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Bounds on the Energy of Zero-divisor Graph of Quotient Ring and Its Topological Indices

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Abstract. In this paper, we study the zero-divisor graph of $\mathbb{Z}_{\wp}[x]/\langle x^5\rangle$ for prime number $\wp$, denoted as $\Gamma(\mathbb{Z}_{\wp}[x]/\langle x^5\rangle)$, including its energy and topological indices. Specifically, we provide bounds of the energy for $\Gamma(\mathbb{Z}_{\wp}[x]/\langle x^5\rangle)$ and show that these bounds are numerically close to the actual energy value. Furthermore, we determine the topological indices of $\Gamma(\mathbb{Z}_{\wp}[x]/\langle x^5\rangle)$, including the topological indices based on distance and degree. We also perform numerical simulations of the topological indices for various prime numbers $\wp$.

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Key Words and Phrases: zero-divisor graph, graph energy, distance-based topological indices, degree-based topological indices

1. Introduction

The concept of graphs regarding different algebraic structures is interesting to investigate because it enables us to explore algebraic properties using graph theory. One notable example is the zero-divisor graph that came from the work done by Beck in 1988 [1]. He defined the vertex set as zero divisors, including zero. In 1999, Anderson and Livingston revised the definition by focusing on only non-zero zero divisors as its vertices, denoted as $\Gamma(\mathcal{R})$ for commutative ring $\mathcal{R}$ [2]. Since then, the zero-divisor graph has been a fast-developing area and widely applied in various fields, including algebraic cryptography [3, 4] and coding theory [5, 6]. For further literature on this topic, see [7–12].

Besides its importance in algebra, graph theory has also seen rapid development in its applications to chemistry, particularly through the study of graph energy. In 1978, Gutman first defined the graph energy as the total of the absolute values of its adjacency matrix's eigenvalues [13]. This concept arose when Erich Huckel developed Huckel molecular orbital theory to estimate the π-electron energy [14]. Beyond graph energy,

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topological indices are also important parameters for analyzing molecular structures and predicting chemical properties. Most of the studies of topological indices currently are based on distance or degree. The Wiener index, proposed by Wiener [15] in 1947, is the earliest distance-based topological indices and is used to approximate the boiling points of alkanes. In 1993, Randic [16] proposed the hyper-Wiener index applied for analyzing the physicochemical properties of organic compounds. Meanwhile, first degree-based topological indices were proposed in 1970s by Gutman and Trinajstic [17]. There are the first and second Zagreb indices that are used for analyzing the thermodynamic stability and reactivity of unsaturated molecules. Furthermore, in 1984, Narumi and Katayama proposed the Narumi-Katayama index, a simpler degree-based topological index used to examine the branching structures of saturated hydrocarbons[18].

In addition some researchers have extended their study to various algebraic graph structures, including zero-divisor graph. In 2011, Ahmadi and Jahani-Nezhad pioneered the examination of the energy and Wiener index of $\Gamma(\mathbb{Z}_{pq})$ and $\Gamma(\mathbb{Z}_{p^2q})$ for every distinct prime p, q [19]. Later, Johnson and Sankar in 2023 studied the energy and topological indices of $\Gamma(\mathbb{Z}_p[x]/\langle x^4\rangle)$ for prime number p [20]. However, Rather [21] revised their results on the energy and second Zagreb index formula. Rayer and Jeyaraj in 2023 studied the topological indices of $\Gamma(\mathbb{Z}_{p^2}[x]/\langle x^2\rangle)$ for every prime $q \geq 3$ and $\Gamma(\mathbb{Z}_{pq}[x]/\langle x^2\rangle)$ for every prime $2 < p < q$ [22], and further investigated the energy of $\Gamma(\mathbb{Z}_{p^2}[x]/\langle x^2\rangle)$ in 2024 [23].

Previous research has primarily focused on determining the energy and topological indices of the zero-divisor graph in quotient rings with principal ideals $\langle x^2\rangle$ and $\langle x^4\rangle$. Recently, Musyarrofah et al. [24] explored a different type of quotient ring structure, specifically $\Gamma(\mathbb{Z}_{\wp}[x]/\langle x^5\rangle)$ for a prime number $\wp$, focusing on the fundamental properties of the graph. However, their study did not examine its energy or topological indices in detail.

In this paper, we address this gap by analyzing the energy and topological indices of $\Gamma(\mathbb{Z}_{\wp}[x]/\langle x^5\rangle)$ for prime number $\wp$. Specifically, we obtain the lower and upper bounds of the energy for the graph and these bounds are numerically close to the actual energy value. This provides a reliable method for estimating the energy of the graph in similar cases. Furthermore, we investigate topological indices, including distance-based topological indices such as the Wiener and hyper-Wiener indices, as well as degree-based topological indices such as the first, second Zagreb, and Narumi-Katayama indices. To validate our theoretical results, we conduct numerical simulations using a computer software MATLAB, comparing the computed graph energy with its theoretical bounds and analyzing the growth patterns of various topological indices.

2. Preliminaries

In this section presents basic concepts, notations, and preliminaries relevant to this paper. All of the graphs mentioned are simple graphs, meaning that they are undirected, do not have loops, and do not contain multiple edges. The fundamental concepts of graph theory discussed in this article are referenced in [25]. Let G be a graph with vertex set $V(G) = \{v_1, v_2, v_3, \ldots, v_n\}$. The adjacency matrix of G, $A(G) = [a_{ij}]$ is a $(0, 1)$-symmetric matrix with entries a_{ij} equal to 1 if v_i is adjacent to v_j and equal to 0 otherwise. The degree of a vertex $v \in V(G)$, $\deg(v)$, indicates the number of vertices that are adjacent to v and the distance $d(v, w)$ indicates the shortest path between the vertices v and w. The distance matrix of G, $D(G) = [d_{ij}]$ is a symmetric matrix, where d_{ij} denotes the distance between two distinct vertices v_i and v_j.

Zero-divisor graph is the subject in this investigation, where the definition used follows Anderson and Livingston [2]. Suppose $\mathcal{R}$ is a commutative ring, with $Z(\mathcal{R})$ being the set of zero-divisors of $\mathcal{R}$. The zero-divisor graph of $\mathcal{R}$, $\Gamma(\mathcal{R})$ is the graph with the set of vertex consisting of the elements of $Z^*(\mathcal{R}) = Z(\mathcal{R}) \setminus \{0\}$ and the set of edges $E(\Gamma(\mathcal{R})) = \{xy \mid xy = 0, \forall x, y \in Z^*(\mathcal{R})\}$.

Furthermore, we present several results related to block and circulant matrices that are utilized. These results are used to calculate the eigenvalues and determinants of the adjacency matrix.

Lemma 1. *[26] Let P, Q, R, S be matrices and suppose that matrix P is invertible. If $M = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$, then $\det(M) = \det(P) \cdot \det(S - RP^{-1}Q)$.*

A circulant matrix is a square matrix where each row is generated by moving the entries of the previous row one position to the right, while the last entry moves to the first position. This matrix C can be represented by the vector $\boldsymbol{c} = [c_0, c_1, c_2, \ldots, c_{m-1}]$, where each row results from a right circular shift of $\boldsymbol{c}$. The circulant matrix of order $m \times m$ with entries $c_0, c_1 \in \mathbb{R}$ is denoted as $C_{(c_0,c_1,m)}$ and has the following form:

$$C_{(c_0,c_1,m)} = \begin{bmatrix} c_0 & c_1 & c_1 & \cdots & c_1 \\ c_1 & c_0 & c_1 & \cdots & c_1 \\ c_1 & c_1 & c_0 & \cdots & c_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_1 & c_1 & \cdots & c_0 \end{bmatrix}_{m \times m}.$$

Proposition 1. *[27] Let $C_{(c_0,c_1,m)}$ be a circulant matrix. The determinant of $C_{(c_0,c_1,m)}$ is*

$$\det(C_{(c_0,c_1,m)}) = [c_0 + (m-1)c_1](c_0 - c_1)^{m-1}.$$

Proposition 2. *[27] Let $C_{(c_0,c_1,m)}$ be a nonsingular circulant matrix. The inverse of $C_{(c_0,c_1,m)}$ is*

$$C^{-1}_{(c_0,c_1,m)} = \frac{1}{\det(C_{(c_0,\, c_1,\, m)})} \begin{bmatrix} \varphi_{m-1} & \vartheta_{m-1} & \vartheta_{m-1} & \cdots & \vartheta_{m-1} \\ \vartheta_{m-1} & \varphi_{m-1} & \vartheta_{m-1} & \cdots & \vartheta_{m-1} \\ \vartheta_{m-1} & \vartheta_{m-1} & \varphi_{m-1} & \cdots & \vartheta_{m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vartheta_{m-1} & \vartheta_{m-1} & \vartheta_{m-1} & \cdots & \varphi_{m-1} \end{bmatrix}$$

$$C^{-1}_{(c_0,c_1,m)} = \frac{1}{\det(C_{(c_0,\, c_1,\, m)})} C_{(\varphi_{m-1},\, \vartheta_{m-1},\, m)},$$

where

$$\varphi_{m-1} = [c_0 + (m-2)c_1](c_0 - c_1)^{m-2} \quad \textit{and} \quad \vartheta_{m-1} = -c_1(c_0 - c_1)^{m-2}.$$

The study of graph energy proposed by Gutman in 1978 [13], is defined as:

$$En(G) = \sum_{i=1}^{n} |\lambda_i|,$$

where λ_i represents the eigenvalues of $A(G)$.

To further analyze graph energy, the Maclaurin symmetric mean inequality is used to establish lower and upper bounds.

Proposition 3. *[28] Let $a_1, a_2, a_3, \ldots, a_s$ be a positive real numbers and the average of the product of all subsets of order k represented by $\prod_k$, where the number of such subsets is given by the number of ways to choose k elements from s elements. The values of $\prod_k$ are defined as follows.*

$$\begin{aligned}
\prod\nolimits_1 &= \frac{a_1 + a_2 + a_3 + \cdots + a_s}{s}, \\
\prod\nolimits_2 &= \frac{1}{\frac{s(s-1)}{2}}(a_1a_2 + a_1a_3 + \cdots + a_1a_s + a_2a_3 + \cdots + a_{s-1}a_s), \\
&\vdots \\
\prod\nolimits_s &= a_1a_2a_3 \ldots a_s.
\end{aligned}$$

The Maclaurin symmetric mean inequality states that

$$\prod\nolimits_1 \geq {\prod\nolimits_2}^{1/2} \geq {\prod\nolimits_3}^{1/3} \geq {\prod\nolimits_4}^{1/4} \geq \cdots \geq {\prod\nolimits_s}^{1/s}, \tag{1}$$

with equalities holding if and only if $a_1 = a_2 = a_3 = \cdots = a_s$

We review several topological indices of graph that are used in this paper. The Wiener index [15] can be described as the following equation.

$$\mathcal{W}(G) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}.$$

Meanwhile, the Hyper-Wiener index [16] is defined as

$$\mathcal{WW}(G) = \frac{1}{2}\mathcal{W}(G) + \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n}(d_{ij})^2.$$

The first Zagreb index and the second Zagreb index [17] are respectively defined as

$$\mathcal{M}_1(G) = \sum_{vw \in E(G)} [\deg(v) + \deg(w)] \quad \text{and} \quad \mathcal{M}_2(G) = \sum_{vw \in E(G)} [\deg(v)\deg(w)].$$

The Narumi-Katayama index [18] is defined as

$$\mathcal{NK}(G) = \prod_{v \in V(G)} \deg(v).$$

3. Energy of zero-divisor graph of $\mathbb{Z}_{\wp}[x]/\langle x^5 \rangle$

In this section, we discuss the energy of $\Gamma(\mathbb{Z}_{\wp}[x]/\langle x^5 \rangle)$. To provide a comprehensive understanding, we first revisit essential results related to the structure of $\Gamma(\mathbb{Z}_{\wp}[x]/\langle x^5 \rangle)$. Next, we explore the calculation of eigenvalues. Following this, we discuss the lower and upper bounds for graph's energy.

Let $\mathbb{Z}_{\wp}[x]$ be a polynomial commutative ring and $\langle x^5 \rangle$ be a principal ideal of $\mathbb{Z}_{\wp}[x]$. A quotient ring $\mathbb{Z}_{\wp}[x]$ is formed from the set of all cosets

$$\mathbb{Z}_{\wp}[x]/\langle x^5 \rangle = \{kx^4 + lx^3 + mx^2 + nx + o + \langle x^5 \rangle \mid k, l, m, n, o \in \mathbb{Z}_{\wp}\}.$$

Further, we write $\mathbb{Z}_{\wp}[x]/\langle x^5 \rangle$ as

$$\mathbb{Z}_{\wp}[x]/\langle x^5 \rangle = \{\overline{kx^4 + lx^3 + mx^2 + nx + o} \mid k, l, m, n, o \in \mathbb{Z}_{\wp}\}.$$

The graph $\Gamma(\mathbb{Z}_{\wp}[x]/\langle x^5 \rangle)$ has $\wp^4 - 1$ vertices and $\frac{1}{2}(4\wp^5 - 5\wp^4 - \wp^2 + 2)$ edges. The structure of this graph was originally defined by Musyarrofah et al. in [24]. The following expression represents the vertex set of $\Gamma(\mathbb{Z}_{\wp}[x]/\langle x^5 \rangle)$.

$$\begin{aligned}
\mathcal{A} &= \{\overline{kx^4 + lx^3 + mx^2 + nx} \mid k, l, m \in \mathbb{Z}_{\wp}, n \in \mathbb{Z}_{\wp} \setminus \{\bar{0}\}\}, && |\mathcal{A}| = \wp^4 - \wp^3,\\
\mathcal{B} &= \{\overline{kx^4 + lx^3 + mx^2} \mid k, l \in \mathbb{Z}_{\wp}, m \in \mathbb{Z}_{\wp} \setminus \{\bar{0}\}\}, && |\mathcal{B}| = \wp^3 - \wp^2,\\
\mathcal{C} &= \{\overline{kx^4 + lx^3} \mid k \in \mathbb{Z}_{\wp}, l \in \mathbb{Z}_{\wp} \setminus \{\bar{0}\}\}, && |\mathcal{C}| = \wp^2 - \wp,\\
\mathcal{D} &= \{\overline{kx^4} \mid k \in \mathbb{Z}_{\wp} \setminus \{\bar{0}\}\}, && |\mathcal{D}| = \wp - 1.
\end{aligned}$$

The adjacency matrix of $\Gamma(\mathbb{Z}_{\wp}[x]/\langle x^5 \rangle)$ is provided by the following lemma

Lemma 2. *[24] Adjacency matrix of graph $G \cong \Gamma(\mathbb{Z}_{\wp}[x]/\langle x^5 \rangle)$ is*

$$A(G) = \begin{array}{c} \\ \mathcal{A} \\ \mathcal{B} \\ \mathcal{C} \\ \mathcal{D} \end{array} \begin{array}{c} \begin{array}{cccc} \quad\mathcal{A}\quad & \quad\mathcal{B}\quad & \quad\mathcal{C}\quad & \quad\mathcal{D}\quad \end{array} \\ \begin{bmatrix} O_{\wp^4-\wp^3} & O_{(\wp^4-\wp^3)\times(\wp^3-\wp^2)} & O_{(\wp^4-\wp^3)\times(\wp^2-\wp)} & N_{(\wp^4-\wp^3)\times(\wp-1)} \\ O_{(\wp^3-\wp^2)\times(\wp^4-\wp^3)} & O_{\wp^3-\wp^2} & N_{(\wp^3-\wp^2)\times(\wp^2-\wp)} & N_{(\wp^3-\wp^2)\times(\wp-1)} \\ O_{(\wp^2-\wp)\times(\wp^4-\wp^3)} & N_{(\wp^2-\wp)\times(\wp^3-\wp^2)} & N_{\wp^2-\wp} - I_{\wp^2-\wp} & N_{(\wp^2-\wp)\times(\wp-1)} \\ N_{(\wp-1)\times(\wp^4-\wp^3)} & N_{(\wp-1)\times(\wp^3-\wp^2)} & N_{(\wp-1)\times(\wp^2-\wp)} & N_{\wp-1} - I_{\wp-1} \end{bmatrix} \end{array}.$$

Here O represents the zero matrix, N represents the matrix of ones, and I represents the identity matrix.

The eigenvalues of $\Gamma(\mathbb{Z}_\wp[x]/\langle x^5\rangle)$ are provided by the following theorem.

Theorem 1. *Let $G \cong \Gamma(\mathbb{Z}_\wp[x]/\langle x^5\rangle)$. Then the following hold for G.*

(i) The eigenvalues of G are 0, with multiplicity $\wp^4 - \wp^2 - 2$, and -1, with multiplicity $\wp^2 - 3$.

(ii) The other eigenvalues of G are solutions to the following polynomial

$$\lambda^4 - \left(\wp^2 - 3\right)\lambda^3 - \left(2\wp^5 - 3\wp^4 + 2\wp^2 - 2\right)\lambda^2 + \wp^3\left(\wp^3 - \wp^2 - 2\wp - 1\right)(\wp - 1)^2\lambda + \wp^6(\wp - 1)^4 = 0.$$

Proof. Suppose that λ be the eigenvalues of $G \cong \Gamma(\mathbb{Z}_\wp[x]/\langle x^5\rangle)$. Based on Lemma 2 has been obtained adjacency matrix of a graph G, then the matrix $A(G) - \lambda I$ can be expressed as:

$$A(G) - \lambda I = \left[\begin{array}{cc|cc} C_{(-\lambda,\,0,\,\wp^4-\wp^3)} & O_{(\wp^4-\wp^3)\times(\wp^3-\wp^2)} & O_{(\wp^4-\wp^3)\times(\wp^2-\wp)} & N_{(\wp^4-\wp^3)\times(\wp-1)} \\ O_{(\wp^3-\wp^2)\times(\wp^4-\wp^3)} & C_{(-\lambda,\,0,\,\wp^3-\wp^2)} & N_{(\wp^3-\wp^2)\times(\wp^2-\wp)} & N_{(\wp^3-\wp^2)\times(\wp-1)} \\ \hline O_{(\wp^2-\wp)\times(\wp^4-\wp^3)} & N_{(\wp^2-\wp)\times(\wp^3-\wp^2)} & C_{(-\lambda,\,1,\,\wp^2-\wp)} & N_{(\wp^2-\wp)\times(\wp-1)} \\ N_{(\wp-1)\times(\wp^4-\wp^3)} & N_{(\wp-1)\times(\wp^3-\wp^2)} & N_{(\wp-1)\times(\wp^2-\wp)} & C_{(-\lambda,\,1,\,\wp-1)} \end{array}\right] = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}.$$

If $\lambda \neq 0$, then P is invertible and according to Lemma 1, the determinant of $A(G) - \lambda I$ is given by

$$\det(A(G) - \lambda I) = \det(P) \cdot \det(S - RP^{-1}Q).$$

Since P is a diagonal matrix, its determinant is

$$\det(P) = (-\lambda)^{\wp^4-\wp^2}. \tag{2}$$

Moreover, because P can be written as circulant matrix $C_{(-\lambda,\,0,\,\wp^4-\wp^2)}$, its inverse can be computed using Proposition 2, that is

$$P^{-1} = \frac{1}{(-\lambda)^{\wp^4-\wp^2}} C_{\left((-\lambda)^{\wp^4-\wp^2-1},\,0,\,\wp^4-\wp^2\right)} = \frac{(-\lambda)^{\wp^4-\wp^2-1}}{(-\lambda)^{\wp^4-\wp^2}} I_{\wp^4-\wp^2} = -\frac{1}{\lambda} I_{\wp^4-\wp^2},$$

Substituting this inverse into $RP^{-1}Q$, we find

$$RP^{-1}Q = -\frac{1}{\lambda}\begin{bmatrix} (\wp^3 - \wp^2)N_{\wp^2-\wp} & (\wp^3 - \wp^2)N_{(\wp^2-\wp)\times(\wp-1)} \\ (\wp^3 - \wp^2)N_{(\wp-1)\times(\wp^2-\wp)} & (\wp^4 - \wp^2)N_{\wp-1} \end{bmatrix},$$

Thus, $S - RP^{-1}Q$ becomes

$$S - RP^{-1}Q = \begin{bmatrix} C_{\left(-\lambda+\frac{b}{\lambda},1+\frac{b}{\lambda},c\right)} & \left(1+\frac{b}{\lambda}\right) N_{c\times d} \\ \left(1+\frac{b}{\lambda}\right) N_{d\times c} & C_{\left(-\lambda+\frac{a}{\lambda},1+\frac{a}{\lambda},d\right)} \end{bmatrix} = \begin{bmatrix} X_1 & Y \\ Y^T & X_2 \end{bmatrix},$$

where $a = \wp^4 - \wp^2$, $b = \wp^3 - \wp^2$, $c = \wp^2 - \wp$, dan $d = \wp - 1$.
Applying Lemma 1 once again,

$$\det(S - RP^{-1}Q) = \det(X_1) \cdot \det(X_2 - Y^T X_1^{-1} Y).$$

By Proposition 1, it can be seen that

$$\det(X_1) = (-\lambda - 1)^{c-1} f(\lambda), \tag{3}$$

where $f(\lambda) = \frac{-\lambda^2+(c-1)\lambda+cb}{\lambda}$.
Also by Proposition 2,

$$X_1^{-1} = \frac{1}{(-\lambda-1)^{c-1} f(\lambda)} C_{(\varphi_{c-1},\, \vartheta_{c-1},\, c)},$$

where $\varphi_{c-1} = \left(-\lambda + c - 2 + \frac{cb-b}{\lambda}\right)(-\lambda-1)^{c-2}$ and $\vartheta_{c-1} = \left(-1 - \frac{b}{\lambda}\right)(-\lambda-1)^{c-2}$.
Let

$$g(\lambda) = -\lambda + c - 2 + \frac{cb-b}{\lambda}, \quad \text{and} \quad h(\lambda) = -1 - \frac{b}{\lambda}.$$

Then

$$\varphi_{c-1} = (-\lambda-1)^{c-2} g(\lambda) \quad \text{and} \quad \vartheta_{c-1} = (-\lambda-1)^{c-2} h(\lambda).$$

Thus,

$$X_1^{-1} = \frac{1}{(-\lambda-1) f(\lambda)} C_{(g(\lambda),\, h(\lambda),\, c)}.$$

Also,

$$X_2 - Y^T X_1^{-1} Y = C_{\left(-\lambda+\frac{a}{\lambda}-\frac{c(b+\lambda)^2}{\lambda^2 f(\lambda)},\, 1+\frac{a}{\lambda}-\frac{c(b+\lambda)^2}{\lambda^2 f(\lambda)},\, d\right)}.$$

By Proposition 1, the determinant of $X_2 - Y^T X_1^{-1} Y$ is

$$\begin{aligned} &\det\left(X_2 - Y^T X_1^{-1} Y\right) \\ &= \left(-\lambda + \frac{a}{\lambda} - \frac{c(b+\lambda)^2}{\lambda^2 f(\lambda)} + (d-1)\left(1 + \frac{a}{\lambda} - \frac{c(b+\lambda)^2}{\lambda^2 f(\lambda)}\right)\right)(-\lambda-1)^{d-1} \end{aligned}$$

$$= \left(\frac{-\lambda^3 f(\lambda) + (d-1)\lambda^2 f(\lambda) + da\lambda f(\lambda) - dc(b+\lambda)^2}{\lambda^2 f(\lambda)} \right) (-\lambda - 1)^{d-1}$$
$$= \frac{\phi(\lambda)}{\lambda^2 f(\lambda)} (-\lambda - 1)^{d-1}. \tag{4}$$

where $\phi(\lambda)$ is

$$\phi(\lambda) = \lambda^4 - \left(\wp^2 - 3\right)\lambda^3 - \left(2\wp^5 - 3\wp^4 + 2\wp^2 - 2\right)\lambda^2 + \wp^3\left(\wp^3 - \wp^2 - 2\wp - 1\right)(\wp - 1)^2\lambda + \wp^6(\wp - 1)^4.$$

Based on equation (2), (3), and (4), we obtain determinant $A(G) - \lambda I$, is

$$\det(A(G) - \lambda I) = \det(P) \cdot \det(X_1) \cdot \det\left(X_2 - Y^T X_1^{-1} Y\right) = (-\lambda)^{\wp^4 - \wp^2 - 2} \cdot (-\lambda - 1)^{\wp^2 - 3} \cdot \phi(\lambda).$$

Thus the characteristic polynomial of $A(G)$ is

$$(-\lambda)^{\wp^4 - \wp^2 - 2} \cdot (-\lambda - 1)^{\wp^2 - 3} \cdot \phi(\lambda) = 0,$$

Hence, 0 and 1 are eigenvalue of G with multiplicity $\wp^4 - \wp^2 - 2$ and $\wp^2 - 3$ respectively. The other eigenvalues of G are solutions to the following polynomial

$$\lambda^4 - \left(\wp^2 - 3\right)\lambda^3 - \left(2\wp^5 - 3\wp^4 + 2\wp^2 - 2\right)\lambda^2 + \wp^3\left(\wp^3 - \wp^2 - 2\wp - 1\right)(\wp - 1)^2\lambda + \wp^6(\wp - 1)^4 = 0.$$

Theorem 2. *Let* $G \cong \Gamma(\mathbb{Z}_\wp[x]/\langle x^5 \rangle)$. *Then*

$$En(G) \geq \wp^2 - 3 + \sqrt{4\wp^5 - 5\wp^4 - 2\wp^2 + 5 + 12(\wp^6(\wp - 1)^4)^{1/2}}$$

and

$$En(G) \leq \wp^2 - 3 + 2\sqrt{4\wp^5 - 5\wp^4 - 2\wp^2 + 5}.$$

Proof. Let $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ be an eigenvalues that satisfy

$$\lambda^4 - \left(\wp^2 - 3\right)\lambda^3 - \left(2\wp^5 - 3\wp^4 + 2\wp^2 - 2\right)\lambda^2 + \wp^3\left(\wp^3 - \wp^2 - 2\wp - 1\right)(\wp - 1)^2\lambda + \wp^6(\wp - 1)^4 = 0. \tag{5}$$

It is obtained that the sum and the product of the eigenvalues from the equation (5) are as follows

$$\sum_{i=1}^{4} \zeta_i = \wp^2 - 3,$$
$$\sum_{1\le i<j\le 4} \zeta_i\zeta_j = -(2\wp^5 - 3\wp^4 + 2\wp^2 - 2) = -2\wp^5 + 3\wp^4 - 2\wp^2 + 2,$$
$$\prod_{i=1}^{4} \zeta_i = \wp^6(\wp - 1)^4.$$

Next, we can write the energy of G as

$$En(G) = \sum_{i=1}^{\wp^4-1} \mid \lambda_i \mid = \wp^2 - 3 + \sum_{i=1}^{4} |\zeta_i| \tag{6}$$

Then, using the first inequality of (1) to the set $\{|\zeta_1|, |\zeta_2|, |\zeta_3|, |\zeta_4|\}$, we have

$$\left(\frac{\sum_{i=1}^{4} |\zeta_i|}{4}\right)^2 \ge \frac{1}{\frac{4(4-1)}{2}} \left(\sum_{1\le i<j\le 4} |\zeta_i||\zeta_j|\right). \tag{7}$$

Thus

$$\left(\sum_{i=1}^{4} |\zeta_i|\right)^2 \ge \frac{16}{6} \left(\sum_{1\le i<j\le 4} |\zeta_i||\zeta_j|\right). \tag{8}$$

It can be seen that

$$\left(\sum_{i=1}^{4} |\zeta_i|\right)^2 = \sum_{i=1}^{4} |\zeta_i|^2 + 2 \sum_{1\le i<j\le 4} |\zeta_i||\zeta_j|$$
$$\Longleftrightarrow \sum_{1\le i<j\le 4} |\zeta_i||\zeta_j| = \frac{1}{2}\left(\left(\sum_{i=1}^{4} |\zeta_i|\right)^2 - \sum_{i=1}^{4} \zeta_i^2\right). \tag{9}$$

Next, by substituting equation (9) into equation (8), we obtain

$$\left(\sum_{i=1}^{4} |\zeta_i|\right)^2 \ge \frac{4}{3}\left(\left(\sum_{i=1}^{4} |\zeta_i|\right)^2 - \sum_{i=1}^{4} \zeta_i^2\right)$$
$$\Longleftrightarrow \left(\sum_{i=1}^{4} |\zeta_i|\right)^2 \le 4\sum_{i=1}^{4} \zeta_i^2$$

$$\iff \left(\sum_{i=1}^{4}|\zeta_i|\right)^2 \leq 4\left(\left(\sum_{i=1}^{4}\zeta_i\right)^2 - 2\left(\sum_{1\leq i<j\leq 4}\zeta_i\zeta_j\right)\right)$$
$$\iff \left(\sum_{i=1}^{4}|\zeta_i|\right)^2 \leq 4\left((\wp^2-3)^2 - 2(-2\wp^5+3\wp^4-2\wp^2+2)\right)$$
$$\iff \sum_{i=1}^{4}|\zeta_i| \leq 2\sqrt{4\wp^5-5\wp^4-2\wp^2+5}$$

Based on equation (6), we have

$$En(G) \leq \wp^2 - 3 + 2\sqrt{4\wp^5-5\wp^4-2\wp^2+5}. \tag{10}$$

The equality holds if and only if equality holds in equation (7), that is, $|\zeta_1| = |\zeta_2| = |\zeta_3| = |\zeta_4|$.

Next, we determine the lower bound of energy of G. Again using the inequality of (1), we have

$$\frac{1}{\frac{4(4-1)}{2}}\left(\sum_{1\leq i<j\leq 4}|\zeta_i||\zeta_j|\right) \geq \left(\prod_{i=1}^{4}|\zeta_i|\right)^{1/2}. \tag{11}$$

Thus,

$$\frac{1}{6}\left(\sum_{1\leq i<j\leq 4}|\zeta_i||\zeta_j|\right) \geq \left(\prod_{i=1}^{4}|\zeta_i|\right)^{1/2} \tag{12}$$

Next, by substituting equation (9) into equation (12), we obtain

$$\frac{1}{12}\left(\left(\sum_{i=1}^{4}|\zeta_i|\right)^2 - \sum_{i=1}^{4}\zeta_i^2\right) \geq \left(\prod_{i=1}^{4}|\zeta_i|\right)^{1/2}$$
$$\iff \left(\sum_{i=1}^{4}|\zeta_i|\right)^2 \geq \sum_{i=1}^{4}\zeta_i^2 + 12\left(\prod_{i=1}^{4}|\zeta_i|\right)^{1/2}$$
$$\iff \left(\sum_{i=1}^{4}|\zeta_i|\right)^2 \geq \left(\sum_{i=1}^{4}\zeta_i\right)^2 - 2\left(\sum_{1\leq i<j\leq 4}\zeta_i\zeta_j\right) + 12\left(\left|\prod_{i=1}^{4}\zeta_i\right|\right)^{1/2}$$
$$\iff \left(\sum_{i=1}^{4}|\zeta_i|\right)^2 \geq (\wp^2-3)^2 - 2(-2\wp^5+3\wp^4-2\wp^2+2) + 12(|\wp^6(\wp-1)^4|)^{1/2}$$

$$\iff \sum_{i=1}^{4} |\zeta_i| \geq \sqrt{4\wp^5 - 5\wp^4 - 2\wp^2 + 5 + 12(\wp^6(\wp-1)^4)^{1/2}}.$$

Based on equation (6), we get

$$En(G) \geq \wp^2 - 3 + \sqrt{4\wp^5 - 5\wp^4 - 2\wp^2 + 5 + 12(\wp^6(\wp-1)^4)^{1/2}}. \tag{13}$$

Equality holds if and only if $|\zeta_1| = |\zeta_2| = |\zeta_3| = |\zeta_4|$.

Therefore, from (13) and (10), we get

$$En(G) \geq \wp^2 - 3 + \sqrt{4\wp^5 - 5\wp^4 - 2\wp^2 + 5 + 12(\wp^6(\wp-1)^4)^{1/2}}$$

and

$$En(G) \leq \wp^2 - 3 + 2\sqrt{4\wp^5 - 5\wp^4 - 2\wp^2 + 5}.$$

For example, If $\wp = 2$, the vertex set of $\Gamma(\mathbb{Z}_2[x]/\langle x^5\rangle)$ is given by

$$\begin{aligned}\mathcal{A} &= \{\overline{x}, \overline{x^2+x}, \overline{x^3+x}, \overline{x^3+x^2+x}, \overline{x^4+x}, \overline{x^4+x^2+x}, \overline{x^4+x^3+x},\\ &\quad \overline{x^4+x^3+x^2+x}\},\\ \mathcal{B} &= \{\overline{x^2}, \overline{x^3+x^2}, \overline{x^4+x^2}, \overline{x^4+x^3+x^2}\}, \mathcal{C} = \{\overline{x^3}, \overline{x^4+x^3}\}, \text{and } \mathcal{D} = \{\overline{x^4}\}.\end{aligned}$$

The zero-divisor graph of $\mathbb{Z}_2[x]/\langle x^5\rangle$ is presented in Figure 1.

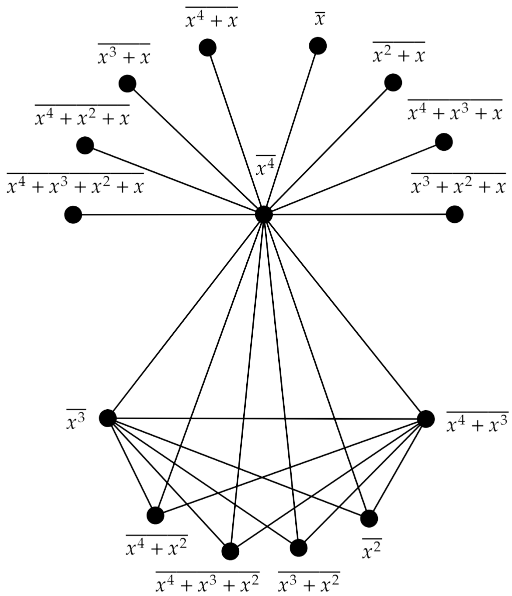

Figure 1: $\Gamma(\mathbb{Z}_2[x]/\langle x^5\rangle)$

The adjacency matrix of $\Gamma(\mathbb{Z}_2[x]/\langle x^5\rangle)$ is

$$A(\Gamma(\mathbb{Z}_2[x]/\langle x^5\rangle)) = \left[\begin{array}{cccccccc|cccc|cc|c} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\ \hline 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1\\ \hline 0&0&0&0&0&0&0&0&1&1&1&1&0&1&1\\ 0&0&0&0&0&0&0&0&1&1&1&1&1&0&1\\ \hline 1&1&1&1&1&1&1&1&1&1&1&1&1&1&0 \end{array}\right]$$

$$= \begin{bmatrix} O_8 & O_{8\times4} & O_{8\times2} & N_{8\times1}\\ O_{4\times8} & O_4 & N_{4\times2} & N_{4\times1}\\ O_{2\times8} & N_{2\times4} & N_2 - I_2 & N_{2\times1}\\ N_{1\times8} & N_{1\times4} & N_{1\times2} & N_1 - I_1 \end{bmatrix}.$$

By using Maple software, the characteristic equation of the matrix is obtained as:

$$(-\lambda)^{10}(-\lambda-1)(\lambda^4-\lambda^3-22\lambda^2-4\lambda+64)=0.$$

and its approximated the zeros are 0,-1,-3.4746, -2.2111, 1.6564, 5.0294. Thus, the spectrum of $\Gamma(\mathbb{Z}_2[x]/\langle x^5\rangle$ is

$$\{0^{[10]}, -1, -3.4746, -2.2111, 1.6564, 5.0294\}.$$

From this spectrum, we calculate the energy of $\Gamma(\mathbb{Z}_2[x]/\langle x^5\rangle)$ and we have

$$\begin{aligned} En(\Gamma(\mathbb{Z}_2[x]/\langle x^5\rangle)) &= \sum_{i=1}^{15} |\lambda_i| \\ &= |-1| + |-3.4746| + |-2.2111| + |1.6564| + |5.0294| \\ &= 13.3715. \end{aligned}$$

To further validate these results, we compute the eigenvalues and energy of $\Gamma(\mathbb{Z}_2[x]/\langle x^5\rangle)$ using Theorem 1 and Theorem 2. If $\wp = 2$, the eigenvalues of $\Gamma(\mathbb{Z}_2[x]/\langle x^5\rangle)$ are 0 with multiplicity 10, and -1 with multiplicity 1, as well as the eigenvalues satisfying the equation $\lambda^4-\lambda^3-22\lambda^2-4\lambda+64=0$, which are -3.4746, -2.2111, 1.6564, 5.0294. Then the spectrum is

$$\{0^{[10]}, -1, -3.4746, -2.2111, 1.6564, 5.0294\}.$$

It can be observed that these eigenvalues match the results from the Maple computation, confirming that Theorem 1 holds.

Further, by Theorem 2, the lower and upper bounds of the energy for $\Gamma(\mathbb{Z}_2[x]/\langle x^5\rangle$ are

$$12.8743 \leq En(\Gamma(\mathbb{Z}_2[x]/\langle x^5\rangle)) \leq 14.4164.$$

It can be seen that the energy of $\Gamma(\mathbb{Z}_2[x]/\langle x^5\rangle)$ falls between these lower and upper bounds. Therefore, Theorem 2 is satisfied.

To clarify the relationships between energy values and the defined bounds, we present a table showing numerical results of the lower bound, energy, and upper bound for several prime numbers. From Table 1, it can be seen that for prime numbers less than 100, the lower and upper bounds are quite close to the energy value. This indicates that the formulas for the lower and upper bounds can be effectively used to estimate the energy of the graph. This suggests that the methods used to calculate these bounds are effective in describing the energy of the graph.

Table 1: Lower Bound, Energy, and Upper Bound for Various Prime Numbers $\wp$

$\wp$	Lower Bound	Graph Energy	Upper Bound
2	12.8743	13.3715	14.4164
3	49.0116	50.2380	53.0744
5	204.5651	207.8708	215.1839
7	496.8947	502.8265	515.5956
$\vdots$	$\vdots$	$\vdots$	$\vdots$
79	225576.2183	225836.4582	226360.3543
83	255190.6650	255471.0792	256035.4105
89	303779.2343	304090.8360	304717.6682
97	376613.0645	376967.9188	377681.4176

4. Topological indices of zero-divisor graph of $\mathbb{Z}_\wp[x]/\langle x^5\rangle$

In this section, we determine the Wiener index, hyper-Wiener index, first Zagreb index, second Zagreb index, Narumi-Katayama index.

Theorem 3. *The Wiener index of graph $G \cong \Gamma(\mathbb{Z}_\wp[x]/\langle x^5\rangle)$, is*

$$\mathcal{W}(G) = \frac{1}{2}(2\wp^8 - 4\wp^5 - \wp^4 + \wp^2 + 2).$$

Proof. Let $G \cong \Gamma(\mathbb{Z}_\wp[x]/\langle x^5\rangle)$. The distances between the vertices of G have been determined by Musyarrofah et al. in [24]. It was shown that the distance between adjacent vertices in G is 1, and for non-adjacent vertices, it is 2. As a result, the distance matrix for G, $D(G)$ is represented as follows:

$$\begin{array}{c} \\ \mathcal{A} \\ \mathcal{B} \\ \mathcal{C} \\ \mathcal{D} \end{array} \begin{array}{c} \begin{array}{cccc} \mathcal{A} & \mathcal{B} & \mathcal{C} & \mathcal{D} \end{array} \\ \begin{bmatrix} 2(N_{\wp^4-\wp^3} - I_{\wp^4-\wp^3}) & 2N_{(\wp^4-\wp^3)\times(\wp^3-\wp^2)} & 2N_{(\wp^4-\wp^3)\times(\wp^2-\wp)} & N_{(\wp^4-\wp^3)\times(\wp-1)} \\ 2N_{(\wp^3-\wp^2)\times(\wp^4-\wp^3)} & 2(N_{\wp^3-\wp^2} - I_{\wp^3-\wp^2}) & N_{(\wp^3-\wp^2)\times(\wp^2-\wp)} & N_{(\wp^3-\wp^2)\times(\wp-1)} \\ 2N_{(\wp^2-\wp)\times(\wp^4-\wp^3)} & N_{(\wp^2-\wp)\times(\wp^3-\wp^2)} & N_{\wp^2-\wp} - I_{\wp^2-\wp} & N_{(\wp^2-\wp)\times(\wp-1)} \\ N_{(\wp-1)\times(\wp^4-\wp^3)} & N_{(\wp-1)\times(\wp^3-\wp^2)} & N_{(\wp-1)\times(\wp^2-\wp)} & N_{\wp-1} - I_{\wp-1} \end{bmatrix} \end{array}.$$

Here O represents the zero matrix, N represents the matrix of ones, and I represents the identity matrix. Thus, the Wiener index of G is

$$\begin{aligned} \mathcal{W}(G) &= \frac{1}{2}\sum_{i=1}^{\wp^4-1}\sum_{j=1}^{\wp^4-1} d_{ij} \\ &= \frac{1}{2}\Big(2\left(\wp^4-\wp^3\right)^2 - 2(\wp^4-\wp^3) + 4\left(\wp^4-\wp^3\right)\left(\wp^3-\wp^2\right) + 4\left(\wp^4-\wp^3\right)\left(\wp^2-\wp\right) \\ &\quad + 2\left(\wp^4-\wp^3\right)(\wp-1) + 2\left(\wp^3-\wp^2\right)^2 - 2(\wp^3-\wp^2) + 2\left(\wp^3-\wp^2\right)\left(\wp^2-\wp\right) \\ &\quad + 2\left(\wp^3-\wp^2\right)(\wp-1) + \left(\wp^2-\wp\right)^2 - (\wp^2-\wp) + 2\left(\wp^2-\wp\right)(\wp-1) + (\wp-1)^2 - (\wp-1)\Big) \\ &= \frac{1}{2}(2\wp^8 - 4\wp^5 - \wp^4 + \wp^2 + 2). \end{aligned}$$

Theorem 4. *The Hyper-Wiener index of graph $G \cong \Gamma(\mathbb{Z}_\wp[x]/\langle x^5\rangle)$ is*

$$\mathcal{WW}(G) = \frac{1}{2}(3\wp^8 - 8\wp^5 + \wp^4 + 2\wp^2 + 2).$$

Proof. Based on the proof of the Theorem 3, the distance matrix of G has been determined, so the Hyper-Wiener index of graph G is

$$\begin{aligned} \mathcal{WW}(G) &= \frac{1}{2}\mathcal{W}(G) + \frac{1}{4}\sum_{i=1}^{\wp^4-1}\sum_{j=1}^{\wp^4-1}(d_{ij})^2 \\ &= \frac{1}{2}\mathcal{W}(G) + \frac{1}{4}\Big(4\left(\wp^4-\wp^3\right)^2 - 4(\wp^4-\wp^3) + 8\left(\wp^4-\wp^3\right)\left(\wp^3-\wp^2\right) \\ &\quad + 8\left(\wp^4-\wp^3\right)\left(\wp^2-\wp\right) + 2\left(\wp^4-\wp^3\right)(\wp-1) + 4\left(\wp^3-\wp^2\right)^2 \\ &\quad - 4(\wp^3-\wp^2) + 2\left(\wp^3-\wp^2\right)\left(\wp^2-\wp\right) + 2\left(\wp^3-\wp^2\right)(\wp-1) \end{aligned}$$

$$
\begin{aligned}
&+ \left(\wp^2-\wp\right)^2 - (\wp^2-\wp) + 2\left(\wp^2-\wp\right)(\wp-1) + (\wp-1)^2 - (\wp-1)\Big) \\
&= \frac{1}{2}\left(\frac{2\wp^8-4\wp^5-\wp^4+\wp^2+2}{2}\right) + \frac{1}{4}(4\wp^8-12\wp^5+3\wp^4+3\wp^2+2) \\
&= \frac{1}{2}(3\wp^8-8\wp^5+\wp^4+2\wp^2+2).
\end{aligned}
$$

Theorem 5. *The first and second Zagreb indices of graph $G \cong \Gamma(\mathbb{Z}_\wp[x]/\langle x^5\rangle)$ respectively are*

$$
\begin{aligned}
\mathcal{M}_1(G) &= \wp^9 - 13\wp^5 + 13\wp^4 + 3\wp^2 - 4. \\
\mathcal{M}_2(G) &= \frac{1}{2}(\wp-1)(10\wp^9 - 13\wp^8 - 3\wp^7 - 8\wp^6 - 4\wp^5 + 32\wp^4 - 8\wp - 8).
\end{aligned}
$$

Proof. Let $a \in \mathcal{A}$, $b \in \mathcal{B}$, $c, c' \in \mathcal{C}$ for $c \neq c'$, and $d, d' \in \mathcal{D}$ for $d \neq d'$, where $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are the partition of the vertex set of G. Based on Lemma 2, it is clear that the set of edges established of graph G is as follows:

$$
\begin{aligned}
&E_1 = \{ad \in E(G) \mid a \in \mathcal{A}, d \in \mathcal{D}\}, && |E_1| = \wp^5 - 2\wp^4 + \wp^3, \\
&E_2 = \{bc \in E(G) \mid b \in \mathcal{B}, c \in \mathcal{C}\}, && |E_2| = \wp^5 - 2\wp^4 + \wp^3, \\
&E_3 = \{bd \in E(G) \mid b \in \mathcal{B}, d \in \mathcal{D}\}, && |E_3| = \wp^4 - 2\wp^3 + \wp^2, \\
&E_4 = \{cc' \in E(G) \mid c, c' \in \mathcal{C}, c \neq c'\}, && |E_4| = \frac{1}{2}(\wp^4 - 2\wp^3 + \wp), \\
&E_5 = \{cd \in E(G) \mid c \in \mathcal{C}, d \in \mathcal{D}\}, && |E_5| = \wp^3 - 2\wp^2 + \wp, \\
&E_6 = \{dd' \in E(G) \mid d, d' \in \mathcal{D}, d \neq d'\}, && |E_6| = \frac{1}{2}(\wp^2 - 3\wp + 2).
\end{aligned}
$$

In [24], Musyarrofah et al. determined the degree of each vertex of the graph G as follows.

$$
\deg(a) = \wp - 1, \quad \deg(b) = \wp^2 - 1, \quad \deg(c) = \wp^3 - 2, \quad \deg(d) = \wp^4 - 2.
$$

Thus, the first Zagreb of G is

$$
\begin{aligned}
\mathcal{M}_1(G) &= \sum_{vw \in E(G)} [\deg(v) + \deg(w)] \\
&= \left[(\wp-1) + \left(\wp^4-2\right)\right]\left(\wp^5-2\wp^4+\wp^3\right) + \left[\left(\wp^2-1\right) + \left(\wp^3-2\right)\right]\left(\wp^5-2\wp^4+\wp^3\right) + \\
&\quad \left[\left(\wp^2-1\right) + \left(\wp^4-2\right)\right]\left(\wp^4-2\wp^3+\wp^2\right) + \left[\left(\wp^3-2\right) + \left(\wp^3-2\right)\right]\frac{1}{2}\left(\wp^4-2\wp^3+\wp\right) + \\
&\quad \left[\left(\wp^3-2\right) + \left(\wp^4-2\right)\right]\left(\wp^3-2\wp^2+\wp\right) + \left[\left(\wp^4-2\right) + \left(\wp^4-2\right)\right]\frac{1}{2}\left(\wp^2-3\wp+2\right) \\
&= \wp^9 - 13\wp^5 + 13\wp^4 + 3\wp^2 - 4.
\end{aligned}
$$

Moreover, the second Zagreb index of G is

$$\begin{aligned}\mathcal{M}_2(G) &= \sum_{vw\in E(G)} [\deg(v)\deg(v)] \\ &= (\wp-1)\left(\wp^4-2\right)\left(\wp^5-2\wp^4+\wp^3\right)+\left(\wp^2-1\right)\left(\wp^3-2\right)\left(\wp^5-2\wp^4+\wp^3\right)+ \\ &\left(\wp^2-1\right)\left(\wp^4-2\right)\left(\wp^4-2\wp^3+\wp^2\right)+\frac{1}{2}\left(\left(\wp^3-2\right)\left(\wp^3-2\right)\left(\wp^4-2\wp^3+\wp\right)\right)+ \\ &\left(\wp^3-2\right)\left(\wp^4-2\right)\left(\wp^3-2\wp^2+\wp\right)+\frac{1}{2}\left(\left(\wp^4-2\right)\left(\wp^4-2\right)\left(\wp^2-3\wp+2\right)\right) \\ &= \frac{1}{2}(\wp-1)(10\wp^9-13\wp^8-3\wp^7-8\wp^6-4\wp^5+32\wp^4-8\wp-8).\end{aligned}$$

Theorem 6. *The Narumi-Katayama index of graph $G \cong \Gamma(\mathbb{Z}_\wp[x]/\langle x^5\rangle)$ is*

$$\mathcal{NK}(G) = (\wp-1)^{\wp^4-\wp^3}\left(\wp^2-1\right)^{\wp^3-\wp^2}\left(\wp^3-2\right)^{\wp^2-\wp}\left(\wp^4-2\right)^{\wp-1}$$

Proof. Based on Musyarrofah et al. [24], the degrees of the vertices in graph G as follows:

$$\deg(a)=\wp-1,\quad \deg(b)=\wp^2-1,\quad \deg(c)=\wp^3-2,\quad \deg(d)=\wp^4-2.$$

Thus, the Narumi-Katayama index of graph G is

$$\begin{aligned}\mathcal{NK}(G) &= \prod_{v\in V(G)} \deg(v) \\ &= \underbrace{(\wp-1)(\wp-1)\cdots(\wp-1)}_{\wp^4-\wp^3\text{-times}} \times \underbrace{(\wp^2-1)(\wp^2-1)\cdots(\wp^2-1)}_{\wp^3-\wp^2\text{-times}} \\ &\times \underbrace{(\wp^3-2)(\wp^3-2)\cdots(\wp^3-2)}_{\wp^2-\wp\text{-times}} \times \underbrace{(\wp^4-2)(\wp^4-2)\cdots(\wp^4-2)}_{\wp-1\text{-times}} \\ &= (\wp-1)^{\wp^4-\wp^3}\left(\wp^2-1\right)^{\wp^3-\wp^2}\left(\wp^3-2\right)^{\wp^2-\wp}\left(\wp^4-2\right)^{\wp-1}.\end{aligned}$$

After deriving the formulas for each topological index, numerical simulations (except Narumi-Katayama index) are performed for several prime numbers $\wp$ on the graph $G \cong \mathbb{Z}_\wp[x]/\langle x^5\rangle$ using a computer software MATLAB. We exclude the computed value of the Narumi-Katayama index, since the value is close to infinity. The results of these simulations are presented in Table 2 and Figure 2, which display the values of the indices $\mathcal{W}(G)$, $\mathcal{WW}(G)$, $\mathcal{M}_1(G)$, and $\mathcal{M}_2(G)$ for various prime numbers $\wp$.

Table 2: Computed Values of Topological Indices for Various Prime Numbers

$\wp$	$\mathcal{W}(G)$	$\mathcal{WW}(G)$	$\mathcal{M}_1(G)$	$\mathcal{M}_2(G)$
2	1.87×10^2	2.69×10^2	3.12×10^2	6.28×10^2
3	6.04×10^3	8.92×10^3	1.76×10^4	1.0073×10^5
5	3.8408×10^5	5.7378×10^5	1.9207×10^6	2.8202×10^7
7	5.73×10^6	8.5812×10^6	4.0166×10^7	9.7557×10^8
11	2.1403×10^8	3.209×10^8	2.356×10^9	1.036×10^{11}
13	8.1497×10^8	1.2221×10^9	1.06×10^{10}	5.7128×10^{11}
17	6.9729×10^9	1.0458×10^{10}	1.1857×10^{11}	8.7501×10^{12}
19	1.6979×10^{10}	2.5466×10^{10}	3.2266×10^{11}	2.7027×10^{13}
23	7.8298×10^{10}	1.1744×10^{11}	1.8011×10^{12}	1.868×10^{14}
29	5.0021×10^{11}	7.5029×10^{11}	1.4507×10^{13}	1.9392×10^{15}
31	8.5283×10^{11}	1.2792×10^{12}	2.6439×10^{13}	3.7983×10^{15}
37	3.5123×10^{12}	5.2684×10^{12}	1.2996×10^{14}	2.2566×10^{16}

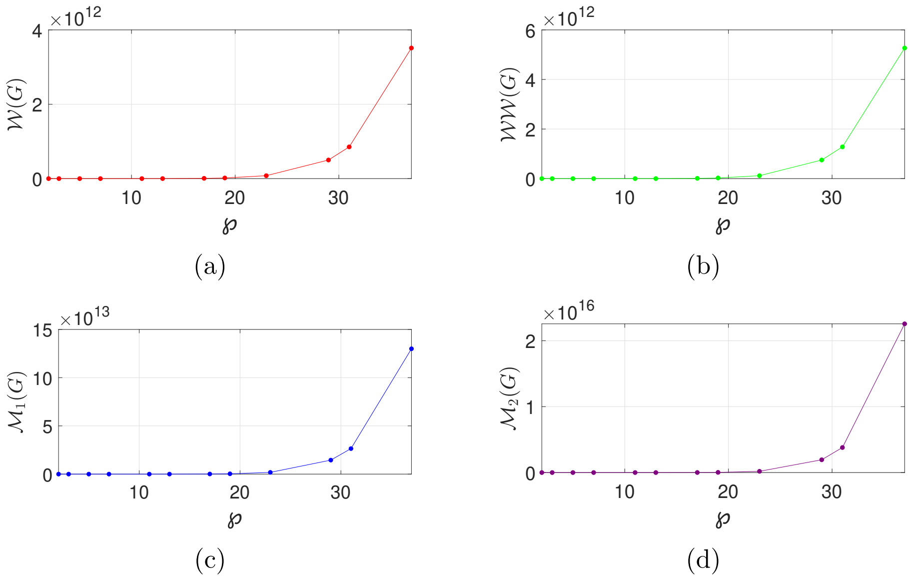

Figure 2: Numerical Simulation of (a) Wiener index (b) Hyper-Wiener index (c) First Zagreb index (d) Second Zagreb index

Based on Table 2 and Figure 2, it is observed that for all indices exhibit the same graphs and a significant increase as $\wp$ increases. In particular, the second Zagreb index shows the most rapid growth compared to the other indices. This rise is followed by the first Zagreb index, the Hyper-Wiener index, and the Wiener index. The observed ordering indicates that degree-based indices tend to grow faster than distance-based indices in this graph structure. This behavior suggests that as the graph expands, the contribution of higher-degree vertices becomes more dominant. Such insights are valuable for understanding molecular stability in chemical graph theory, where these indices play a crucial role in modeling molecular properties.

5. Conclusion

From the results presented earlier, we obtain the upper and lower bounds of the energy of the zero-divisor graph of $\mathbb{Z}_{\wp}[x]/\langle x^5\rangle$ for prime number $\wp$. We conclude that the obtained energy bounds are sharp and close to the actual energy. This result shows that the methods used are effective for estimating the graph's energy accurately. We also acquire the topological indices of $\Gamma(\mathbb{Z}_{\wp}[x]/\langle x^5\rangle)$ based on distance and degree, such as Wiener index, hyper-Wiener index, first Zagreb index, second Zagreb index, and Narumi-Katayama index. Additionally, numerical simulations of topological indices (excluding the Narumi-Katayama index) show that the second Zagreb index grows the fastest, followed by the first Zagreb index, while the Hyper-Wiener and Wiener indices grow much slower. For further research, the distance and Laplacian energy of $\Gamma(\mathbb{Z}_{\wp}[x]/\langle x^5\rangle)$ can be explored, along with an investigation of other graph topological indices.

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