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Supra ϵ-open Sets: Features, Operators and Applications

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Abstract. In supra topological spaces, we provide supra ϵ-open sets, an extremely broad class of open sets. We demonstrate that, the previously comparable concepts of supra regular (respectively, α-, semi-, pre-, b-, β-, and R-) open sets are contained in this new category of open sets. To further illustrate the key concepts discussed in the study, we have included a geometric topological diagram [see Diagram 1]. Also, we outline this class's primary characteristics. Specifically, we show that our new category forms a supra topology rather than a topological space. Utilizing our recently introduced category of supra open sets, we define new kinds of operators called supra ϵ-interior (closure, accumulation, exterior, and boundary, respectively). Moreover, we highlight the deviations between these new operators and their corresponding operators. Furthermore, we also give some key examples and counterexamples to illustrate the importance of our new operators. In addition, we highlight the advantages and distinctions of our work in comparison to similar studies in the field.

2020 Mathematics Subject Classifications: 54A05, 54C10, 54C08.

Key Words and Phrases: Supra ϵ-open set; Supra ϵ-interior operator; Supra ϵ-closure operator; Partition; Applications

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1. Introduction

In the past few decades, a significant focus of topological, supra topological, and soft topological research has been the examination of various generalized open, supra open, and soft open set types as well as their structural characteristics.

Semi-open sets, and semi-continuity of mappings were first proposed by Levine [1] in 1963. Njasta [2] then presented his approach of α-open sets in 1965. Mashhour et al. [3] developed the notion of pre-open set to analyze the pre-continuous mappings. The notion of β-open sets was proposed by Abd-El-Monsef et al. [4] in 1983 as a way to study β-continuous mappings. In [5, 6], the notion of b-open sets was thoroughly examined. Piotrowski [7] defined somewhat open sets to preseent somewhat continuity as stated in [8]. The notion of somewhere dense sets was proposed in [9, 10]. Additional characteristics of this notion were examined in [11]. Recently, Alqahtani and Abd El-latif [12], generalized almost all the previous notions by introducing the approach of $\mathcal{N}$-open sets, in 2024.

Mashhour et al. [13], presented the notion of supra open sets which consider the basic building blocks of supra topology ((abbreviated, STS)). They expanded on some basic topological concepts, including the continuity and separation axioms, as well as interior and closure operators. The notions of of supra α- [14] (respectively, pre- [15], b- [16], β- [17], R- [18], and semi- [19]) open sets have been presented and their primary characteristics have been presented. More operators on supra topological spaces [20–22] have been introduced.

In the field of generalized soft open sets [23, 24], generalized soft continuity [25], soft semi-open sets and soft semi irresolute soft mappings [26, 27], various types of soft open sets and continuity [28], soft somewhere dense sets[29], and nearly soft β-open sets [30] , have been provided. Subsequent studies on soft continuity were carried out [31, 32].

The concept of the soft ideal was initially presented in [33]. Soft compactness [34], soft connectedness [35], soft generalized open sets [36–38], generalized (fuzzy) soft rough sets [39, 40], and soft separation axioms [41] are just a few of the topological properties that are generalized using this concept.

El-Sheikh et al. proposed the concept of supra soft topological spaces [42]. Additionally, they presented the notions of supra soft pre- (respectively, α, semi, β, and γ-) open sets. Subsequent research has examined numerous generalized supra soft operators through supra soft-b-open sets [43], supra soft-δ_i-open sets [44, 45], supra soft somewhere dense sets [46], and supra soft somewhat open sets [47].

We aim in this paper to present the approach of supra ϵ-open sets to supra topological spaces. Also, we go over the connections between our novel approach and the earlier related studies. In order to properly demonstrate the main ideas covered in the paper, we have included a geometric topological diagram [see Diagram 1].

$$\begin{array}{ccccccccccc}
SO_{regular}(\chi) & \longrightarrow & SO(\chi) & \longrightarrow & S\alpha O(\chi) & \longrightarrow & SSO(\chi) & \longrightarrow & S\beta O(\chi) & \longrightarrow & SRO(\chi) \\
 & & & & \downarrow & & \downarrow & \nearrow & & & \downarrow \\
 & & & & SPO(\chi) & \longrightarrow & SBO(\chi) & & \longrightarrow & & SO_{\epsilon}(\chi)
\end{array}$$

DIAGRAM 1. The connections between the new category and other preceding studies.

Moreover, we provide the main features of this new category. In particular, we show that supra ϵ-open sets in an STS and their subspaces generally do not relate to one another. In addition to, the intersection of finite numbers of supra ϵ-open sets is not supra ϵ-open, generally. After that, we investigate new types of operators known as supra ϵ-interior (closure, accumulation, exterior, and boundary, respectively) operator using our recently established category of supra open sets. In order to demonstrate the significance of our new operators, we also provide some important examples and counterexamples.

2. Preliminaries and background

Let (χ, ν) be an STS, the categories of supra (respectively, regular-, pre-, semi, β-, α-, b-, and R-) open sets will represented by $SO(\chi)$ (respectively, $SO_{regular}(\chi)$, $SPO(\chi)$, $SSO(\chi)$, $S\beta O(\chi)$,$S\alpha O(\chi)$, $SBO(\chi)$, and $SRO(\chi)$) across this paper.

Definition 1. *[13] The collection $\nu \subseteq P(\chi)$ is called supra topology (or STS) on χ if ν contains χ and $\emptyset$ and closed under arbitrary union. Also, if $G \in \nu$, then G is called supra open set and G^c is called supra closed set. Moreover, $SO(\chi)$ will denote the class of all supra open sets.*

Definition 2. *[13] For the subset K of an STS (χ, ν), the $int(K)$ or K° (respectively, $cl(K)$ or $\overline{K}$, and $b(K)$) will denote the supra interior (respectively, closure, and boundary) of K, where*
$int(K) = \cup\{G : G \in \nu$ and $G \subseteq K\}$, $cl(K) = \cap\{N : N \in \nu^c$ and $K \subseteq N\}$, and $b(K) = cl(K)\backslash int(K)$.

Theorem 1. *[13] Regarding a subset T of an STS (χ, ν), we have*

(1) $cl(T^c) = [int(T)]^c$.

(2) $int(T^c) = [cl(T)]^c$.

Definition 3. *[15–19] Let H be a subset of an STS (χ, ν). Then,*

(1) *If $H = int(cl(H))$, then $H \in SO_{regular}(\chi)$.*

(2) *If $H \subseteq int(cl(H))$, then $H \in SPO(\chi)$.*

(3) *If $H \subseteq cl(int(H))$, then $H \in SSO(\chi)$.*

(4) *If $H \subseteq int(cl(int(H)))$, then $H \in S\alpha O(\chi)$.*

(5) *If $H \subseteq cl(int(cl(H)))$, then $H \in S\beta O(\chi)$.*

(6) *If $H \subseteq cl(int(H))\tilde{\cup} int(cl(H))$, then $H \in SBO(\chi)$.*

(7) *If $int(cl(H)) \neq \emptyset$, then $H \in SRO(\chi)$.*

(8) *If $int(cl(H)) = \emptyset$, then $H \in SND(\chi)$.*

Definition 4. *For the subset K of an STS (χ, ν), the class*

$$\nu_K = \{K \cap G : G \in \nu\}$$

defines an STS on K, and it is called a subspace of (χ, ν).

3. Supra ϵ-open sets and relationships

This part begins by presenting the definitions of supra ϵ-open and supra ϵ-closed sets and the properties based on them. We show that, this new category of supra open sets includes the previously comparable concepts of supra regular (α-, semi-, pre-, b-, β-, and R-) open sets. In addition, we have included a geometric topological diagram [see Diagram 1] to further demonstrate the basic concepts covered in the study. Moreover, we discuss the main features of this class. In particular, we demonstrate that instead of forming a topological space, our new category forms a supra topology.

Definition 5. *Let H be a subset of an STS (χ, ν). Then, H is called supra ϵ-open set if either $H = \emptyset$ or*

$$H \subseteq \begin{cases} b(H) \cup \overline{H}^{\circ}, & H \in SRO(\chi), \\ b(H), & H \in SND(\chi) \text{ and } b(H) \text{ is infinite.} \end{cases}$$

Also, H^c is called supra ϵ-closed-set. The category of all supra ϵ-open (closed) sets will be indicated by $SO_\epsilon(\chi)$ $(SC_\epsilon(\chi))$.

Proposition 1. *Every singleton $\{c\}$ subset of an STS (χ, ν) is either supra ϵ-open or supra nowhere dense.*

Proof. Let $\{c\} \nsubseteq SND(\chi)$. Then, $\overline{\{c\}}^{\circ} \neq \tilde{\emptyset}$ and so $\{c\} \in SRO(\chi)$. Hence, $\{c\} \in SO_\epsilon(\chi)$.
On the other way, suppose that $\{c\} \notin SO_\epsilon(\chi)$, then $\{c\} \nsubseteq \overline{\{c\}}^{\circ} \cup b(\{c\})$ and so $\{c\} \nsubseteq \overline{\{c\}}^{\circ}$. Therefore, $\{c\} \in SND(\chi)$.

Remark 1. *For the subset K of an STS (χ, ν), we have:*

(1) *If K is a non-empty finite, closed and nowhere dense set, then $K \notin SO_\epsilon(\chi)$.*

(2) *If K is a infinite and nowhere dense set, then $K \in SO_\epsilon(\chi)$.*

Remark 2. *As the authors demonstrated in [18], each supra regular (respectively, α-, semi-, pre -, b-, and β-) open set is a supra-R-open. Consequently from Definition 5, the reader can notice that, they are all supra ϵ-open.*

The following counterexample will demonstrate that our perspective on the aforementioned remark is non-reversible, generally.

Example 1. *Consider the supra topology $\nu = \{\emptyset, T \subseteq \mathbb{R} : -1 \in T \text{ or } 0 \in T\}$, on the set of real numbers $\mathbb{R}$. Regarding the set of natural numbers $\mathbb{N}$, we have $\overline{\mathbb{N}}^{\circ} = \mathbb{N}^{\circ} = \emptyset$, and hence $\mathbb{N} \notin SRO(\chi)$. However, $b(\mathbb{N}) = \mathbb{N}$ is infinite. Hence, $\mathbb{N} \in SO_\epsilon(R)$.*

Corollary 1. *The next implications are hold for an STS (χ, ν), which are not reversible.*

$$\begin{array}{ccccccccccc} SO_{regular}(\chi) & \longrightarrow & SO(\chi) & \longrightarrow & S\alpha O(\chi) & \longrightarrow & SSO(\chi) & \longrightarrow & S\beta O(\chi) & \longrightarrow & SRO(\chi) \\ & & & & \downarrow & & \downarrow & \nearrow & & & \downarrow \\ & & & & SPO(\chi) & \longrightarrow & SBO(\chi) & & & \longrightarrow & SO_\epsilon(\chi) \end{array}$$

DIAGRAM 1. *The connections between the new category and other preceding studies.*

Supra ϵ-open sets in an STS (χ, ν) and their subspaces generally do not relate to one another, as shown in the upcoming example.

Example 2. *Let $\nu = \{\chi, \emptyset, \{i, l\}, \{i, j, k\}, \{i, j, l\}\}$ be an STS on $\chi = \{i, j, k, l\}$. Regarding the set $W = \{k, l\}$, we have $\nu_W = \{W, \emptyset, \{k\}, \{l\}\}$. The set $\{k\}$ is supra ϵ-open in (W, ν_W) whereas $\{k\}$ is not supra ϵ-open in (χ, ν).*

Definition 6. *For the subset K of an STS (χ, ν), the class*

$$\nu_K = \{K \cap G : G \in SO_\epsilon(\chi)\}$$

defines an STS on K, and it is called an ϵ-subspace of (χ, ν).

Proposition 2. *Let (W, ν_W) be an ϵ-subspace of an STS (χ, ν) and F be a subset of χ. Then, $F \in SC_\epsilon(W)$ if and only if there is $B \in SC_\epsilon(\chi)$ such that $F = W \cap B$.*

Proof. Obvious.

Theorem 2. *If $Y \in SBO(\chi)$ such that $int(Y) = \emptyset$, for a proper subset Y of an STS (χ, ν), then both of Y and Y^c are supra ϵ-open.*

Proof. Let $Y \in SBO(\chi)$, then $Y \subseteq cl(int(Y))\tilde{\cup}int(cl(Y))$. Since $int(Y) = \emptyset$, $Y \subseteq int(cl(Y))$. Hence, $Y \in SRO(\chi)$. Given Remark 2, $Y \in SO_\epsilon(\chi)$. Furthermore, we have $[int(Y)]^c = cl(G^c) = \emptyset^c = \chi$. This implies, $int(cl(G^c)) = int(\chi) = \chi \neq \emptyset$ and so $G^c \in SRO(\chi)$. Thus, $G^c \in SO_\epsilon(\chi)$.

Proposition 3. *Each supra neighbourhood of any point in an STS (χ, ν) is supra ϵ-open.*

Proof. Suppose that S is a supra neighbourhood for $x \in \chi$. Then, $\exists\, G \in \nu$ such that $x \in G \subseteq S$. Hence, $G \subseteq cl(G) \subseteq cl(S)$ and so $S \in SRO(\chi)$. Therefore, $S \subseteq \overline{S}^{\circ} \cup b(S) = cl(S)$. Thus, $S \in SO_\epsilon(\chi)$.

Remark 3. *Generally, the converse of Proposition 3 is untrue. Let $\nu = \{\chi, \emptyset, \{1, 2\}, \{1, 3, 4\}\}$ be an STS on $\chi = \{1, 2, 3, 4\}$. Then, $\{1\} \in SO_\epsilon(\chi)$, however $\{1\}$ is not supra neighbourhood for any point in χ.*

Theorem 3. **(1)** *If $\psi = \{H_\jmath, \jmath \in \pi\} \subseteq SO_\epsilon(\chi)$, then $\bigcup_{\jmath\in\pi} H_\jmath \in SO_\epsilon(\chi)$.*

(2) *If $\psi = \{H_\jmath, \jmath \in \pi\} \subseteq SC_\epsilon(\chi))$, then $\tilde{\bigcap}_{\jmath\in\pi} H_\jmath \in SC_\epsilon(\chi)$.*

Proof.

(1) Let $\psi = \{H_\jmath, \jmath \in \pi\} \subseteq SO_\epsilon(\chi)$. If for all $\jmath \in \pi$, $H_\jmath = \emptyset$, then we get our result. Now, if some members of ψ are non-empty, then we have to cases.
Case (1): If $\bigcup_{\jmath\in\pi} H_\jmath \in SND(\chi)$, then

$$\bigcup_{\jmath\in\pi} \overline{H_\jmath}^{\circ} \subseteq \overline{\bigcup_{\jmath\in\pi} H_\jmath}^{\circ} = \emptyset.$$

This means,
$H_\jmath \in SND(\chi)$ for each $\jmath \in \pi$, which follows $b(H_\jmath)$ is infinite for all $\jmath \in \pi$ and hence

$$b(\bigcup_{\jmath\in\pi} H_\jmath) = \bigcup_{\jmath\in\pi} b(H_\jmath) \text{ is infinite.}$$

Therefore, we get our result.
Case (2): If $(\bigcup_{\jmath\in\pi} H_\jmath \in SRO(\chi)$, then $(\overline{\bigcup_{\jmath\in\pi} H_\jmath}^{\circ}) \neq \emptyset$. Hence,

$$\bigcup_{\jmath\in\pi} H_\jmath \subseteq (\overline{\bigcup_{\jmath\in\pi} H_\jmath}^{\circ}) \cup b(\bigcup_{\jmath\in\pi} H_\jmath) = \overline{(\bigcup_{\jmath\in\pi} H_\jmath)}.$$

Therefore, $\bigcup_{\jmath\in\pi} H_\jmath \in SO_\epsilon(\chi)$.

(2) By a similar way to (1).

Remark 4. *The next example shall prove that:*

(1) *The intersection of finite numbers of supra ϵ-open sets is not supra ϵ-open, generally.*

(2) *The union of finite numbers of supra ϵ-closed sets is not supra ϵ-closed, generally.*

Example 3. *Let $\nu = \{\chi, \emptyset, \{200, 300\}, \{100, 300\}\}$ be an STS on $\chi = \{100, 200, 300\}$. Then, $A = \{100, 300\}$ and $B = \{100, 200\}$ are supra ϵ-open sets, however $A \cap B = \{100\}$ is not supra ϵ-open. Also, $C = \{200\}$ and $D = \{300\}$ are supra ϵ-closed sets, however $C \cup D = \{200, 300\}$ is not supra ϵ-closed.*

Remark 5. *It is evident from Theorem 3 and Remark 4 that our new category does not form a topological space and instead forms a supra topology.*

4. Applications of supra ϵ-open for new supra operators

The objective of this section, is to outline novel kinds of operators, called supra ϵ-interior (respectively, closure, accumulation, exterior, and boundary) operator, using our new category of supra open sets. The primary characteristics of every operator are listed. We also give the distinctions between these new operators and the operators that correspond to them. Furthermore, in order to demonstrate the significance of our new operators, we provide several essential examples and counterexamples.

Definition 7. *For the subset K of an STS (χ, ν), the $int_{\epsilon}(K)$ will denote the supra ϵ-interior of K, where*

$$int_{\epsilon}(K) = \cup\{G : G \in SO_{\epsilon}(\chi) \text{ and } G \subseteq K\}.$$

The proof of the next lemma is obvious from Definition 7, so it is omitted.

Lemma 1. *For the subsets K and I of an STS (χ, ν), we have the following:*

(1) $u \in int_{\epsilon}(K) \Leftrightarrow$ *if there is $I \in SO_{\epsilon}(\chi)$ such that $u \in I \subseteq K$.*

(2) $K \in SO_{\epsilon}(\chi) \Leftrightarrow int_{\epsilon}(K) = K$.

Theorem 4. *For the supra ϵ-interior operator $int_{\epsilon} : P(\chi) \longrightarrow P(\chi)$ and $E \in P(\chi)$, we have*

$$int_{\epsilon}(E) = \begin{cases} \emptyset, & E \in SND(\chi) \text{ and } b(E) \text{ is finite.} \\ E \cap b(E), & E \in SND(\chi) \text{ and } b(E) \text{ is infinite.} \\ E, & E \in SRO(\chi). \end{cases}$$

Proof. Assume contrary that, $s \in E$, where as $E \in SND(\chi)$ and $b(E)$ is finite. Given Lemma 1 (1), there is $I \in SO_{\epsilon}(\chi)$ such that $s \in I \subseteq E$. Since $E \in SND(\chi)$, $I \in SND(\chi)$ and so $s \in b(I) \subseteq b(E)$. Given $I \in SO_{\epsilon}(\chi)$, $b(I)$ is infinite, then $b(E)$ is also infinite, which contradicts our assumption. Hence, $int_{\epsilon}(E) = \emptyset$.
Now, Assume contrary that $s \in E$, where as $E \in SND(\chi)$ and $b(E)$ is infinite. By the same technique, we can get

$$int_{\epsilon}(E) \subseteq E \cap b(E) \tag{1}$$

On the other way, assume that $s \in E \cap b(E)$. Since $E \in SND(\chi)$ and $b(E)$ is infinite, $E \in SO_{\epsilon}(\chi)$, given Definition 7. By Lemma 1 (2), $s \in E = int_{\epsilon}(E)$. Hence,

$$E \cap b(E) \subseteq int_{\epsilon}(E) \tag{2}$$

From Eqs (1) and (2), $int_{\epsilon}(E) = E \cap b(E)$.
Finally, If $E \in SRO(\chi)$, given Definition 7 and Lemma 1 (2), $int_{\epsilon}(E) = E$.

In the example that follows, we illustrate the previously mentioned theorem.

Example 4. *Consider the sets $B = \{0, 1, 2\}$, $C = \{2, 3, 4, 5\}$ and the natural number set $\mathbb{N}$, in Example 1, we have*

(1) $\mathbb{N} \in SND(\chi)$ *and $b(\mathbb{N})$ is infinite, and hence* $int_{\epsilon}(\mathbb{N}) = \mathbb{N} \cap b(\mathbb{N}) = \mathbb{N}$.

(2) $B \in SRO(\chi)$, *and so* $int_{\epsilon}(B) = B$.

(3) $C \in SND(\chi)$ *and $b(C)$ is finite, and then* $int_{\epsilon}(C) = \emptyset$.

Theorem 5. *For the subsets K and I of an STS (χ, ν), we have the following:*

(1) *If $K \subseteq I$, then* $int_{\epsilon}(K) \subseteq int_{\epsilon}(I)$.

(2) $int(K) \subseteq int_\epsilon(K)$.

Proof.

(1) Suppose that $u \in int_\epsilon(K)$. Given Theorem 4, either $K \in SND(\chi)$ and $b(K)$ is infinite or $K \in SRO(\chi)$, and in both situations, results in $int_\epsilon(K) = K$. Hence, $u \in int_\epsilon(I)$, and therefore $int_\epsilon(K) \subseteq int_\epsilon(I)$.

(2) suppose that $u \in int(K)$, then there exists $G \in \nu$ such that $u \in G \subseteq K$. Given (1),

$$G \in SO_\epsilon(\chi) \text{ and } u \in int_\epsilon(G) = G \subseteq int_\epsilon(K).$$

Thus, $u \in int_\epsilon(K)$.

Theorem 6. *Let (χ, ν) be an STS and $V, U \in P(\chi)$. Then,*

(1) $int_\epsilon(\chi) = \chi$ *and* $int_\epsilon(\emptyset) = \emptyset$.

(2) $int_\epsilon(V) \subseteq (V)$.

(3) $int_\epsilon(int_\epsilon(V)) = int_\epsilon(V)$.

(4) $int_\epsilon[V \cap U] \subseteq int_\epsilon(V) \cap int_\epsilon(U)$.

(5) $int_\epsilon(V) \cup int_\epsilon(U) \subseteq int_\epsilon[V \cup U]$.

Proof. Follows from Definition 7.

Remark 6. *The equality of Theorem 5 and Theorem 6 parts (2), (4) and (5) are not satisfied as shall shown in the provided counterexamples.*

Examples 1. *Consider the sets $B = \{0, 1, 2, 3\}$, $C = \{2, 3, 4, 6\}$, $D = \{-1, 2, 4, 5\}$ and the natural number set $\mathbb{N}$, in Example 1, we have:*

(1) $int_\epsilon(C) = \emptyset \subseteq int_\epsilon(B) = B$, *however* $C \nsubseteq B$.

(2) $int_\epsilon(\mathbb{N}) = \mathbb{N} \nsubseteq int(\mathbb{N}) = \emptyset$.

(3) $C \nsubseteq int_\epsilon(C) = \emptyset$.

(4) $int_\epsilon(B) \cap int_\epsilon(D) = \{2\} \nsubseteq int_\epsilon[B \cap D] = int_\epsilon(\{2\}) = \emptyset$.

(5) $int_\epsilon(B \cup C) = int_\epsilon(\{0, 1, 2, 3, 4, 6\}) = \{0, 1, 2, 3, 4, 6\} \nsubseteq int_\epsilon(B) \cup int_\epsilon(C) = \{0, 1, 2, 3\}$.

Definition 8. *Let $C \in P(\chi)$ be a subset of an STS (χ, ν), then $cl_\epsilon(C)$ will denote the supra ϵ-closure of C, where*

$$cl_\epsilon(C) = \cap\{N : N \in SC_\epsilon(\chi) \text{ and } C \subseteq N\}.$$

Theorem 7. *Given a subset J of an STS (χ, ν), then J has the following characteristics:*

(1) $cl_\epsilon(J) = J \Leftrightarrow J \in SC_\epsilon(\chi))$.

(2) $u \in cl_\epsilon(J) \Leftrightarrow J \cap G \neq \emptyset$ *for each* $G_u \in SO_\epsilon(\chi)$.

(3) $cl_\epsilon(J) \subseteq cl(J)$.

Proof.

(1) Follows from Definition 8.

(2) "*Necessity*" Assume that there is $G_u \in SO_\epsilon(\chi)$ such that $J \cap G = \emptyset$, whereas $u \in cl_\epsilon(J)$. Then,

$$J \subseteq G^c.$$

Given (1),

$$cl_\epsilon(J) \subseteq G^c \text{ and } u \notin G^c.$$

Therefore,

$$u \notin cl_\epsilon(J), \text{ which is a contradiction.}$$

"*Sufficient*" Assume contrary that, $u \notin cl_\epsilon(J)$, then there is $V \in SC_\epsilon(\chi)$ such that $u \notin V$ and $J \subseteq V$, and so

$$u \in V^c \text{ and } V^c \cap J = \emptyset, \text{ where}$$

$V^c \in SO_\epsilon(\chi)$, which is a contradiction.

(3) Assume that $u \notin cl(J)$. Then, $J \cap G = \emptyset$, for some $G_u \in \nu$. Hence, $J \cap G = \emptyset$, for some $G_u \in SO_\epsilon(\chi)$. Given (2), $u \notin cl_\epsilon(J)$.

Theorem 8. *For the supra* ϵ*-closure operator* $cl_\epsilon : P(\chi) \longrightarrow P(\chi)$ *and* $E \in P(\chi)$, *we have*

$$cl_\epsilon(E) = \begin{cases} \chi, & E^c \in SND(\chi) \text{ and } b(E^c) \text{ is finite.} \\ E, & E^c \in SND(\chi) \text{ and } b(E^c) \text{ is infinite.} \\ E, & E^c \in SRO(\chi). \end{cases}$$

Proof. Much like the proof of Theorem 4.

The relationship between the supra ϵ-closure operator and the supra ϵ-closure operator is examined in the forthcoming theorem.

Theorem 9. *Regarding a subset T of an STS (χ, ν), we have*

(1) $cl_\epsilon(T^c) = [int_\epsilon(T)]^c$.

(2) $int_\epsilon(T^c) = [cl_\epsilon(T)]^c$.

Proof.

(1) Suppose that $u \notin [int_\epsilon(T)]^c$. Then, $u \in int_\epsilon(T)$. This implies that,

$$\exists\ G \in SO_\epsilon(\chi) \text{ such that } u \in G \subseteq T, \text{ given Lemma 1,}$$

which follows

$$T^c \cap G = \emptyset.$$

Hence,

$$u \notin cl_\epsilon(T^c) \text{ from Theorem 7 (2).}$$

Thus,

$$cl_\epsilon(T^c) \subseteq [int_\epsilon(T)]^c \tag{3}$$

Now, assume that $u \notin cl_\epsilon(T^c)$. Given Theorem 7 (2),

$$\text{there is } G_u \in SO_\epsilon(\chi) \text{ such that } T^c \cap G = \emptyset$$

That means,

$$u \in G \subseteq T$$

Given Lemma 1,

$$u \in int_\epsilon(T), \text{ and so } u \notin [int_\epsilon(T)]^c$$

Therefore,

$$[int_\epsilon(T)]^c \subseteq cl_\epsilon(T^c) \tag{4}$$

From Eqs (3) and (4), $cl_\epsilon(T^c) = [int_\epsilon(T)]^c$.

(2) Through a method akin to (1).

Proposition 4. *Regarding subsets J and I of an STS (χ, ν). Then,*

(1) $cl_\epsilon(\emptyset) = \emptyset$ *and* $cl_\epsilon(\chi) = \chi$.

(2) $J \subseteq cl_\epsilon(J)$.

(3) $cl_\epsilon(cl_\epsilon(J)) = cl_\epsilon(J)$.

(4) *If* $J \subseteq (I)$, *then* $cl_\epsilon(J) \subseteq cl_\epsilon(I)$.

(5) $cl_\epsilon(J \cap I) \subseteq cl_\epsilon(J) \cap cl_\epsilon(I)$.

(6) $cl_\epsilon(J) \cup cl_\epsilon(I) \subseteq cl_\epsilon(J \cup I)$.

Proof. Straightforward.

Remark 7. *The inclusions of part (3) in Theorem 7 and parts (2), (4), (5) and (6) in Proposition 4 are proper as the upcoming examples will demonstrate.*

Examples 2. *Let* $\nu = \{\chi, \emptyset, \{2,3\}, \{1,3\}\}$ *be an STS on* $\chi = \{1,2,3\}$. *Consider the sets* $A = \{1,3\}$, $C = \{2\}$, $D = \{3\}$ *and* $E = \{2,3\}$. *We have*

(1) $cl(D) = \chi \nsubseteq cl_\epsilon(D) = D$.

(2) $cl_\epsilon(E) = \chi \nsubseteq E$.

(3) $E \nsubseteq A$ *whereas* $cl_\epsilon(E) = \chi \subseteq cl_\epsilon(A) = \chi$.

(4) $cl_\epsilon(A) \cap cl_\epsilon(E) = \chi \nsubseteq cl_\epsilon[A \cap E] = cl_\epsilon(D) = D$.

(5) $cl_\epsilon(C \cup D) = cl_\epsilon(E) = \chi \nsubseteq cl_\epsilon(C) \cup cl_\epsilon(D) = E$.

Definition 9. *Given a subset* T *of an STS* (χ, ν) *with arbitrary point* $s \in \chi$. *Then,* s *called a supra* ϵ*-accumulation point of* T *if each supra* ϵ*-open set* G_s, *we have*

$$[T \backslash \{s\}] \cap G \neq \emptyset.$$

The set of all supra ϵ*-accumulation points of* T *will denoted by* $acc_\epsilon(T)$.

Theorem 10. *Let* (χ, ν) *be an STS and* $T \in P(\chi)$. *Then,*

(1) $acc_\epsilon(T) \subseteq acc(T)$.

(2) $acc_\epsilon(T) \subseteq T \Leftrightarrow T$ *is a proper supra* ϵ*-closed set.*

Proof.

(1) Let's pretend that $s \notin acc(T)$, then $\exists\ G_s \in \nu$ such that $[T \backslash \{s\}] \cap G = \emptyset$. Then, $G_s \in SO_\epsilon(\chi)$ such that $[T \backslash \{s\}] \cap G = \emptyset$. Hence , $s \notin acc_\epsilon(T)$.

(2) ($\Rightarrow$) Pretend that $s \notin T$ for a proper subset T. Considering the condition, $s \notin acc_\epsilon(T)$ and so there is $G_s \in SO_\epsilon(\chi)$ such that $[G_s \backslash \{s\}] \cap T = \emptyset$. Since $s \notin T$, $G_s \cap T = \emptyset$ and so $s \notin cl_\epsilon(T)$. Hence, $cl_\epsilon(T) \subseteq T$. Nevertheless, we have $T \subseteq cl_\epsilon(T)$. Therefore, $T = cl_\epsilon(T)$. Thus, T is a proper supra ϵ-closed set.
($\Leftarrow$) Let $s \notin T$ for a proper supra ϵ-closed set T and then $s \in T^c$ for $T^c \in SO_\epsilon(\chi)$. Since $T \cap [T^c \backslash \{s\}] = \emptyset$ for $T^c \in SO_\epsilon(\chi)$, $s \notin acc_\epsilon(T)$. Therefore, $acc_\epsilon(T) \subseteq T$.

Proposition 5. *Let (χ, ν) be an STS and $T, H \in P(\chi)$, then*

(1) *If $T \subseteq H$, then $acc_\epsilon(T) \subseteq acc_\epsilon(H)$.*

(2) *$acc_\epsilon[T \cap H] \subseteq acc_\epsilon(T) \cap acc_\epsilon(H)$.*

(3) *$acc_\epsilon(T) \cup acc_\epsilon(H) \subseteq acc_\epsilon[T \cup H]$.*

Proof. Follows from Definition 9 and Theorem 10.

Remark 8. *In Proposition 5, the reverse inclusions aren't hold in general, as demonstrated by the upcoming examples.*

Examples 3. *In Examples 2, consider the sets $A = \{1, 3\}$ and $B = \{1, 2\}$. We have*

(1) *$acc_\epsilon(B) = \{3\} \subseteq acc_\epsilon(A) = \{2, 3\}$, whereas $B \nsubseteq A$.*

(2) *$acc_\epsilon(A) \cap acc_\epsilon(B) = \{3\} \nsubseteq acc_\epsilon[A \cap B] = \emptyset$.*

(3) *$acc_\epsilon[A \cup B] = \chi \nsubseteq acc_\epsilon(A) \cup acc_\epsilon(B) = \{2, 3\}$.*

Lemma 2. *Given a subset T of an STS (χ, ν) with arbitrary point $x \in \chi$. Then, $x \in acc_\epsilon(T)$ if and only if $x \in acc_\epsilon(T \backslash \{x\})$.*

Proof. Follows from Lemma 5.

Theorem 11. *For any subset Z of an STS (χ, ν).*

(1) *$Z \in SC_\epsilon(\chi))$ if and only if $acc_\epsilon(Z) \subseteq Z$.*

(2) *$Z \cup acc_\epsilon(Z) \in SC_\epsilon(\chi)$.*

Proof.

(1) Assume that Z is a supra ϵ-closed set and $z \notin Z$. Then, Z^c is a supra ϵ-open set with $z \in Z^c$, and hence $[Z \backslash \{z\}] \cap Z^c = \emptyset$. Therefore, $z \notin acc_\epsilon(Z)$, and thus $acc_\epsilon(Z) \subseteq Z$. Presently, we prove that $Z \in SC_\epsilon(\chi))$ which sufficient to prove that $Z^c \in SO_\epsilon(\chi))$. So, let $z \in Z^c$. Then, $z \notin Z$. Given the condition, $z \notin acc_\epsilon(Z)$, and hence $[Z \backslash \{z\}] \cap G_z \neq \emptyset$, for some supra ϵ-open set G_z. Since $z \notin Z$, $Z \cap G_z \neq \emptyset$ and thus $G_z \subseteq Z^c$. Therefore, $Z^c \in SO_\epsilon(\chi))$, and consequently $Z \in SC_\epsilon(\chi))$.

(2) Suppose that $s \notin Z \cup acc_\epsilon(Z)$ and so $s \notin Z$ and $s \notin acc_\epsilon(Z)$. Hence, there is $G_s \in SO_\epsilon(\chi)$ such that

$$[Z\backslash\{s\}] \cap G_s = \emptyset \text{ and then } s \notin cl_\epsilon(Z).$$

Hence,

$$cl_\epsilon(Z) \subseteq Z \cup acc_\epsilon(Z) \tag{5}$$

Now, suppose that $s \notin cl_\epsilon(Z)$. Given Theorem 7 (2), $N_u \cap Z \neq \emptyset$ for some $N_u \in SO_\epsilon(\chi)$. Since, $Z \subseteq cl_\epsilon(Z)$, $s \notin Z$ and hence $[Z\backslash\{s\}] \cap N_u = \emptyset$. Hence, $s \notin acc_\epsilon(Z)$. Therefore,

$$Z \cup acc_\epsilon(Z) \subseteq cl_\epsilon(Z) \tag{6}$$

According to Eqs 5 and 6, $Z \cup acc_\epsilon(Z) = cl_\epsilon(Z)$. Given Theorem 7 (1), $Z \cup acc_\epsilon(Z) \in SC_\epsilon(\chi)$.

Corollary 2. *Given a subset Z of an STS (χ, ν). Then, $cl_\epsilon(Z) = Z \cup acc_\epsilon(Z)$.*

Proof. It is derived from Theorem 11.

Definition 10. *If $s \in [cl_\epsilon(Z)\backslash int_\epsilon(Z)]$ for an arbitrary point s and subset Z of an STS (χ, ν), then s is called a supra-ϵ-boundary point of Z. The supra-ϵ-boundary set of Z is the set of all upper-so-boundary points of Z, and it is represented by $b_\epsilon(Z)$.*
Also, the supra-ϵ-exterior of Z is also represented by $ext_\epsilon(Z)$, where $ext_\epsilon(Z) = int_\epsilon(Z^c)$.

Theorem 12. *Regarding a subset J of an STS (χ, ν), we have*

(1) $b_\epsilon(J) = cl_\epsilon(J) \cap [int_\epsilon(J)]^c = cl_\epsilon(J) \cap cl_\epsilon(J^c) = [int_\epsilon(J) \tilde{\cup} ext_\epsilon(J)]^c$.

(2) $b_\epsilon(J) = b_\epsilon(J^c)$.

Proof.

(1) $[int_\epsilon(J) \tilde{\cup} ext_\epsilon(J)]^c = [int_\epsilon(J)]^c \cap [int_\epsilon(J^c)]^c$

$$\begin{aligned} &= cl_\epsilon(J) \cap [int_\epsilon(J)]^c \quad \text{from Theorem 9 (1)} \\ &= cl_\epsilon(J) \cap cl_\epsilon(J^c) \\ &= cl_\epsilon(J)\backslash int_\epsilon(J) \\ &= b_\epsilon(J). \end{aligned}$$

(2) $b_\epsilon(J^c) = cl_\epsilon(J^c) \cap [int_\epsilon(J^c)]^c = [int_\epsilon(J)]^c \cap cl_\epsilon(J) = b_\epsilon(J)$.

Theorem 13. *Regarding a subset J of an STS (χ, ν), we have*

(1) $cl_\epsilon(J) = int_\epsilon(J) \tilde{\cup} b_\epsilon(J)$.

(2) $cl_\epsilon(J) = J \tilde{\cup} b_\epsilon(J)$.

(3) $int_\epsilon(J) = J\backslash b_\epsilon(J)$.

Proof.

$$\textbf{(1)}\ int_\epsilon(J)\tilde{\cup}b_\epsilon(J) = int_\epsilon(J)\tilde{\cup}[cl_\epsilon(J)\cap[int_\epsilon(J)]^c] \text{ from Theorem 12 (1)}$$
$$= [int_\epsilon(J)\tilde{\cup}cl_\epsilon(J)]\cap[int_\epsilon(J)\tilde{\cup}[int_\epsilon(J)]^c]$$
$$= cl_\epsilon(J)\cap\chi$$
$$= cl_\epsilon(J).$$

(2) By a similar way to (1).

$$\textbf{(3)}\ J\backslash b_\epsilon(J) = J\cap[cl_\epsilon(J)\cap[int_\epsilon(J)]^c]^c$$
$$= J\cap[[cl_\epsilon(J)]^c\tilde{\cup}[int_\epsilon(J)]]$$
$$= [J\cap[cl_\epsilon(J)]^c]\tilde{\cup}[J\cap int_\epsilon(J)]$$
$$= \emptyset\tilde{\cup}int_\epsilon(J)$$
$$= int_\epsilon(J).$$

Proposition 6. *Regarding a subset J of an STS (χ,ν), the class $\{b_\epsilon(J), int_\epsilon(J), ext_\epsilon(J)\}$ forms a partition for χ.*

Proof. $b_\epsilon(J)\cup int_\epsilon(J)\cup ext_\epsilon(J) = [cl_\epsilon(J)\cap[int_\epsilon(J)]^c]\cup int_\epsilon(J)\cup[cl_\epsilon(J)]^c = \chi$. Moreover, $b_\epsilon(J)\cap int_\epsilon(J)\cap ext_\epsilon(J) = [cl_\epsilon(J)\cap[int_\epsilon(J)]^c]\cap int_\epsilon(J)\cap[cl_\epsilon(J)]^c = \emptyset$.

Proposition 7. *Regarding subsets T and J of an STS (χ,ν), we have*

(1) $b_\epsilon[int_\epsilon(T)]\subseteq b_\epsilon(T)$.

(2) $b_\epsilon[cl_\epsilon(T)]\subseteq b_\epsilon(T)$.

(3) $b_\epsilon[T\cup J]\subseteq b_\epsilon(T)\cup b_\epsilon(J)$.

(4) $b_\epsilon[T\cap J]\subseteq b_\epsilon(T)\cup b_\epsilon(J)$.

Proof.

(1) $b_\epsilon[int_\epsilon(T)] = cl_\epsilon(int_\epsilon(T))\cap[int_\epsilon(int_\epsilon(T))]^c\subseteq cl_\epsilon(T)\cap[int_\epsilon(T)]^c = b_\epsilon(T)$.

(2) $b_\epsilon[cl_\epsilon(T)] = cl_\epsilon(cl_\epsilon(T))\cap[int_\epsilon(cl_\epsilon(T))]^c = cl_\epsilon(T)\cap cl_\epsilon[cl_\epsilon(T)]^c$

$$\subseteq cl_\epsilon(T)\cap[int_\epsilon(T)]^c = b_\epsilon(T).$$

(3)-(4) Follows from Theorem 12.

Remark 9. *The inclusions of Proposition 7 are proper as shown in the next example.*

Example 5. *Regarding the sets $A=\{1,3\}$, $C=\{2\}$, $D=\{3\}$ and $E=\{2,3\}$, in Example 2, we have:*

(1) $b_\epsilon(C) = C\nsubseteq b_\epsilon[int_\epsilon(C)] = b_\epsilon(\emptyset) = \emptyset$.

(2) $b_\epsilon(E) = \chi\backslash\{2,3\} = \{1\}\nsubseteq b_\epsilon[cl_\epsilon(E)] = b_\epsilon(\chi) = \emptyset$.

(3) $b_\epsilon(A)\cup b_\epsilon(E) = \{1,2\}\cup\{2\} = \{1,2\}\nsubseteq b_\epsilon[A\cup E] = b_\epsilon(\chi) = \emptyset$.

(4) $b_\epsilon(A) \cup b_\epsilon(E) = \{1,2\} \nsubseteq b_\epsilon[A \cap E] = b_\epsilon(\{3\}) = \emptyset$.

Proposition 8. *The following holds for a subset H of an STS (χ, ν):*

(1) $b_\epsilon(H) \cap H = \emptyset$ *if and only if* $H \in SO_\epsilon(\chi)$.

(2) $b_\epsilon(H) \subseteq H$ *if and only if H is a supra ϵ-closed set.*

(3) $b_\epsilon(H) = \emptyset$ *if and only if H is both supra ϵ-closed and supra ϵ-open set.*

Proof.

(1) " $\Rightarrow$ " Let $b_\epsilon(H) \cap (H) = \emptyset$, then

$$[cl_\epsilon(H) \cap [int_\epsilon(H)]^c] \cap (H) = [int_\epsilon(H)]^c \cap H = \emptyset.$$

Hence,

$$H \subseteq int_\epsilon(H). \text{ But, we have } int_\epsilon(H) \subseteq H.$$

Thus, $int_\epsilon(H) = H$ and so $H \in SO_\epsilon(\chi)$, given Proposition 1.

" $\Leftarrow$ " Obvious.

(2) Clear.

(3) " $\Rightarrow$ " Assume that $b_\epsilon(H) = \emptyset$, then $cl_\epsilon(H) \cap [int_\epsilon(H)]^c = \emptyset$. Hence, $cl_\epsilon(H) \subseteq int_\epsilon(H)$. However, we have that $int_\epsilon(H) \subseteq cl_\epsilon(H)$. Thus, $int_\epsilon(H) = cl_\epsilon(H)$. Therefore, H is both supra ϵ-closed and supra ϵ-open set, given Theorem 6 (2) and Proposition 4 (2).

" $\Leftarrow$ " Obvious

Theorem 14. $b_\epsilon(H) \in SC_\epsilon(\chi)$ *for a subset H of an STS (χ, ν).*

Proof. If either $cl_\epsilon(H) = \chi$ or $cl_\epsilon(H^c) = \chi$, given Theorem 12 (1), we get the our proof. If $cl_\epsilon(H) \neq \chi$ and $cl_\epsilon(H^c) \neq \chi$, given Theorem 3 (2), $b_\epsilon(H) = cl_\epsilon(H) \cap cl_\epsilon(H^c) \in SC_\epsilon(\chi)$.

5. Conclusion

In this project, we introduce a novel weaker form of supra open sets, named supra ϵ-open sets and provide its essential features. The notions of supra regular (α-, semi-, pre-, b-, β-, and R-) open sets, which were previously similar, are shown to be included in this new supra open set category. Moreover, we provide new types of operators named supra ϵ-interior (closure, accumulation, exterior, and boundary, respectively) using our recently established category of supra open sets. We also describe the differences between these new operators and their corresponding operators. Moreover, we prove that supra ϵ-interior operator, supra ϵ-boundary operator and supra ϵ-exterior operator form a partition for χ. Finally, we complement our investigations with many examples and counterexamples that highlight the significance of our innovative operators. Further research on the theoretical aspects of these generalized concepts might be conducted from the specific approaches presented in this work by examining the following topics:

- Study some topological properties inspired by the specific approaches presented in this work, like supra continuity (separation axioms, connectedness, and compactness).
- Examine whether these notions, in particular the separation axiom, may be applied to information systems.
- Apply these approaches to soft ideal topological spaces [33, 48, 49], and supra soft topological spaces [42].

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Conflict of Interest

There are no conflicts of interest disclosed by the authors.

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