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Second Hankel Determinant for a Bi-univalent Function Subclass Based Gegenbauer (Ultraspherical) Polynomials

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Abstract. In this paper, we aim to establish a new upper bound approximation for the second Hankel determinant utilizing a certain subclass of the class of normalized analytic and bi-univalent functions in the open unit disk $\mathbb{U}$. These functions have inverses with a bi-univalent analytic continuation to $\mathbb{U}$ and are associated with orthogonal polynomials; namely, Gegenbauer polynomials that satisfy subordination conditions on $\mathbb{U}$. Finally, we introduce new essential results derived by specializing the parameter τ employed in our foundational finding.

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1. Introduction

The n^{th} - degree Gegenbauer (or ultraspherical) polynomials (GPS), denoted here by $\mathcal{U}_k^{(\beta)}(t)$, with parameter β at the point t are recursively defined by

$$\begin{aligned} &\mathcal{U}_0^{(\beta)}(t) = 1, \;\; \mathcal{U}_1^{(\beta)}(t) = 2\beta t, \\ &\mathcal{U}_k^{(\beta)}(t) = \frac{1}{k}\left[2t\,(k+\beta-1)\,\mathcal{U}_{k-1}^{(\beta)}(t) - (k+2\beta-2)\,\mathcal{U}_{k-2}^{(\beta)}(t)\right], \quad k \geq 2. \end{aligned} \tag{1}$$

These polynomials are orthogonal on the interval $\mathrm{I} = [-1, 1]$ with respect to the weight function $(1-t^2)^{\beta-\frac{1}{2}}$, where $\beta > -\frac{1}{2}$. That is, for any two GPS, $\mathcal{U}_k^{(\beta)}(t)$ and $\mathcal{U}_l^{(\beta)}(t)$, with $k \neq l$, we have

$$\int_{-1}^{1} \mathcal{U}_k^{(\beta)}(t)\,\mathcal{U}_l^{(\beta)}(t)\left(1-t^2\right)^{\beta-\frac{1}{2}} dt = 0,$$

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and with the condition that $l = k$, we have

$$\int_{-1}^{1} \left(\mathcal{U}_k^{(\beta)}(t)\right)^2 \left(1 - t^2\right)^{\beta - \frac{1}{2}} dt = \frac{\sqrt{\pi}\,\Gamma(k + 2\beta - 1)}{2^{1-2\beta}\, k!\,\Gamma(\beta)\,\Gamma(k + \beta - \frac{1}{2})}.$$

For $\beta > 0$, a generating function of GPS, $\mathcal{G}_\beta(t,\zeta)$, is defined by the form

$$\mathcal{G}_\beta(t,\zeta) = \frac{1}{(1 - 2t\zeta + \zeta^2)^\beta} = \sum_{k=0}^{\infty} \mathcal{U}_k^{(\beta)}(t)\ \zeta^k, \tag{2}$$

where $t \in \mathrm{I}$, ζ is in the open unit disk $\mathbb{U} = \{\zeta : \zeta \in \mathbb{C} \text{ and } |\zeta| < 1\}$, and $\mathbb{C}$ is, as usual, the set of complex numbers. For a fixed $t \in \mathrm{I}$, $\mathcal{G}_\beta(t,\zeta)$ is analytic in $\mathbb{U}$ that has a Taylor series expansion given by (2). Evidently, we see that $\mathcal{G}_\beta(t,\zeta)$ produces no values when $\beta = 0$. Therefore, the generating function of GPS is set to be of the form

$$\mathcal{G}_0(t,\zeta) = 1 - \log\left(1 - 2\,t\,\zeta + \zeta^2\right) = \sum_{k=0}^{\infty} \mathcal{U}_k^{(0)}(t)\ \zeta^k. \tag{3}$$

Note that $\mathcal{U}_k^{(\beta)}(t)$ are particular solutions of the Gegenbauer differential equation given by

$$(1 - t^2)\,\frac{d^2\mathrm{y}}{dt^2} - (2\,\beta + 1)\,t\,\frac{d\mathrm{y}}{dt} + k\,(k + 2\,\beta)\,\mathrm{y} = 0, \tag{4}$$

and when setting

(i) $\beta = 1/2$, equation (4) reduces to the Legendre differential equations, and the GPS reduce to the Legendre polynomials.
(ii) $\beta = 1$, equation (4) reduces to the Chebyshev differential equations, and the GPS reduce to the Chebyshev polynomials of the second kind.

Let $\mathfrak{A}$ denote the class of all functions of the form

$$\mathrm{f}(\zeta) = \zeta + \sum_{n=2}^{\infty} a_n\,\zeta^n, \quad (\zeta \in \mathbb{U}), \tag{5}$$

which are analytic in $\mathbb{U}$ and normalized by these two conditions $\mathrm{f}(0) = 0$ and $\mathrm{f}'(0) = 1$. Moreover, let $\mathfrak{S}$ be the subclass of $\mathfrak{A}$ consisting of all normalized univalent functions of the form (5) which are also univalent in $\mathbb{U}$. Two functions, f and g, are said to be subordinate ($\mathrm{f} \prec \mathrm{g}$) if there is an analytic function $\mathrm{h}(\zeta)$ (namely; a Schwarz function) in $\mathbb{U}$, such that $\mathrm{f}(\zeta) = \mathrm{g}(\mathrm{h}(\zeta))$ with $\mathrm{h}(0) = 0$ and $|\,\mathrm{h}(\zeta)\,| \le 1$. Especially, if the function g is univalent in $\mathbb{U}$, then the following equivalence is valid [1]

$$\mathrm{f}(\zeta) \prec \mathrm{g}(\zeta) \iff \mathrm{f}(0) = \mathrm{g}(0)$$

and

$$\mathrm{f}(\mathbb{U}) \subset \mathrm{g}(\mathbb{U}).$$

The Koebe One-Quarter Theorem [2] states that the image of $\mathbb{U}$ under every function $\mathrm{f} \in \mathfrak{S}$ contains a disk of radius $\frac{1}{4}$ and center at the origin; i.e., $\mathbb{U}_{\frac{1}{4}}(0) \in \mathrm{f}(\mathbb{U})$. Therefore, every univalent function $\mathrm{f} \in \mathfrak{S}$ has an inverse $\mathrm{f}^{-1} : \mathrm{f}(\mathbb{U}) \to \mathbb{U}$ which satisfies the following conditions:

$$(\mathrm{f}^{-1} \circ \mathrm{f})_{(\zeta)} = \zeta \qquad (\zeta \in \mathbb{U})$$

and

$$(\mathrm{f} \circ \mathrm{f}^{-1})_{(\eta)} = \eta \qquad \left(|\eta| < \mathrm{r}_0(\mathrm{f}); \quad \mathrm{r}_0(\mathrm{f}) \geq \frac{1}{4}\right),$$

where f^{-1} has the series expansion of the form

$$\mathrm{f}^{-1}(\eta) = \eta - a_2\,\eta^2 + (2a_2^2 - a_3)\,\eta^3 - (5a_2^3 - 5a_2 a_3 + a_4)\,\eta^4 + \cdots . \tag{6}$$

A function $\mathrm{f} \in \mathfrak{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Ξ be denoting the class of bi-univalent functions in $\mathbb{U}$ given by (5). Herein, we recall the following examples of functions in the bi-univalent function class Ξ that have apparently revived the study of bi-univalent functions in recent years:

$$\mathrm{f}_1(\zeta) = \frac{\zeta}{1-\zeta}, \quad \mathrm{f}_2(\zeta) = -\log(1-\zeta), \quad \text{and} \quad \mathrm{f}_3(\zeta) = \frac{1}{2}\log\left(\frac{1+\zeta}{1-\zeta}\right),$$

where their inverses are respectively given by

$$\mathrm{f}_1^{-1}(\eta) = \frac{\eta}{1+\eta}, \quad \mathrm{f}_2^{-1}(\eta) = \frac{e^{\eta}-1}{e^{\eta}}, \quad \text{and} \quad \mathrm{f}_3^{-1}(\eta) = \frac{e^{2\eta}-1}{e^{2\eta}+1}.$$

However, the familiar Koebe function, $\mathrm{K}(\zeta) = \frac{\zeta}{(1-\zeta)^2}$, is not a member of the bi-univalent function class Ξ since it maps the open unit disk $\mathbb{U} \subset \mathbb{C}$ onto the set $\mathrm{K}(\mathbb{U}) = \mathbb{C}\backslash(-\infty, -\frac{1}{4}]$, which does not contain $\mathbb{U}$ (i.e., $\{\eta : \eta \in \mathbb{C} \text{ and } |\eta| \leq \frac{1}{4}\} \subseteq \mathrm{K}(\mathbb{U})$) (see [3–11]). Other common univalent functions in $\mathfrak{S}$ that are not members of Ξ are

$$\vartheta_1(\zeta) = \frac{\zeta}{1-\zeta^2} \quad \text{and} \quad \vartheta_2(\zeta) = \zeta - \frac{\zeta^2}{2}.$$

Historically speaking, certain subclasses of Ξ were introduced by Brannan and Taha (see [12]) similar to the familiar subclasses $\mathfrak{S}^*(\varepsilon)$ and $\mathcal{K}(\varepsilon)$ of star-like and convex functions of order $\varepsilon \in [0,1)$ in the open unit disk $\mathbb{U}$, which are respectively defined by

$$\mathfrak{S}^*(\varepsilon) = \left\{\mathrm{f} : \mathrm{f} \in \mathfrak{S} \quad \text{and} \quad \Re\left\{\frac{\zeta\,\mathrm{f}'(\zeta)}{\mathrm{f}(\zeta)}\right\} > \varepsilon, \quad \zeta \in \mathbb{U}\right\},$$

and

$$\mathcal{K}(\varepsilon) = \left\{\mathrm{f} : \mathrm{f} \in \mathfrak{S} \quad \text{and} \quad \Re\left\{1 + \frac{\zeta\,\mathrm{f}''(\zeta)}{\mathrm{f}'(\zeta)}\right\} > \varepsilon, \quad \zeta \in \mathbb{U}\right\}.$$

Analogously, the bi-star-like and bi-convex function classes $\mathfrak{S}^*_{\Xi}(\varepsilon)$ and $\mathcal{K}_{\Xi}(\varepsilon)$ of order $\varepsilon \in [0,1)$ in the open unit disk $\mathbb{U}$, corresponding to $\mathfrak{S}^*(\varepsilon)$ and $\mathcal{K}(\varepsilon)$, were introduced and studied, and non-sharp upper bound estimations of the initial Taylor-Maclaurin coefficients were obtained as well. In 1976, Noonan and Thomas defined the $\mathfrak{q}^{\text{th}}$ Hankel determinant of the function $\mathrm{f} \in \mathfrak{A}$ of the form (5) for integers $\mathfrak{n}, \mathfrak{q} \in \mathbb{N} = \{1, 2, 3, \dots\}$ by [13]

$$\mathrm{H}_{\mathrm{f}}(\mathfrak{n}, \mathfrak{q}) = \begin{vmatrix} a_{\mathfrak{n}} & a_{\mathfrak{n}+1} & \cdots & a_{\mathfrak{n}+\mathfrak{q}-1} \\ a_{\mathfrak{n}+1} & a_{\mathfrak{n}+2} & \cdots & a_{\mathfrak{n}+\mathfrak{q}} \\ \vdots & \vdots & \vdots & \vdots \\ a_{\mathfrak{n}+\mathfrak{q}-1} & a_{\mathfrak{n}+\mathfrak{q}} & \cdots & a_{\mathfrak{n}+2\mathfrak{q}-2} \end{vmatrix}, \qquad a_1 := 1.$$

In particular, it is observed that, for $\mathfrak{n} = 1, 2$ and $\mathfrak{q} = 2$, the Hankel determinants are given by

$$\mathrm{H}_{\mathrm{f}}(1,\, 2) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2 \qquad \text{and} \qquad \mathrm{H}_{\mathrm{f}}(2,\, 2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2\, a_4 - a_3^2,$$

which are respectively referred to as the well-known Fekete-Szegö functional and second Hankel determinant functional. Several other authors have considered the determinant $\mathrm{H}_{\mathrm{f}}(\mathfrak{n}, \mathfrak{q})$ in their studies. In [14], for instance, Noor found that the rate of growth of $\mathrm{H}_{\mathrm{f}}(\mathfrak{n}, \mathfrak{q})$ as $\mathfrak{n} \to \infty$ occurs when functions $\mathrm{f} \in \mathfrak{A}$ of the form (5) with bounded boundary. The authors in [14] and [15], in particular, achieved sharp upper bounds on $\mathrm{H}_{\mathrm{f}}(2,\, 2)$ for several types of function classes. For $\mathrm{f} \in \mathfrak{S}$ in the open unit disk $\mathbb{U}$, the authors in [16] obtained the sharp upper inequality for the functional $\mathrm{H}_{\mathrm{f}}(1,\, 2)$ that is given by $|\,\mathrm{H}_{\mathrm{f}}(1,\, 2)\,| = |\,a_3 - a_2^2\,| \le 1$. Several authors have recently investigated the upper bounds of $\mathrm{H}_{\mathrm{f}}(1,\, 2)$ and Taylor-Maclaurin coefficients for various subclasses of bi-univalent functions (see, for examples, [17–30]). Furthermore, the subclass of $\mathfrak{S}$ consisting of all functions whose derivatives have positive real part, introduced in [31], was considered by the authors of [32] in order to derive the sharp bounds for the functional $\mathrm{H}_{\mathrm{f}}(2,\, 2)$ that is given by $|\,\mathrm{H}_{\mathrm{f}}(2,\, 2)\,| = |\,a_2 a_4 - a_3^2\,| \le \frac{4}{9}$ for each function belongs to that subclass. In addition, they discovered the sharp second Hankel determinant, $\mathrm{H}_{\mathrm{f}}(2,\, 2)$, in (see [32]) for star-like and convex function subclasses, $\mathfrak{S}^*$ and $\mathcal{K}$, of $\mathfrak{S}$ with bounds of $|\,\mathrm{H}_{\mathrm{f}}(2,\, 2)\,| = |\,a_2 a_4 - a_3^2\,| \le \frac{1}{8}$ and $|\,\mathrm{H}_{\mathrm{f}}(2,\, 2)\,| = |\,a_2 a_4 - a_3^2\,| \le 1$, respectively. In recent times, several researchers have explored upper bounds for the coefficients and Hankel determinant of functions within different subclasses of univalent functions (see, for examples, [33–36]).

Definition 1. *Let* $\tau \in [0,1]$ *and* $t \in (\frac{1}{2}, 1]$. *A function* $\mathrm{f} \in \Xi$ *of the form* (5) *is said to be in the class* $\Omega^{\beta}_{\Xi}(t, \tau)$ *with a nonzero real constant* β *if the following subordinations are satisfied*

$$\tau\left(1 + \frac{\zeta\, \mathrm{f}''(\zeta)}{\mathrm{f}'(\zeta)}\right) + (1-\tau)\left(\frac{\zeta\, \mathrm{f}'(\zeta)}{\mathrm{f}(\zeta)}\right) \prec \mathcal{G}_{\beta}(t, \zeta) \tag{7}$$

and

$$\tau\left(1 + \frac{\eta\, \mathrm{g}''(\eta)}{\mathrm{g}'(\eta)}\right) + (1-\tau)\left(\frac{\eta\, \mathrm{g}'(\eta)}{\mathrm{g}(\eta)}\right) \prec \mathcal{G}_{\beta}(t, \eta), \tag{8}$$

where the function $\mathrm{g} = \mathrm{f}^{-1}$ *is defined by* (6) *and* $\mathcal{G}_{\beta}$ *is the GPS - generating function given by* (2).

Remark 1. *[37] By setting* $\tau = 0$ *in* (1), *we obtain the class* $\Omega^{\beta}_{\Xi}(t, 0) = \Sigma^{\beta}_{\Xi}(t)$ *that consists of functions* $\mathrm{f} \in \Xi$ *satisfying the conditions*

$$\frac{\zeta\, \mathrm{f}'(\zeta)}{\mathrm{f}(\zeta)} \prec \mathcal{G}_{\beta}(t, \zeta) \tag{9}$$

and

$$\frac{\eta\, \mathrm{g}'(\eta)}{\mathrm{g}(\eta)} \prec \mathcal{G}_{\beta}(t, \eta), \tag{10}$$

where the function $\mathrm{g} = \mathrm{f}^{-1}$ *is defined by* (6).

Remark 2. *[37] By setting* $\tau = 1$ *in* (1), *we obtain the class* $\Omega^{\beta}_{\Xi}(t, 1) = \mathfrak{D}^{\beta}_{\Xi}(t)$ *that consists of functions* $\mathrm{f} \in \Xi$ *satisfying the conditions*

$$1 + \frac{\zeta\, \mathrm{f}''(\zeta)}{\mathrm{f}'(\zeta)} \prec \mathcal{G}_{\beta}(t, \zeta) \tag{11}$$

and

$$1 + \frac{\eta\, \mathrm{g}''(\eta)}{\mathrm{g}'(\eta)} \prec \mathcal{G}_{\beta}(t, \eta), \tag{12}$$

where the function $\mathrm{g} = \mathrm{f}^{-1}$ *is defined by* (6).

Let $\mathcal{Q} : \mathbb{U} \to \mathbb{C}$ be the class of functions $\mathfrak{s}(\zeta)$ with positive real part consisting of all analytic functions satisfying the conditions that $\mathfrak{s}(0) = 1$ and $\Re\left(\mathfrak{s}(\zeta)\right) > 0$. To derive our desirable upper bounds estimation for the second Hankel determinant $\mathrm{H}_{\mathrm{f}}(2, 2) = \mathrm{a}_2 \mathrm{a}_4 - \mathrm{a}_3^2$ associated with the class $\Omega^{\beta}_{\Xi}(t, \tau)$ in (1), we state the necessary lemmas:

Lemma 1. *[38] If the function* $\mathfrak{s} \in \mathcal{Q}$ *is defined by*

$$\mathfrak{s}(\zeta) = 1 + \sum_{k=1}^{\infty} \mathfrak{s}_k\, \zeta^k, \tag{13}$$

then

$$|\, \mathfrak{s}_k \,| \leq 2, \qquad k = 1,\, 2,\, ...$$

Lemma 2. *[39] If the function* $\mathfrak{s} \in \mathcal{Q}$ *is of the form* (13), *then*

$$2\mathfrak{s}_2 = \mathfrak{s}_1^2 + (4 - \mathfrak{s}_1^2)\, \xi \tag{14}$$

and

$$4\mathfrak{s}_3 = \mathfrak{s}_1^3 + 2(4 - \mathfrak{s}_1^2)\, \mathfrak{s}_1\, \xi - \mathfrak{s}_1 (4 - \mathfrak{s}_1^2)\, \xi^2 + 2(4 - \mathfrak{s}_1^2)(1 - |\xi|^2)\, \zeta \tag{15}$$

for some ξ *and* ζ *with* $|\, \xi \,| \leq 1$ *and* $|\, \zeta \,| \leq 1$.

Also by considering the class Δ that consists of all analytic functions $\omega \in \mathbb{U}$ satisfying the conditions that $\omega(0) = 0$ and $|\omega(\zeta)| < 1$ for all $\zeta \in \mathbb{U}$, we state the following lemma:

Lemma 3. *[2] Let $\omega \in \Delta$ with $\omega(\zeta) = \sum_{k=1}^{\infty} \omega_k \zeta^k$, $\zeta \in \mathbb{U}$. Then*

$$|\omega_1| \leq 1 \quad and \quad |\omega_k| \leq 1 - |\omega_1|^2 \quad for \quad k \geq 2.$$

2. Second Hankel Determinant

Theorem 1. *Let the function $\mathrm{f} \in \Xi$ of the form (5) be in the class $\Omega_{\Xi}^{\beta}(t,\tau)$ in (1). Then*

$$|a_2 a_4 - a_3^2| \leq \begin{cases} \mathcal{T}(2^-, t) & \mathcal{E}_1 \geq 0 \quad and \quad \mathcal{E}_2 \geq 0; \\ \max\left\{\frac{4\beta^2 t^2}{(1+2\tau)^2},\ \mathcal{T}(2^-, t)\right\} & \mathcal{E}_1 > 0 \quad and \quad \mathcal{E}_2 < 0; \\ \frac{4\beta^2 t^2}{(1+2\tau)^2} & \mathcal{E}_1 \leq 0 \quad and \quad \mathcal{E}_2 \leq 0; \\ \max\left\{\mathcal{T}(\mathfrak{c}_0, t),\ \mathcal{T}(2^-, t)\right\} & \mathcal{E}_1 < 0 \quad and \quad \mathcal{E}_2 > 0, \end{cases} \tag{16}$$

where

$$\mathcal{T}(2^-,\, t) = \frac{\left(\mathcal{U}_1^{(\beta)}(t)\right)^2}{(1+2\tau)^2} + \frac{\mathcal{E}_1 + \mathcal{E}_2}{3(1+3\tau)(1+\tau)^3(1+2\tau)^2}, \tag{17}$$

$$\mathcal{T}(\mathfrak{c}_0, t) = \frac{\left(\mathcal{U}_1^{(\beta)}(t)\right)^2}{(1+2\tau)^2} - \frac{\mathcal{E}_2^2}{12\,\mathcal{E}_1(1+3\tau)(1+\tau)^3(1+2\tau)^2}; \quad \mathfrak{c}_0 = \sqrt{-\frac{2\,\mathcal{E}_2}{\mathcal{E}_1}}, \tag{18}$$

$$\begin{aligned} \mathcal{E}_1 = {} & 16(1+2\tau)^2\, \mathcal{U}_1^{(\beta)}(t) \left| \left(\mathcal{U}_3^{(\beta)}(t) + \mathcal{U}_2^{(\beta)}(t) + \frac{1}{4}\,\mathcal{U}_1^{(\beta)}(t)\right)(1+\tau)^2 - \left(\mathcal{U}_1^{(\beta)}(t)\right)^3 \right| \\ & + \left(\mathcal{U}_1^{(\beta)}(t)\right)^2 \left[3(1+3\tau)(1+\tau)^3 - 12(1+\tau)^2(1+2\tau)^2\right] \\ & - 2(1+\tau)(1+2\tau)\,\mathcal{U}_1^{(\beta)}(t) \left[3(1+3\tau)\left(\mathcal{U}_1^{(\beta)}(t)\right)^2 + 8(1+\tau)(1+2\tau)\,\mathcal{U}_2^{(\beta)}(t)\right], \end{aligned} \tag{19}$$

$$\begin{aligned} \mathcal{E}_2 = {} & 12(1+2\tau)^2(1+\tau)^2\left(\mathcal{U}_1^{(\beta)}(t)\right)^2 + 6(1+\tau)(1+2\tau)(1+3\tau)\left(\mathcal{U}_1^{(\beta)}(t)\right)^3 \\ & + 16(1+\tau)^2(1+2\tau)^2\,\mathcal{U}_1^{(\beta)}(t)\,\mathcal{U}_2^{(\beta)}(t) - 6(1+3\tau)(1+\tau)^3\left(\mathcal{U}_1^{(\beta)}(t)\right)^2, \end{aligned} \tag{20}$$

and $\mathcal{U}_1^{(\beta)}(t)$, $\mathcal{U}_2^{(\beta)}(t)$, and $\mathcal{U}_3^{(\beta)}(t)$ are defined by (1).

Proof. Suppose $\mathrm{f} \in \Omega_{\Xi}^{\beta}(t,\tau)$ for some $\tau \in [0,1]$. Then from (7) and (8) we have

$$\tau\left(1+\frac{\zeta\,\mathrm{f}''(\zeta)}{\mathrm{f}'(\zeta)}\right)+(1-\tau)\left(\frac{\zeta\,\mathrm{f}'(\zeta)}{\mathrm{f}(\zeta)}\right)\prec \mathcal{G}_{\beta}(t,\,\mathfrak{u}(\zeta)) \tag{21}$$

and

$$\tau\left(1+\frac{\eta\,\mathrm{g}''(\eta)}{\mathrm{g}'(\eta)}\right)+(1-\tau)\left(\frac{\eta\,\mathrm{g}'(\eta)}{\mathrm{g}(\eta)}\right)\prec \mathcal{G}_{\beta}(t,\,\mathfrak{v}(\eta)), \tag{22}$$

where $\mathrm{g}=\mathrm{f}^{-1}$ and $\mathfrak{u},\,\mathfrak{v}\in\Delta$ are given by

$$\mathfrak{u}(\zeta)=\sum_{\mathfrak{n}=1}^{\infty}\mathfrak{c}_{\mathfrak{n}}\,\zeta^{\mathfrak{n}}\qquad \textit{and}\qquad \mathfrak{v}(\eta)=\sum_{\mathfrak{n}=1}^{\infty}\mathfrak{d}_{\mathfrak{n}}\,\eta^{\mathfrak{n}}.$$

Then by using $\mathcal{G}_{\beta}(t,\,\zeta)$ given in (2), we can write the right hand sides of (21) and (22) as follows:

$$\begin{aligned}\mathcal{G}_{\beta}(t,\,\mathfrak{u}(\zeta)) = 1 &+ \mathcal{U}_1^{(\beta)}(t)\,\mathfrak{c}_1\,\zeta+\left[\mathcal{U}_1^{(\beta)}(t)\,\mathfrak{c}_2+\mathcal{U}_2^{(\beta)}(t)\,\mathfrak{c}_1^2\right]\zeta^2\\ &+\left[\mathcal{U}_1^{(\beta)}(t)\,\mathfrak{c}_3+2\mathcal{U}_2^{(\beta)}(t)\,\mathfrak{c}_1\,\mathfrak{c}_2+\mathcal{U}_3^{(\beta)}(t)\,\mathfrak{c}_1^3\right]\zeta^3+\cdots\end{aligned} \tag{23}$$

and

$$\begin{aligned}\mathcal{G}_{\beta}(t,\,\mathfrak{u}(\eta)) = 1 &+ \mathcal{U}_1^{(\beta)}(t)\,\mathfrak{d}_1\,\eta+\left[\mathcal{U}_1^{(\beta)}(t)\,\mathfrak{d}_2+\mathcal{U}_2^{(\beta)}(t)\,\mathfrak{d}_1^2\right]\eta^2\\ &+\left[\mathcal{U}_1^{(\beta)}(t)\,\mathfrak{d}_3+2\mathcal{U}_2^{(\beta)}(t)\,\mathfrak{d}_1\,\mathfrak{d}_2+\mathcal{U}_3^{(\beta)}(t)\,\mathfrak{d}_1^3\right]\eta^3+\cdots.\end{aligned} \tag{24}$$

Therefore, (21) and (22) become

$$\begin{aligned}&\tau\left[1+2\,\mathrm{a}_2\,\zeta+(6\,\mathrm{a}_3-4\,\mathrm{a}_2^2)\,\zeta^2+2\,(4\,\mathrm{a}_2^3-9\,\mathrm{a}_2\,\mathrm{a}_3+6\,\mathrm{a}_4)\zeta^3+\cdots\right]\\ &\quad+(1-\tau)\left[1+\mathrm{a}_2\,\zeta+(2\,\mathrm{a}_3-\mathrm{a}_2^2)\,\zeta^2+(\mathrm{a}_2^3-3\,\mathrm{a}_2\,\mathrm{a}_3+3\,\mathrm{a}_4)\zeta^3+\cdots\right]\\ &\qquad=1+\mathcal{U}_1^{(\beta)}(t)\,\mathfrak{c}_1\,\zeta+\left[\mathcal{U}_1^{(\beta)}(t)\,\mathfrak{c}_2+\mathcal{U}_2^{(\beta)}(t)\,\mathfrak{c}_1^2\right]\zeta^2\\ &\qquad\quad+\left[\mathcal{U}_1^{(\beta)}(t)\,\mathfrak{c}_3+2\mathcal{U}_2^{(\beta)}(t)\,\mathfrak{c}_1\,\mathfrak{c}_2+\mathcal{U}_3^{(\beta)}(t)\,\mathfrak{c}_1^3\right]\zeta^3+\cdots,\end{aligned} \tag{25}$$

and

$$\begin{aligned} &\tau\left[1-2\,\mathrm{a}_2\,\eta+(8\,\mathrm{a}_2^2-6\,\mathrm{a}_3)\eta^2+(-32\,\mathrm{a}_2^3+42\,\mathrm{a}_2\,\mathrm{a}_3-12\,\mathrm{a}_4)\,\eta^3+\cdots\right] \\ &+(1-\tau)\left[1-\mathrm{a}_2\,\eta+(3\,\mathrm{a}_2^2-2\,\mathrm{a}_3)\,\eta^2+(-10\,\mathrm{a}_2^3+12\,\mathrm{a}_2\,\mathrm{a}_3-3\,\mathrm{a}_4)\,\eta^3+\cdots\right] \\ &=1+\mathcal{U}_1^{(\beta)}(t)\,\mathfrak{d}_1\,\eta+\left[\mathcal{U}_1^{(\beta)}(t)\,\mathfrak{d}_2+\mathcal{U}_2^{(\beta)}(t)\,\mathfrak{d}_1^2\right]\eta^2 \\ &+\left[\mathcal{U}_1^{(\beta)}(t)\,\mathfrak{d}_3+2\mathcal{U}_2^{(\beta)}(t)\,\mathfrak{d}_1\,\mathfrak{d}_2+\mathcal{U}_3^{(\beta)}(t)\,\mathfrak{d}_1^3\right]\eta^3+\cdots \end{aligned} \tag{26}$$

At this point, the corresponding coefficients in (25) and (26) can be equated to obtain

$$(1+\tau)\,\mathrm{a}_2=\mathcal{U}_1^{(\beta)}(t)\,\mathfrak{c}_1, \tag{27}$$

$$2\,(1+2\,\tau)\,\mathrm{a}_3-(1+3\tau)\,\mathrm{a}_2^2=\mathcal{U}_1^{(\beta)}(t)\,\mathfrak{c}_2+\mathcal{U}_2^{(\beta)}(t)\,\mathfrak{c}_1^2, \tag{28}$$

$$(1+7\,\tau)\,\mathrm{a}_2^3-3\,(1+5\,\tau)\,\mathrm{a}_2\,\mathrm{a}_3+3\,(1+3\,\tau)\,\mathrm{a}_4=\mathcal{U}_1^{(\beta)}(t)\,\mathfrak{c}_3+2\mathcal{U}_2^{(\beta)}(t)\,\mathfrak{c}_1\,\mathfrak{c}_2+\mathcal{U}_3^{(\beta)}(t)\,\mathfrak{c}_1^3, \tag{29}$$

$$-(1+\tau)\,\mathrm{a}_2=\mathcal{U}_1^{(\beta)}(t)\,\mathfrak{d}_1, \tag{30}$$

$$(3+5\,\tau)\,\mathrm{a}_2^2-2\,(1+2\,\tau)\,\mathrm{a}_3=\mathcal{U}_1^{(\beta)}(t)\,\mathfrak{d}_2+\mathcal{U}_2^{(\beta)}(t)\,\mathfrak{d}_1^2, \tag{31}$$

and

$$-2\,(5+11\,\tau)\,\mathrm{a}_2^3+6\,(2+5\,\tau)\,\mathrm{a}_2\,\mathrm{a}_3-3\,(1+3\,\tau)\,\mathrm{a}_4=\mathcal{U}_1^{(\beta)}(t)\,\mathfrak{d}_3+2\mathcal{U}_2^{(\beta)}(t)\,\mathfrak{d}_1\,\mathfrak{d}_2+\mathcal{U}_3^{(\beta)}(t)\,\mathfrak{d}_1^3. \tag{32}$$

From (27) and (30), we obtain that

$$\mathfrak{c}_1=-\,\mathfrak{d}_1 \tag{33}$$

and

$$\mathrm{a}_2=\frac{\mathcal{U}_1^{(\beta)}(t)}{1+\tau}\,\mathfrak{c}_1. \tag{34}$$

Upon subtracting (31) from (28), we have that

$$\mathrm{a}_3=\mathrm{a}_2^2+\frac{\mathcal{U}_1^{(\beta)}(t)\,(\mathfrak{c}_2-\mathfrak{d}_2)}{4\,(1+2\,\tau)}=\frac{\left(\mathcal{U}_1^{(\beta)}(t)\right)^2}{(1+\tau)^2}\mathfrak{c}_1^2+\frac{\mathcal{U}_1^{(\beta)}(t)\,(\mathfrak{c}_2-\mathfrak{d}_2)}{4\,(1+2\,\tau)}. \tag{35}$$

Furthermore, if we subtract (32) from (29), together with (27), (33) and (35), we have

$$\begin{aligned} \mathrm{a}_4=&\frac{5\left(\mathcal{U}_1^{(\beta)}(t)\right)^2(\mathfrak{c}_2-\mathfrak{d}_2)\,\mathfrak{c}_1}{8\,(1+\tau)\,(1+2\,\tau)}+\frac{\mathcal{U}_1^{(\beta)}(t)\,(\mathfrak{c}_3-\mathfrak{d}_3)}{6(1+3\tau)} \\ &+\frac{\mathcal{U}_2^{(\beta)}(t)\,(\mathfrak{c}_2+\mathfrak{d}_2)\,\mathfrak{c}_1}{3\,(1+3\,\tau)}+\left[\frac{\mathcal{U}_3^{(\beta)}(t)}{3\,(1+3\,\tau)}+\frac{2\,(1+4\,\tau)\left(\mathcal{U}_1^{(\beta)}(t)\right)^3}{3\,(1+\tau)^3\,(1+3\,\tau)}\right]\mathfrak{c}_1^3. \end{aligned} \tag{36}$$

Thus, when applying (27), (35), and (36), we can simply establish that

$$\begin{aligned} a_2\, a_4 - a_3^2 = &\frac{\left(\mathcal{U}_1^{(\beta)}(t)\right)^3 (\mathfrak{c}_2 - \mathfrak{d}_2)\, \mathfrak{c}_1^2}{8\,(1+\tau)^2\,(1+2\,\tau)} + \frac{\left(\mathcal{U}_1^{(\beta)}(t)\right)^2 (\mathfrak{c}_3 - \mathfrak{d}_3)\, \mathfrak{c}_1}{6\,(1+\tau)\,(1+3\,\tau)} \\ &+ \frac{\mathcal{U}_1^{(\beta)}(t)\, \mathcal{U}_2^{(\beta)}(t)\, (\mathfrak{c}_2 + \mathfrak{d}_2)\, \mathfrak{c}_1^2}{3\,(1+\tau)\,(1+3\,\tau)} - \frac{\left(\mathcal{U}_1^{(\beta)}(t)\right)^2 (\mathfrak{c}_2 - \mathfrak{d}_2)^2}{16\,(1+2\,\tau)^2} \\ &+ \frac{\mathcal{U}_1^{(\beta)}(t) \left[\mathcal{U}_3^{(\beta)}(t)\,(1+\tau)^2 - \left(\mathcal{U}_1^{(\beta)}(t)\right)^3\right] \mathfrak{c}_1^4}{3\,(1+3\,\tau)\,(1+\tau)^3}. \end{aligned} \tag{37}$$

Next, according to Lemma (2), we now have that

$$\mathfrak{c}_2 - \mathfrak{d}_2 = \frac{4 - \mathfrak{c}^2}{2}\, (\mathfrak{x} - \mathfrak{y}), \tag{38}$$

$$\mathfrak{c}_2 + \mathfrak{d}_2 = \mathfrak{c}_1^2 + \frac{4 - \mathfrak{c}^2}{2}\, (\mathfrak{x} + \mathfrak{y}), \tag{39}$$

and

$$\begin{aligned} \mathfrak{c}_3 - \mathfrak{d}_3 = &\frac{\mathfrak{c}_1^3}{2} + \frac{(4 - \mathfrak{c}^2)\, \mathfrak{c}_1}{2}\, (\mathfrak{x} + \mathfrak{y}) - \frac{(4 - \mathfrak{c}_1^2)\, \mathfrak{c}_1}{4} (\mathfrak{x}^2 + \mathfrak{y}^2) \\ &+ \frac{4 - \mathfrak{c}_1^2}{2} \left[\left(1 - |\mathfrak{x}|^2\right) \zeta - \left(1 - |\mathfrak{y}|^2\right) \eta\right], \end{aligned} \tag{40}$$

for some $\mathfrak{x}$, $\mathfrak{y}$, ζ, and η with $|\mathfrak{x}| \le 1$, $|\mathfrak{y}| \le 1$, $|\zeta| \le 1$, and $|\eta| \le 1$. Then, by substituting (38), (39), and (40) into (37), we obtain that

$$\begin{aligned} \left| a_2\, a_4 - a_3^2 \right| \le &\frac{\mathcal{U}_1^{(\beta)}(t) \left| \left(\mathcal{U}_3^{(\beta)}(t) + \mathcal{U}_2^{(\beta)}(t) + \frac{1}{4} \mathcal{U}_1^{(\beta)}(t)\right) (1+\tau)^2 - \left(\mathcal{U}_1^{(\beta)}(t)\right)^3 \right|}{3\,(1+3\,\tau)\,(1+\tau)^3}\, \mathfrak{c}_1^4 + \frac{\left(\mathcal{U}_1^{(\beta)}(t)\right)^2 (4 - \mathfrak{c}_1^2)\, \mathfrak{c}_1}{6\,(1+\tau)\,(1+3\,\tau)} \\ &+ \left[\frac{\left(\mathcal{U}_1^{(\beta)}(t)\right)^3 (4 - \mathfrak{c}_1^2)\, \mathfrak{c}_1^2}{16\,(1+\tau)^2\,(1+2\,\tau)} + \frac{\left(\mathcal{U}_1^{(\beta)}(t)\right)^2 (4 - \mathfrak{c}_1^2)\, \mathfrak{c}_1^2}{12\,(1+\tau)\,(1+3\,\tau)} + \frac{\mathcal{U}_1^{(\beta)}(t)\, \mathcal{U}_2^{(\beta)}(t)\, (4 - \mathfrak{c}_1^2)\, \mathfrak{c}_1^2}{6\,(1+\tau)\,(1+3\,\tau)}\right] \left(|\mathfrak{x}| + |\mathfrak{y}|\right) \\ &+ \left[\frac{\left(\mathcal{U}_1^{(\beta)}(t)\right)^2 (4 - \mathfrak{c}_1^2)\, \mathfrak{c}_1^2}{24\,(1+\tau)\,(1+3\,\tau)} - \frac{\left(\mathcal{U}_1^{(\beta)}(t)\right)^2 (4 - \mathfrak{c}_1^2)\, \mathfrak{c}_1}{12\,(1+\tau)\,(1+3\,\tau)}\right] \left(|\mathfrak{x}|^2 + |\mathfrak{y}|^2\right) \\ &+ \left[\frac{\left(\mathcal{U}_1^{(\beta)}(t)\right)^2 (4 - \mathfrak{c}_1^2)^2}{64\,(1+2\,\tau)^2}\right] \left(|\mathfrak{x}| + |\mathfrak{y}|\right)^2. \end{aligned} \tag{41}$$

Lemma (1) allows us to assume, without any loss of generality, that $\mathfrak{c} \in [0, 2]$ where $\mathfrak{c} = |\mathfrak{c}_1|$. Thus, for $\delta_1 = |\, \mathfrak{x}\,| \le 1$ and $\delta_2 = |\, \mathfrak{y}\,| \le 1$, we can rewrite (41) to be in the following form:

$$\left|a_2\,a_4 - a_3^2\right| \leq \Upsilon_1 + \Upsilon_2\,(\delta_1+\delta_2) + \Upsilon_3\left(\delta_1^2+\delta_2^2\right) + \Upsilon_4\left(\delta_1+\delta_2\right)^2 = \varphi(\delta_1,\,\delta_2), \tag{42}$$

where

$$\Upsilon_1 = \frac{\mathcal{U}_1^{(\beta)}(t)\left|\left(\mathcal{U}_3^{(\beta)}(t)+\mathcal{U}_2^{(\beta)}(t)+\frac{1}{4}\mathcal{U}_1^{(\beta)}(t)\right)(1+\tau)^2-\left(\mathcal{U}_1^{(\beta)}(t)\right)^3\right|}{3\,(1+3\,\tau)\,(1+\tau)^3}\,\mathfrak{c}^4 + \frac{\left(\mathcal{U}_1^{(\beta)}(t)\right)^2(4-\mathfrak{c}^2)\,\mathfrak{c}}{6\,(1+\tau)\,(1+3\,\tau)} \geq 0, \tag{43}$$

$$\Upsilon_2 = \left[\frac{\left(\mathcal{U}_1^{(\beta)}(t)\right)^3(4-\mathfrak{c}^2)\,\mathfrak{c}^2}{16\,(1+\tau)^2\,(1+2\,\tau)} + \frac{\left(\mathcal{U}_1^{(\beta)}(t)\right)^2(4-\mathfrak{c}^2)\,\mathfrak{c}^2}{12\,(1+\tau)\,(1+3\,\tau)} + \frac{\mathcal{U}_1^{(\beta)}(t)\,\mathcal{U}_2^{(\beta)}(t)\,(4-\mathfrak{c}^2)\,\mathfrak{c}^2}{6\,(1+\tau)\,(1+3\,\tau)}\right] \geq 0, \tag{44}$$

$$\Upsilon_3 = \left[\frac{\left(\mathcal{U}_1^{(\beta)}(t)\right)^2(4-\mathfrak{c}^2)\,(\mathfrak{c}-2)\,\mathfrak{c}}{24\,(1+\tau)\,(1+3\,\tau)}\right] \leq 0, \tag{45}$$

and

$$\Upsilon_4 = \left[\frac{\left(\mathcal{U}_1^{(\beta)}(t)\right)^2\left(4-\mathfrak{c}^2\right)^2}{64\,(1+2\,\tau)^2}\right] \geq 0. \tag{46}$$

Now, we have to maximize the function $\varphi(\delta_1,\,\delta_2)$ in (42) on the closed square $\mathbb{S}=[0,\,1]\times[0,\,1]$ by investigating the maximum values of $\varphi(\delta_1,\,\delta_2)$ in accordance with $0<\mathfrak{c}<2$, $\mathfrak{c}=0$, and $\mathfrak{c}=2$. For the case that $0<\mathfrak{c}<2$, since $\Upsilon_3<0$ and $\Upsilon_3+2\,\Upsilon_4>0$ for all $t\in(\frac{1}{2},1)$, we deduce that

$$\varphi_{\delta_1\delta_1}\,\varphi_{\delta_2\delta_2} - \varphi^2_{\delta_1\delta_2} < 0, \quad \text{for all} \quad \delta_1,\delta_2\in\mathbb{S}.$$

Therefore, as a result of this, the function φ cannot have a local maximum in the interior of the square $\mathbb{S}$. Now, we will explore the maximum value of φ on the boundary of $\mathbb{S}$.
(1) for $\delta_1=0$ and $0\leq\delta_2\leq1$ (similarly, for $\delta_2=0$ and $0\leq\delta_1\leq1$), $\varphi(\delta_1,\,\delta_2)$ takes the form

$$\psi_1(\delta_2) := \varphi(0,\,\delta_2) = \Upsilon_1 + \Upsilon_2\,\delta_2 + (\Upsilon_3+\Upsilon_4)\,\delta_2^2.$$

Next, we will separately discuss the following two cases.

Case (i): When $\Upsilon_3+\Upsilon_4\geq0$, for $0<\delta_2<1$, for any fixed $\mathfrak{c}\in(0,2)$, and for all $t\in(\frac{1}{2},1)$, it is obvious that $\psi_1'(\delta_2)=\Upsilon_2+2\,(\Upsilon_3+\Upsilon_4)\,\delta_2>0$.

Case (ii): When $\Upsilon_3+\Upsilon_4<0$ and since $\Upsilon_2+2\,(\Upsilon_3+\Upsilon_4)\geq0$, for $0<\delta_2<1$, for any fixed $\mathfrak{c}\in(0,2)$, and for all $t\in(\frac{1}{2},1)$, it is obvious that $\Upsilon_2+2\,(\Upsilon_3+\Upsilon_4)<\Upsilon_2+2\,(\Upsilon_3+\Upsilon_4)\,\delta_2<\Upsilon_2$ and thus $\psi_1'(\delta_2)=\Upsilon_2+2\,(\Upsilon_3+\Upsilon_4)\,\delta_2>0$.

In both cases, $\psi_1(\delta_2)$ is an increasing function and; therefore, for any fixed $\mathfrak{c} \in (0,2)$ and $t \in (\frac{1}{2},1)$, the maximum value of $\psi_1(\delta_2)$ occurs at $\delta_2 = 1$ and

$$\max_{\delta_2}\Big\{\psi_1(\delta_2)\Big\} = \psi_1(1) = \Upsilon_1 + \Upsilon_2 + \Upsilon_3 + \Upsilon_4. \tag{47}$$

For $\mathfrak{c} = 0$ and $\mathfrak{c} = 2$, we respectively obtain that

$$\varphi(\delta_1,\, \delta_2) = \Upsilon_4\Big|_{\mathfrak{c}=0} = \frac{\Big(\mathcal{U}_1^{(\beta)}(t)\Big)^2}{4\,(1+2\,\tau)^2}\,(\delta_1+\delta_2)^2$$

and

$$\varphi(\delta_1,\, \delta_2) = \Upsilon_1\Big|_{\mathfrak{c}=2} = \frac{16\,\mathcal{U}_1^{(\beta)}(t)\Big|\Big(\mathcal{U}_3^{(\beta)}(t) + \mathcal{U}_2^{(\beta)}(t) + \frac{1}{4}\mathcal{U}_1^{(\beta)}(t)\Big)\,(1+\tau)^2 - \Big(\mathcal{U}_1^{(\beta)}(t)\Big)^3\Big|}{3\,(1+3\,\tau)\,(1+\tau)^3}. \tag{48}$$

By taking equation (48) and the two mentioned cases in account, for $0 \le \delta_2 < 1$, for any fixed $\mathfrak{c} \in [0,2]$, and for all $t \in (\frac{1}{2},1)$, the maximum value of $\psi_1(\delta_2)$ is

$$\max_{\delta_2}\Big\{\psi_1(\delta_2)\Big\} = \psi_1(1) = \Upsilon_1 + \Upsilon_2 + \Upsilon_3 + \Upsilon_4.$$

(2) for $\delta_1 = 1$ and $0 \le \delta_2 \le 1$ (similarly, for $\delta_2 = 1$ and $0 \le \delta_1 \le 1$), $\varphi(\delta_1,\, \delta_2)$ takes the form

$$\psi_2(\delta_2) := \varphi(1,\, \delta_2) = (\Upsilon_3 + \Upsilon_4)\,\delta_2^2 + (\Upsilon_2 + 2\,\Upsilon_4)\,\delta_2 + \Upsilon_1 + \Upsilon_2 + \Upsilon_3 + \Upsilon_4.$$

Analogous to the previously mentioned cases of $\Upsilon_3 + \Upsilon_4$, we conclude that

$$\max_{\delta_2}\Big\{\psi_2(\delta_2)\Big\} = \psi_2(1) = \Upsilon_1 + 2\,\Upsilon_2 + 2\,\Upsilon_3 + 4\,\Upsilon_4. \tag{49}$$

Since $\psi_1(1) \le \psi_2(1)$ for $\mathfrak{c} \in (0,2)$ and $t \in (\frac{1}{2},1)$, we see that

$$\max_{\delta_1,\,\delta_2}\Big\{\varphi(\delta_1,\, \delta_2)\Big\} = \varphi(1,\, 1) \tag{50}$$

on the boundary of $\mathbb{S}$. Therefore, the maximum value of $\varphi(\delta_1,\, \delta_2)$ occurs at $\delta_1 = 1$ and $\delta_2 = 1$ in the closed square $\mathbb{S}$. Now, for a fixed value of t, let $\mathcal{T} : [0,2] \to \mathbb{R}$ be the function defined by

$$\mathcal{T}(\mathfrak{c},\, t) = \max_{\delta_1,\,\delta_2}\Big(\varphi(\delta_1,\, \delta_2)\Big) = \varphi(1,1) = \Upsilon_1 + 2\,\Upsilon_2 + 2\,\Upsilon_3 + 4\,\Upsilon_4 \tag{51}$$

Upon substituting the expressions of Υ_1, Υ_2, Υ_3, and Υ_4 into (51), we obtain that

$$\mathcal{T}(\mathfrak{c},\, t) = \frac{\Big(\mathcal{U}_1^{(\beta)}(t)\Big)^2}{(1+2\,\tau)^2} + \frac{\mathcal{E}_1\,\mathfrak{c}^4 + 4\,\mathcal{E}_2\,\mathfrak{c}^2}{48\,(1+3\,\tau)\,(1+\tau)^3\,(1+2\,\tau)^2}, \tag{52}$$

where

$$\mathcal{E}_1 = 16\,(1+2\,\tau)^2\,\mathcal{U}_1^{(\beta)}(t)\left|\left(\mathcal{U}_3^{(\beta)}(t)+\mathcal{U}_2^{(\beta)}(t)+\frac{1}{4}\mathcal{U}_1^{(\beta)}(t)\right)(1+\tau)^2-\left(\mathcal{U}_1^{(\beta)}(t)\right)^3\right|$$
$$+\left(\mathcal{U}_1^{(\beta)}(t)\right)^2\left[3\,(1+3\,\tau)\,(1+\tau)^3-12\,(1+\tau)^2\,(1+2\,\tau)^2\right]$$
$$-2\,(1+\tau)\,(1+2\,\tau)\,\mathcal{U}_1^{(\beta)}(t)\left[3\,(1+3\,\tau)\left(\mathcal{U}_1^{(\beta)}(t)\right)^2+8\,(1+\tau)\,(1+2\,\tau)\,\mathcal{U}_2^{(\beta)}(t)\right] \tag{53}$$

and

$$\mathcal{E}_2 = 12\,(1+2\,\tau)^2\,(1+\tau)^2\left(\mathcal{U}_1^{(\beta)}(t)\right)^2+6\,(1+\tau)\,(1+2\,\tau)\,(1+3\,\tau)\left(\mathcal{U}_1^{(\beta)}(t)\right)^3$$
$$+16\,(1+\tau)^2\,(1+2\,\tau)^2\,\mathcal{U}_1^{(\beta)}(t)\,\mathcal{U}_2^{(\beta)}(t)-6\,(1+3\,\tau)\,(1+\tau)^3\left(\mathcal{U}_1^{(\beta)}(t)\right)^2. \tag{54}$$

By assuming that the function $\mathcal{T}(\mathfrak{c},\,t)$ has a maximum value at an interior point $0<\mathfrak{c}<2$, we obtain

$$\frac{d\,\mathcal{T}}{d\,\mathfrak{c}} = \frac{\mathcal{E}_1\,\mathfrak{c}^3+2\,\mathcal{E}_2\,\mathfrak{c}}{12\,(1+3\,\tau)\,(1+\tau)^3\,(1+2\,\tau)^2}. \tag{55}$$

With some calculations, we can examine the sign of $\frac{d\,\mathcal{T}}{d\,\mathfrak{c}}$ taking into account the following four cases.

(i) Suppose that $\mathcal{E}_1 \geq 0$ and $\mathcal{E}_2 \geq 0$, then $\frac{d\,\mathcal{T}}{d\,\mathfrak{c}} \geq 0$; indicating that $\mathcal{T}(\mathfrak{c},\,t)$ is an increasing function. Therefore, we get that

$$\max_{0<\mathfrak{c}<2}\left\{\mathcal{T}(\mathfrak{c},\,t)\right\} = \mathcal{T}(2^-,\,t) = \frac{\left(\mathcal{U}_1^{(\beta)}(t)\right)^2}{(1+2\,\tau)^2}+\frac{\mathcal{E}_1+\mathcal{E}_2}{3\,(1+3\,\tau)\,(1+\tau)^3\,(1+2\,\tau)^2}, \tag{56}$$

which means:

$$\max_{0<\mathfrak{c}<2}\left\{\max_{\mathbb{S}}\left\{\varphi(\delta_1,\delta_2)\right\}\right\} = \mathcal{T}(2^-,\,t).$$

(ii) Suppose that $\mathcal{E}_1 > 0$ and $\mathcal{E}_2 < 0$, then $\mathfrak{c}_0 = \sqrt{-\frac{2\,\mathcal{E}_2}{\mathcal{E}_1}}$ is a critical value of the function $\mathcal{T}(\mathfrak{c},\,t)$. By assuming $\mathfrak{c}_0 \in (0,2)$, we get that $\left.\frac{d^2\,\mathcal{T}}{d\,\mathfrak{c}^2}\right|_{\mathfrak{c}=\mathfrak{c}_0} > 0$, that is, $\mathfrak{c}=\mathfrak{c}_0$ is a local minimum value of $\mathcal{T}(\mathfrak{c},\,t)$. Thus, the function $\mathcal{T}(\mathfrak{c},\,t)$ can not possess a local maximum.

(iii) Suppose that $\mathcal{E}_1 \leq 0$ and $\mathcal{E}_2 \leq 0$, then $\frac{d\mathcal{T}}{d\mathfrak{c}} \leq 0$; indicating that $\mathcal{T}(\mathfrak{c},\, t)$ is a decreasing function. Thus,

$$\max_{0<\mathfrak{c}<2}\left\{\mathcal{T}(\mathfrak{c},\, t)\right\} = \mathcal{T}(0^+,\, t) = 4\,\Upsilon_4 = \frac{\left(\mathcal{U}_1^{(\beta)}(t)\right)^2}{(1+2\,\tau)^2}. \tag{57}$$

(iv) Suppose that $\mathcal{E}_1 < 0$ and $\mathcal{E}_2 > 0$, then $\mathfrak{c}_0$ is a critical value of the function $\mathcal{T}(\mathfrak{c},\, t)$. By assuming $\mathfrak{c}_0 \in (0,2)$, we obtain that $\left.\frac{d^2\,\mathcal{T}}{d\,\mathfrak{c}^2}\right|_{\mathfrak{c}=\mathfrak{c}_0} < 0$, which means that the function $\mathcal{T}(\mathfrak{c},\, t)$ has a local maximum occurring at $\mathfrak{c} = \mathfrak{c}_0$. Thus,

$$\max_{0<\mathfrak{c}<2}\left\{\mathcal{T}(\mathfrak{c},\, t)\right\} = \mathcal{T}(\mathfrak{c}_0,\, t), \tag{58}$$

where

$$\mathcal{T}(\mathfrak{c}_0,\, t) = \frac{\left(\mathcal{U}_1^{(\beta)}(t)\right)^2}{(1+2\,\tau)^2} - \frac{\mathcal{E}_2^2}{12\,\mathcal{E}_1\,(1+3\,\tau)\,(1+\tau)^3\,(1+2\,\tau)^2}.$$

Therefore, the proof of the above Theorem is evidently completed.

Ultimately, we introduce two essential corollaries that obtained from the classes $\Sigma_{\Xi}^{\beta}(t)$ and $\Lambda_{\Xi}^{\beta}(t)$.

Corollary 1. *Let* $\mathrm{f} \in \Xi$ *of the form* (5) *be in the class* $\Omega_{\Xi}^{\beta}(t,\,0) = \Lambda_{\Xi}^{\beta}(t)$. *Then*

$$\left|\, a_2\, a_4 - a_3^2 \,\right| \leq \begin{cases} \mathcal{T}(2^-, t) & \mathcal{E}^*{}_1 \geq 0 \quad \textit{and} \quad \mathcal{E}^*{}_2 \geq 0; \\[2ex] \max\limits_{t}\left\{4\,\beta^2\,t^2,\;\; \mathcal{T}(2^-, t)\right\} & \mathcal{E}^*{}_1 > 0 \quad \textit{and} \quad \mathcal{E}^*{}_2 < 0; \\[2ex] 4\,\beta^2\,t^2 & \mathcal{E}^*{}_1 \leq 0 \quad \textit{and} \quad \mathcal{E}^*{}_2 \leq 0; \\[2ex] \max\limits_{t}\left\{\mathcal{T}(\mathfrak{c}_0, t),\;\; \mathcal{T}(2^-, t)\right\} & \mathcal{E}^*{}_1 < 0 \quad \textit{and} \quad \mathcal{E}^*{}_2 > 0, \end{cases} \tag{59}$$

where

$$\mathcal{T}(2^-,\, t) = 4\,\beta^2\,t^2 + \frac{\mathcal{E}^*{}_1 + \mathcal{E}^*{}_2}{3}, \tag{60}$$

$$\mathcal{T}(\mathfrak{c}_0, t) = 4\,\beta^2\,t^2 - \frac{{\mathcal{E}^*{}_2}^2}{12\,\mathcal{E}^*{}_1}; \quad \mathfrak{c}_0 = \sqrt{-\frac{2\,\mathcal{E}^*{}_2}{\mathcal{E}^*{}_1}}, \tag{61}$$

$$\begin{aligned} \mathcal{E}^*{}_1 = 16\,\mathcal{U}_1^{(\beta)}(t)\,&\left|\mathcal{U}_3^{(\beta)}(t) + \mathcal{U}_2^{(\beta)}(t) + \frac{1}{4}\,\mathcal{U}_1^{(\beta)}(t) - \left(\mathcal{U}_1^{(\beta)}(t)\right)^3\right| \\ &- 9\left(\mathcal{U}_1^{(\beta)}(t)\right)^2 - 2\,\mathcal{U}_1^{(\beta)}(t)\left[3\left(\mathcal{U}_1^{(\beta)}(t)\right)^2 + 8\,\mathcal{U}_2^{(\beta)}(t)\right], \end{aligned} \tag{62}$$

$$\mathcal{E}^*{}_2 = 2\,\mathcal{U}_1^{(\beta)}(t)\left[3\,\mathcal{U}_1^{(\beta)}(t) + 3\left(\mathcal{U}_1^{(\beta)}(t)\right)^2 + 8\,\mathcal{U}_2^{(\beta)}(t)\right], \tag{63}$$

and $\mathcal{U}_1^{(\beta)}(t)$, $\mathcal{U}_2^{(\beta)}(t)$, *and* $\mathcal{U}_3^{(\beta)}(t)$ *are defined by* (1).

Corollary 2. *Let* $\mathrm{f} \in \Xi$ *of the form* (5) *be in the class* $\Omega_{\Xi}^{\beta}(t, 1) = \Sigma_{\Xi}^{\beta}(t)$. *Then*

$$\left| a_2 a_4 - a_3^2 \right| \leq \begin{cases} \mathcal{T}(2^-, t) & \mathcal{D}^*{}_1 \geq 0 \quad and \quad \mathcal{D}^*{}_2 \geq 0; \\ \max\limits_{t}\left\{\frac{4\beta^2 t^2}{9},\ \mathcal{T}(2^-, t)\right\} & \mathcal{D}^*{}_1 > 0 \quad and \quad \mathcal{D}^*{}_2 < 0; \\ \frac{4\beta^2 t^2}{9} & \mathcal{D}^*{}_1 \leq 0 \quad and \quad \mathcal{D}^*{}_2 \leq 0; \\ \max\limits_{t}\left\{\mathcal{T}(\mathfrak{c}_0, t),\ \mathcal{T}(2^-, t)\right\} & \mathcal{D}^*{}_1 < 0 \quad and \quad \mathcal{D}^*{}_2 > 0, \end{cases} \tag{64}$$

where

$$\mathcal{T}(2^-, t) = \frac{4\beta^2 t^2}{9} + \frac{\mathcal{D}^*{}_1 + \mathcal{D}^*{}_2}{864}, \tag{65}$$

$$\mathcal{T}(\mathfrak{c}_0, t) = \frac{4\beta^2 t^2}{9} - \frac{\mathcal{D}^{*2}_2}{3456\,\mathcal{D}^*{}_1}, \quad \mathfrak{c}_0 = \sqrt{-\frac{2\,\mathcal{D}^*{}_2}{\mathcal{D}^*{}_1}}, \tag{66}$$

$$\begin{aligned} \mathcal{D}^*{}_1 = 144\,\mathcal{U}_1^{(\beta)}(t) & \left| 4\left(\mathcal{U}_3^{(\beta)}(t) + \mathcal{U}_2^{(\beta)}(t) + \frac{1}{4}\,\mathcal{U}_1^{(\beta)}(t)\right) - \left(\mathcal{U}_1^{(\beta)}(t)\right)^3 \right| \\ & - 336\left(\mathcal{U}_1^{(\beta)}(t)\right)^2 - 144\,\mathcal{U}_1^{(\beta)}(t)\left[\left(\mathcal{U}_1^{(\beta)}(t)\right)^2 + 4\,\mathcal{U}_2^{(\beta)}(t)\right], \end{aligned} \tag{67}$$

$$\mathcal{D}^*{}_2 = 48\,\mathcal{U}_1^{(\beta)}(t)\left[5\,\mathcal{U}_1^{(\beta)}(t) + 3\left(\mathcal{U}_1^{(\beta)}(t)\right)^2 + 12\,\mathcal{U}_2^{(\beta)}(t)\right], \tag{68}$$

and $\mathcal{U}_1^{(\beta)}(t)$, $\mathcal{U}_2^{(\beta)}(t)$, *and* $\mathcal{U}_3^{(\beta)}(t)$ *are defined by* (1).

3. Conclusion

In our present study, we have derived the new upper bound estimates and inequalities for the second Hankel determinant, $\mathrm{H_f}(2, 2)$, of a certain subclass of normalized bi-univalent functions in the open unit disk $\mathbb{U}$. The upper bound estimates are determined by using orthogonal ultraspherical polynomials, which provide information about the properties and characteristics of these functions in the context of $\mathrm{H_f}(2, 2)$. Furthermore, we provide new findings acquired by specializing the parameter τ utilized in our analysis.

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