows

$$H(\delta,\epsilon)=$$

$$\frac{\left(A_1\frac{1}{\delta^2}+A_2\frac{1}{\delta\epsilon}+A_4\frac{1}{\delta}\right)F_1+A_1\frac{1}{\delta}F_2+\left(A_2\frac{1}{\delta\epsilon^2}+A_3\frac{1}{\epsilon^3}+A_5\frac{1}{\epsilon^2}\right)G_1+A_3\frac{1}{\epsilon^2}G_2-A_2\frac{1}{\delta\epsilon^2}\Phi+K}{A_1\frac{1}{\delta^2}+A_2\frac{1}{\delta\epsilon}+A_3\frac{1}{\epsilon^2}+A_4\frac{1}{\delta}+A_5\frac{1}{\epsilon}+A_6} \tag{21}$$

Example 1. *Consider the Klein-Gordon equation*

$$2h_{\xi\xi}-h_{\chi\chi}-h(\xi,\chi)=5\sinh\xi\cos 2\chi,\ \textit{where}\ \xi,\chi\geq 0,$$

With ICs
$h(\xi,0)=\sinh\xi,\ h_\chi(\xi,0)=0,$
and BCs
$h\left(0,\chi\right)=0,\ h_\xi\left(0,\chi\right)=\cos 2\chi.$

Solution 1. *By applying the single Sumudu transform to the ICs and the single Sawi transform to the BCs, I get*
$G_1=\frac{\delta}{1-\delta^2}$, $G_2=0$, $F_1=0$, $F_2=\frac{1}{\epsilon(1+4\epsilon^2)}$, *and* $K=S_\xi W_\chi\left(5\sinh\xi\cos 2\chi\right)=\frac{5\delta}{\epsilon(1-\delta^2)(1+4\epsilon^2)}$.
Substitute in Equation (21) $A_1=2$, $A_3=-1$, $A_6=-1$, $A_2=A_4=A_5=0$ *and the values of* G_1, G_2, F_1, F_2 *and* K, *we get*

$$H(\delta,\epsilon)=\frac{\frac{2}{\delta\epsilon(1+4\epsilon^2)}-\frac{\delta}{\epsilon^3(1-\delta^2)}+\frac{5\delta}{\epsilon(1-\delta^2)(1+4\epsilon^2)}}{\frac{2}{\delta^2}-\frac{1}{\epsilon^2}-1} \tag{22}$$

$$=\frac{\frac{2\epsilon^2\left(1-\delta^2\right)-\delta^2\left(1+4\epsilon^2\right)+5\delta^2\epsilon^2}{\delta\epsilon^3(1-\delta^2)(1+4\epsilon^2)}}{\frac{2\epsilon^2-\delta^2-\delta^2\epsilon^2}{\delta^2\epsilon^2}}. \tag{23}$$

By simplify, we get

$$H(\delta,\epsilon)=\frac{\delta}{\epsilon\left(1-\delta^2\right)\left(1+4\epsilon^2\right)}.$$

So,

$$h(\xi,\chi)=S_\xi^{-1}W_\chi^{-1}\left(\frac{\delta}{\epsilon\left(1-\delta^2\right)\left(1+4\epsilon^2\right)}\right)=\sinh\xi\cos 2\chi.$$

Its graph is

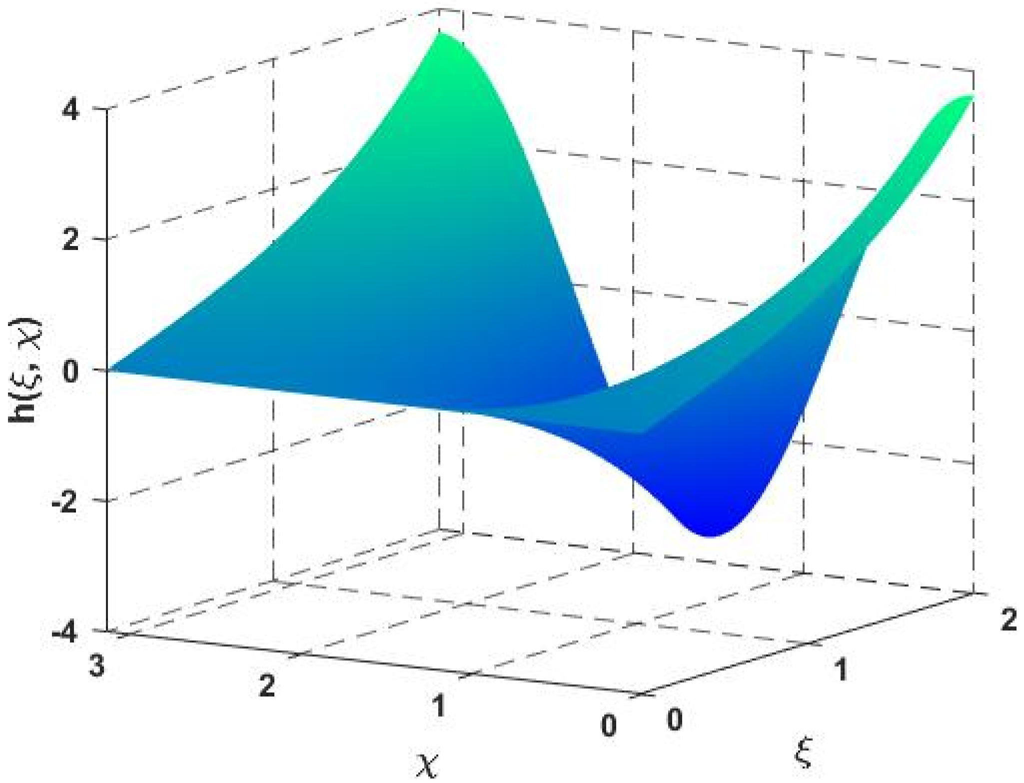

Figure 1: The solution of Example 4.1

Example 2. *Consider the telegraph equation*

$$h_{\xi\xi} - 2h_{\chi\chi} - h_{\xi} = 4h(\xi,\chi), \text{ where } \xi,\chi \geq 0,$$

With ICs
$h(\xi,0) = 0,\ h_{\chi}(\xi,0) = e^{2\xi},$
and BCs
$h\,(0,\chi) = \sin\chi,\ h_{\xi}\,(0,\chi) = 2\sin\chi.$

Solution 2. *By applying the single Sumudu transform to the ICs and the single Sawi transform to the BCs, we get*
$G_1 = 0,\ G_2 = \frac{1}{1-2\delta},\ F_1 = \frac{1}{1+\epsilon^2},\ F_2 = \frac{1}{1+4\epsilon^2}.$
Substitute in Equation (21) $A_1 = 1,\ A_3 = -2,\ A_4 = -1,\ A_6 = -4,\ A_2 = A_5 = 0$ *and the values of* G_1, G_2, F_1 *and* F_2, *we get*

$$H(\delta,\epsilon) = \frac{\left(\frac{1}{\delta^2} - \frac{1}{\delta}\right)\left(\frac{1}{1+\epsilon^2}\right) + \frac{2}{\delta(1+\epsilon^2)} - \frac{2}{\epsilon^2(1-2\delta)}}{\frac{1}{\delta^2} - \frac{2}{\epsilon^2} - \frac{1}{\delta} - 4} \tag{24}$$

$$= \frac{\frac{\epsilon^2(1-\delta)(1-2\delta)+\delta\epsilon^2(1-2\delta)-2\delta^2\left(1+\epsilon^2\right)}{\delta^2\epsilon^2(1-2\delta)(1+\epsilon^2)}}{\frac{\epsilon^2-2\delta^2-\delta\epsilon^2-4\delta^2\epsilon^2}{\delta^2\epsilon^2}}.$$

By simplify, we get

$$H(\delta,\epsilon) = \frac{1}{(1-2\delta)\,(1+\epsilon^2)}.$$

So,

$$h(\xi,\chi) = S_\xi^{-1} W_\chi^{-1}\left(\frac{1}{(1-2\delta)\,(1+\epsilon^2)}\right) = e^{2\xi}\sin\chi.$$

Its graph is

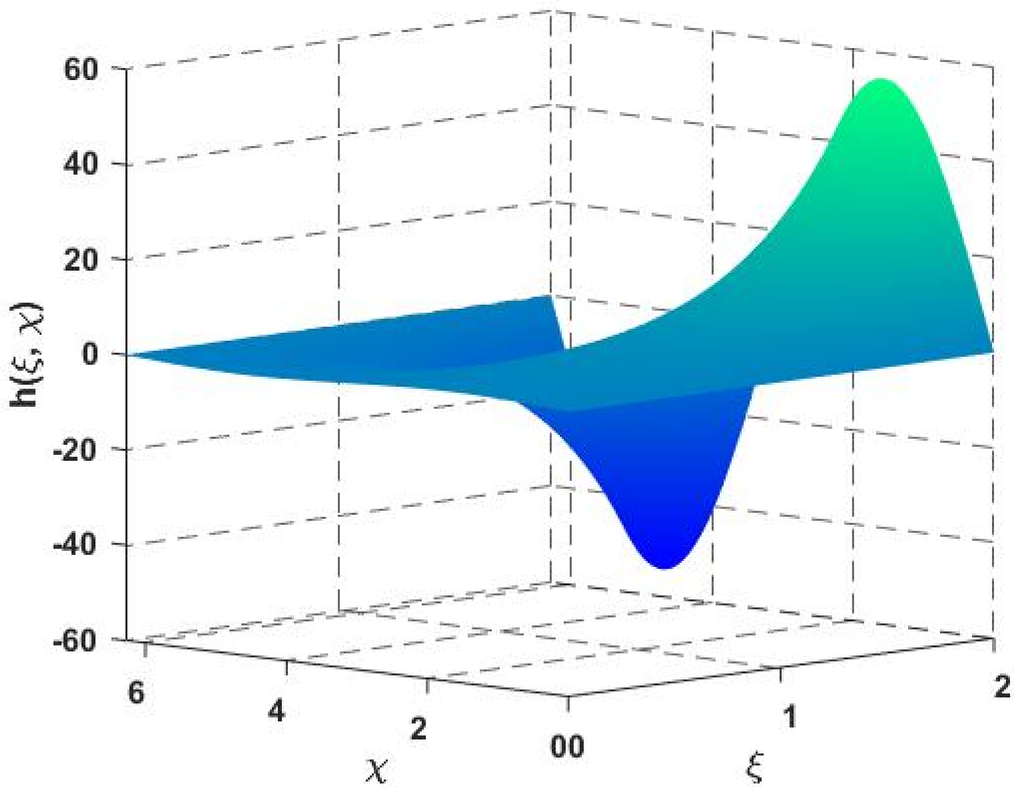

Figure 2: The solution of Example 4.2

4.2. DS-SWT for solving Integro PDEs

Example 3. *Consider the equation of Volterra Integro PDE.*

$$h_\xi + h_\chi - \cos\chi + \xi\sin\chi + 2\xi^2\sin\chi = 4\int_0^\xi\int_0^\chi h(u,v))dudv,\ \textit{where}\ \xi,\chi \geq 0, \tag{25}$$

With ICs
$h(\xi,0)=\xi,\ h(0,\chi)=0.$

Solution 3. *By applying the single Sumudu transform and the single Sawi transform to the ICs, we get*
$G_1=\delta,\ F_1=0.$
By Definition 4 and Theorem 2, we have

$$\int_0^{\xi}\int_0^{\chi} h(u,v))dudv=(1**h)(\xi,\chi). \tag{26}$$

Apply the DS-SWT to Equation 26*, we get*

$$\begin{aligned}&\frac{1}{\delta}H(\delta,\epsilon)+\frac{1}{\epsilon}H(\delta,\epsilon)-\frac{\delta}{\epsilon^2}-\frac{1}{\epsilon(1+\epsilon^2)}\\+\frac{\delta}{(1+\epsilon^2)}+\frac{4\delta^2}{(1+\epsilon^2)} &= 4\delta\epsilon H(\delta,\epsilon).\end{aligned}$$

So,

$$\begin{aligned}\frac{\epsilon+\delta-4\delta^2\epsilon^2}{\delta\epsilon}\times H(\delta,\epsilon) &= \\ &\frac{\delta\left(1+\epsilon^2\right)+\epsilon-\delta\epsilon^2-4\delta^2\epsilon^2}{\epsilon^2(1+\epsilon^2)}.\end{aligned}$$

Thus,

$$\begin{aligned}H(\delta,\epsilon) &= \frac{\delta\left(\epsilon+\delta-4\delta^2\epsilon^2\right)}{\epsilon(1+\epsilon^2)(\epsilon+\delta-4\delta^2\epsilon^2)}\\ &= \frac{\delta}{\epsilon(1+\epsilon^2)}.\end{aligned}$$

Therefore,

$$h(\xi,\chi)=S_{\xi}^{-1}W_{\chi}^{-1}\left(\frac{\delta}{\epsilon(1+\epsilon^2)}\right)=\xi\cos\chi.$$

Its graph is

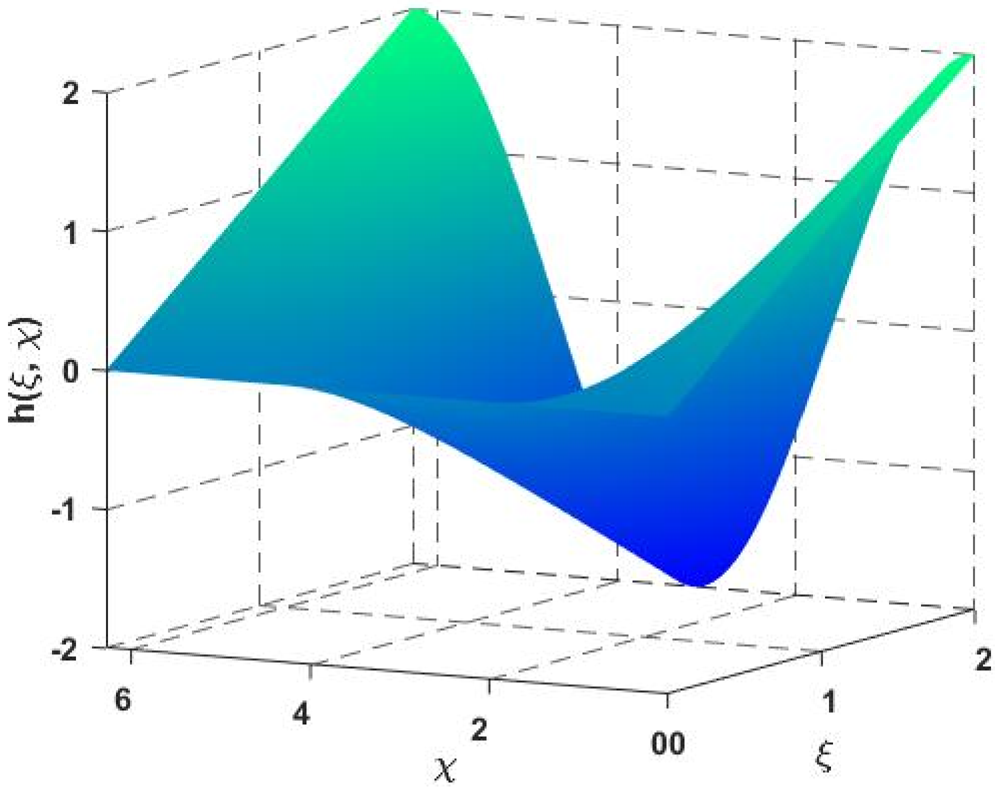

Figure 3: The solution of Example 4.3

Example 4. *Consider the equation of Integro PDE.*

$$h_{\xi\chi} - 2h_{\chi} + 2e^{\chi}\sinh\xi - \cosh\xi - e^{\chi} + 1 = \int_0^{\xi}\int_0^{\chi} h(u,v))dudv, \text{ where } \xi,\chi \geq 0, \tag{27}$$

With ICs
$h(\xi,0) = \sinh\xi,\ h(0,\chi) = 0$

Solution 4. *By applying the single Sumudu transform and the single Sawi transform to the ICs, we get*
$G_1 = \frac{\delta}{1-\delta^2},\ F_1 = 0$
Apply the DS-SWT to Equation 27*, we get*

$$\begin{aligned} & \frac{1}{\delta}H(\delta,\epsilon) - \frac{2}{\epsilon}H(\delta,\epsilon) + \frac{2\delta}{\epsilon^2(1-\delta^2)} + \frac{2\delta}{\epsilon(1-\epsilon)(1-\delta^2)} \\ -\frac{1}{\epsilon(1-\delta^2)} - \frac{1}{\epsilon(1-\epsilon)} + \frac{1}{\epsilon} \quad = & \quad \delta\epsilon H(\delta,\epsilon). \end{aligned}$$

So,

$$\frac{\epsilon-2\delta-\delta^2\epsilon^2}{\delta\epsilon}\times H(\delta,\epsilon) = \frac{-2\delta(1-\epsilon)-2\delta\epsilon+\epsilon(1-\epsilon)+\epsilon\left(1-\delta^2\right)-\epsilon(1-\epsilon)\left(1-\delta^2\right)}{\epsilon^2(1-\epsilon)(1-\delta^2)}$$

Thus,

$$\begin{aligned} H(\delta,\epsilon) &= \frac{\delta\left(\epsilon-2\delta-\delta^2\epsilon^2\right)}{\epsilon(1-\epsilon)(1-\delta^2)(\epsilon-2\delta-\delta^2\epsilon^2)} \\ &= \frac{\delta}{\epsilon(1-\epsilon)(1-\delta^2)} \end{aligned}$$

Therefore,

$$h(\xi,\chi) = S_\xi^{-1}W_\chi^{-1}\left(\frac{\delta}{\epsilon(1-\epsilon)(1-\delta^2)}\right) = \xi e^\chi$$

Its graph is

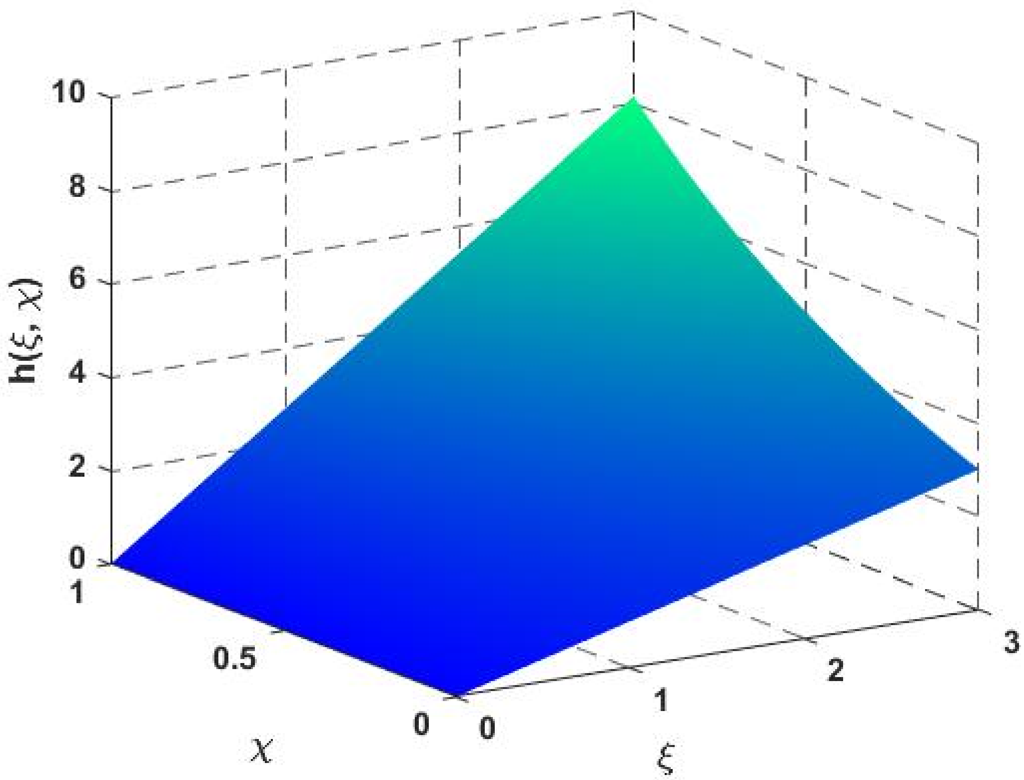

Figure 4: The solution of Example 4.4

5. Conclusion

In this paper, we introduce the Double Sumudu-Sawi Transform (DS-SWT) and explore its fundamental properties, shedding light on the key features that define this novel double transform. Several examples are presented to demonstrate the successful application of the DS-SWT in solving various partial and integral equations exactly. Our discussion is grounded in practical applications, where, where appropriate, we reference earlier numerical procedures that benefited from our previous research, while emphasizing the key advantages of the DS-SWT in solving complex problems. We believe that the future of the DS-SWT holds significant promise in the realm of conformable partial differential equations and integro-PDEs, particularly those with varying coefficients. Additional results related to conformable PDEs and Integro PDEs are available in references [10, 11].

Author contribution statement

All authors listed have significantly contributed to the development and the writing of this article.

Data availability statement

No data was used for the research described in the article.

Conflict of interest

The authors declare that they have no conflict of interest.

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