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Quasi-Ideals and $\mathcal{H}$-Classes on the Direct Product of Two Semigroups

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Abstract. Let S be a semigroup and $x \in S$. The principal quasi-ideal of S containing x is denoted by $Q(x)$. An $\mathcal{H}$-class of S containing x is denoted by H_x. Let S_1, S_2 be semigroups. The direct product $S_1 \times S_2$ is defined as the Cartesian product of S_1 and S_2 equipped with the componentwise binary operation. Let $(a, b) \in S_1 \times S_2$. The direct product of $Q(a) \times Q(b)$ need not to be $Q((a, b))$. In this paper, we provide necessary and sufficient conditions when $Q(a) \times Q(b) = Q((a, b))$ and the conditions when $H_{(a,b)} = H_a \times H_b$.

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1. Introduction

Let S be a semigroup. A nonempty subset Q of S is called a *quasi-ideal* of S if $QS \cap SQ \subseteq Q$. The concept of quasi-ideals was introduced by Steinfeld [1]. Quasi-ideals have been studied extensively in semigroup theory and have been found to have many important applications and connections with other algebraic structures. One such connection is with congruences on semigroups. It was proved by Steinfeld [2] that for any $a, b \in S$, $a\mathcal{H}b$ if and only if $Q(a) = Q(b)$. This result shows that a congruence $\mathcal{H}$ on a semigroup can be described in terms of the associated quasi-ideal.

Given semigroups S_1 and S_2, the direct product $S_1 \times S_2$ is defined as the Cartesian product of S_1 and S_2, equipped with the componentwise binary operation. That is, for any $(s_1, s_2), (t_1, t_2) \in S_1 \times S_2$, we define $(s_1, s_2)(t_1, t_2) = (s_1t_1, s_2t_2)$. Let $(a, b) \in S_1 \times S_2$. Fabrici investigated the structure of $\mathcal{L}$-classes in the direct product of S_1 and S_2 [3]. The author showed that $L(a) \times L(b)$ need not be equal to $L((a, b))$ and give the conditions when $L(a) \times L(b) = L((a, b))$. Fabrici also investigated the conditions under which the direct product of $\mathcal{L}$-classes $L_a \times L_b$ is an $\mathcal{L}$-class $L_{(a,b)}$. Conditions for the direct product of $\mathcal{J}$-classes $J_a \times J_b = J_{(a,b)}$ were investigated as well [4].

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Let m, n be nonnegative integers. A subsemigroup A of S is called an (m, n)-*ideal* of S if $A^m S A^n \subseteq A$ [5]. Here, $A^0 S = SA^0 = S$. The smallest (m, n)-ideal of S containing a is denoted by $[a]_{(m,n)}$. Luangchaisri and Changphas provided necessary and sufficient conditions for $[a]_{(m,n)} \times [b]_{(m,n)} = [(a, b)]_{(m,n)}$ [6]. Moreover, they determined an equivalence class on a semigroup S by for any $x \in S$,

$$J_{(m,n),x} = \{y \in S \mid [x]_{(m,n)} = [y]_{(m,n)}\}$$

and provided the condition for $J_{(m,n),a} \times J_{(m,n),b} = J_{(m,n),(a,b)}$.

In this paper, we show that the direct product $Q(a) \times Q(b)$ need not to be $Q((a, b))$. In addition, we provide necessary and sufficient conditions when $Q(a) \times Q(b) = Q((a, b))$ and the condition when $H_a \times H_b = H_{(a,b)}$.

2. Main Results

Let S be a semigroup. For nonempty subsets A, B of S, the set product of A and B, denoted by AB, is defined by

$$AB = \{ab \mid a \in A \text{ and } b \in B\}.$$

In particular, we write Ab instead of $A\{b\}$ and write aB instead of $\{a\}B$.

Definition 1. *A nonempty subset Q of a semigroup S is called a* quasi-ideal *of S if $QS \cap SQ \subseteq Q$.*

Definition 2. *Let S be a semigroup and $a \in S$. The smallest quasi-ideal of S containing a, denoted by $Q(a)$, is called the* principal quasi-ideal *of S generated by a.*

Remark 1. *Let $a \in S$. Then $Q(a) = \{a\} \cup (aS \cap Sa)$.*

Theorem 1. *If Q_1 and Q_2 are quasi-ideals of semigroups S_1 and S_2, respectively, then $Q_1 \times Q_2$ is a quasi-ideal of $S_1 \times S_2$.*

Proof. Assume that Q_1 and Q_2 be quasi-ideals of S_1 and S_2, respectively. Then

$$\begin{aligned}(Q_1 \times Q_2)(S_1 \times S_2) \cap (S_1 \times S_2)(Q_1 \times Q_2) &= (Q_1S_1 \times Q_2S_2) \cap (S_1Q_1 \times S_2Q_2) \\ &= (Q_1S_1 \cap S_1Q_1) \times (Q_2S_2 \cap S_2Q_2) \\ &\subseteq Q_1 \times Q_2.\end{aligned}$$

Therefore, $Q_1 \times Q_2$ is a quasi-ideal of $S_1 \times S_2$.

The above theorem states that the direct product of two quasi-ideals is a quasi-ideal. However, it is important to note that the direct product of two principal quasi-ideals does not necessarily result in a principal quasi-ideal. This can be seen in the following example:

Example 1. *([7]) Consider two semigroups $(S_1, *)$ and $(S_2, \cdot)$, where $S_1 = \{a_1, a_2, a_3, a_4\}$ and $S_2 = \{b_1, b_2, b_3, b_4\}$. The binary operation $*$ on S_1 and the binary operation $\cdot$ on S_2 are defined as follows:*

$*$	a_1	a_2	a_3	a_4
a_1	a_1	a_1	a_1	a_1
a_2	a_1	a_2	a_2	a_4
a_3	a_1	a_2	a_2	a_4
a_4	a_1	a_4	a_4	a_2

$\cdot$	b_1	b_2	b_3	b_4
b_1	b_1	b_1	b_1	b_4
b_2	b_1	b_2	b_2	b_4
b_3	b_1	b_3	b_3	b_4
b_4	b_4	b_4	b_4	b_1

We observe that $(a_3, b_4) \in Q(a_3) \times Q(b_3)$, but $(a_3, b_4) \notin Q((a_3, b_3))$. Thus, we have $Q(a_3) \times Q(b_3) \neq Q((a_3, b_3))$. Furthermore, for any $(u, v) \in S_1 \times S_2$ such that $(u, v) \neq (a_3, b_4)$, we have $(a_3, b_4) \in Q(a_3) \times Q(b_3)$, but $(a_3, b_4) \notin \{(u, v)\} \cup ((uS_1 \cap S_1u) \times (vS_2 \cap S_2v))$. Therefore, we obtain $Q(a_3) \times Q(b_3) \neq Q((u, v))$, which implies that $Q(a_3) \times Q(b_3)$ is not the principal quasi-ideal of $S_1 \times S_2$.

From Example 1, we see that $Q(a) \times Q(b)$ need not to be $Q((a, b))$. Next, we present the inclusion of $Q(a) \times Q(b)$ and $Q((a, b))$ and give necessary and sufficient conditions when $Q(a) \times Q(b) = Q((a, b))$.

Theorem 2. *Let $(a, b) \in S_1 \times S_2$. Then $Q((a, b)) \subseteq Q(a) \times Q(b)$.*

Proof. Assume that $(a, b) \in S_1 \times S_2$. Then

$$\begin{aligned} Q((a, b)) &= \{(a, b)\} \cup ((a, b)(S_1 \times S_2) \cap (S_1 \times S_2)(a, b)) \\ &= \{(a, b)\} \cup ((aS_1 \times bS_2) \cap (S_1a \times S_2b)) \\ &= \{(a, b)\} \cup ((aS_1 \cap S_1a) \times (bS_2 \cap S_2b)) \\ &\subseteq (\{a\} \cup (aS_1 \cap S_1a)) \times (\{b\} \cup (bS_2 \cap S_2b)) \\ &= Q(a) \times Q(b). \end{aligned}$$

Therefore, $Q((a, b)) \subseteq Q(a) \times Q(b)$.

Theorem 3. *Let $(a, b) \in S_1 \times S_2$. Then $Q((a, b)) = Q(a) \times Q(b)$ if and only if at least one of the following conditions holds:*

(1) $aS_1 \cap S_1a = \{a\}$;

(2) $bS_2 \cap S_2b = \{b\}$;

(3) $a \in aS_1 \cap S_1a$ *and* $b \in bS_2 \cap S_2b$.

Proof. Assume that (1), (2), and (3) do not hold. Then we have two cases to consider:

(i) $a \notin aS_1 \cap S_1a$ and $bS_2 \cap S_2b \neq \{b\}$;

(ii) $b \notin bS_2 \cap S_2b$ and $aS_1 \cap S_1a \neq \{a\}$.

If (i) holds, then there exists $v \in bS_2 \cap S_2b$ such that $v \neq b$. Then $(a, v) \neq (a, b)$ and $(a, v) \notin (aS_1 \cap S_1a) \times (bS_2 \cap S_2b)$. Thus, $(a, v) \notin Q((a, b))$. Since $(a, v) \in Q(a) \times Q(b)$, it follows that $Q((a, b)) \neq Q(a) \times Q(b)$. We can prove similarly that $Q((a, b)) \neq Q(a) \times Q(b)$ when (ii) holds.

Conversely, assume that (1) holds. Then we get

$$\begin{aligned}
Q(a) \times Q(b) &= (\{a\} \cup (aS_1 \cap S_1 a)) \times (\{b\} \cup (bS_2 \cap S_2 b)) \\
&= \{a\} \times (\{b\} \cup (bS_2 \cap S_2 b)) \\
&= \{(a,b)\} \cup (\{a\} \times (bS_2 \cap S_2 b)) \\
&= \{(a,b)\} \cup ((aS_1 \cap S_1 a) \times (bS_2 \cap S_2 b)) \\
&= Q((a,b)).
\end{aligned}$$

On the same way, we have $Q(a) \times Q(b) = Q((a,b))$ when the condition (2) is satisfied. Assume that the condition (3) occurs. Then we have

$$\begin{aligned}
Q(a) \times Q(b) &= (\{a\} \cup (aS_1 \cap S_1 a)) \times (\{b\} \cup (bS_2 \cap S_2 b)) \\
&= (aS_1 \cap S_1 a) \times (bS_2 \cap S_2 b) \\
&= Q((a,b)).
\end{aligned}$$

Therefore, $Q(a) \times Q(b) = Q((a,b))$.

The following theorem shows that if $Q(a) \times Q(b) \neq Q((a,b))$, then $Q(a) \times Q(b)$ is not the principal quasi ideal.

Theorem 4. *If $Q(a) \times Q(b) \neq Q((a,b))$, then $Q(a) \times Q(b) \neq Q((u,v))$ for all $(u,v) \in S_1 \times S_2$.*

Proof. Assume that $Q(a) \times Q(b) \neq Q((a,b))$ and $Q(a) \times Q(b) = Q((u,v))$ for some $(u,v) \in S_1 \times S_2$. Then we obtain the following:

$$\begin{aligned}
(a,b) &\in Q(a) \times Q(b) = Q((u,v)) \subseteq Q(u) \times Q(v) \\
(u,v) &\in Q((u,v)) = Q(a) \times Q(b).
\end{aligned}$$

This implies that

$$\begin{aligned}
(aS_1 \cap S_1 a) \times (bS_2 \cap S_2 b) &\subseteq (Q(u)S_1 \cap S_1 Q(u)) \times (Q(v)S_2 \cap S_2 Q(v)) \\
&= (uS_1 \cap S_1 u) \times (vS_2 \cap S_2 v) \\
&\subseteq (Q(a)S_1 \cap S_1 Q(a)) \times (Q(b)S_2 \cap S_2 Q(b)) \\
&= (aS_1 \cap S_1 a) \times (bS_2 \cap S_2 b).
\end{aligned}$$

Since $(a,b) \neq (u,v)$ and $(a,b) \in Q(a) \times Q(b) = Q((u,v))$, it follows that

$$(a,b) \in (uS_1 \cap S_1 u) \times (vS_2 \cap S_2 v) = (aS_1 \cap S_1 a) \times (bS_2 \cap S_2 b).$$

By Theorem 3, we have

$$Q(a) \times Q(b) = Q((a,b)),$$

which contradicts the hypothesis.

The concept of Green's relations were introduced by Green [8]. It can be found in many textbooks on semigroup theory (e.g. [9], [10]). Equivalence relations $\mathcal{L}$ and $\mathcal{R}$ on a semigroup S are defined by

$$a\mathcal{L}b \Leftrightarrow L(a) = L(b)$$
$$a\mathcal{R}b \Leftrightarrow R(a) = R(b),$$

where $L(u)$ (respectively, $R(u)$) is the principal left (respectively, right) ideal of S containing u. An equivalence relation $\mathcal{H}$ on S is define by

$$\mathcal{H} = \mathcal{L} \cap \mathcal{R}.$$

For any $a \in S$, let H_a denote an $\mathcal{H}$-class of S containing a, that is,

$$H_a = \{b \in S \mid a\mathcal{H}b\}.$$

Lemma 1. *([2]) Let $a, b \in S$. Then $a\mathcal{H}b$ if and only if $Q(a) = Q(b)$. This implies that $H_a = H_b$ if and only if $Q(a) = Q(b)$.*

We give an example to show that the direct product of two $\mathcal{H}$-classes need not to be an $\mathcal{H}$-class as the following illustrative example.

Example 2. *([7]) Let $(S_1, *)$ and $(S_2, \cdot)$ be semigroups where $S_1 = \{a_1, a_2, a_3, a_4\}$ and $S_2 = \{b_1, b_2, b_3, b_4\}$ together with $* : S_1 \times S_1 \to S_1$ and $\cdot : S_2 \times S_2 \to S_2$ defined by*

$*$	a_1	a_2	a_3	a_4
a_1	a_1	a_1	a_1	a_1
a_2	a_1	a_2	a_3	a_4
a_3	a_1	a_3	a_2	a_4
a_4	a_4	a_4	a_4	a_4

$\cdot$	b_1	b_2	b_3	b_4
b_1	b_1	b_1	b_3	b_4
b_2	b_1	b_1	b_3	b_4
b_3	b_3	b_3	b_4	b_1
b_4	b_4	b_4	b_1	b_3

Since $(a_3, b_2) \in H_{a_2} \times H_{b_2}$ and $(a_3, b_2) \notin H_{(a_2,b_2)}$, we have that $H_{a_2} \times H_{b_2} \nsubseteq H_{(a_2,b_2)}$. Thus, $H_{a_2} \times H_{b_2}$ is not an $\mathcal{H}$-class of $S_1 \times S_2$.

Theorem 5. *Let $(a, b) \in S_1 \times S_2$. Then*

(1) *$H_{(a,b)} \subseteq H_a \times H_b$;*

(2) *If $H_{(a,b)} \subsetneq H_a \times H_b$, then $H_a \times H_b$ is a union of at least two $\mathcal{H}$-classes.*

Proof. (1). Let $(u, v) \in H_{(a,b)}$. Then,

$$Q((u, v)) = Q((a, b)).$$

We obtain that $Q(u) = Q(a)$ and $Q(v) = Q(b)$. Indeed,

$$(u, v) \in Q((u, v)) = Q((a, b)) \subseteq Q(a) \times Q(b)$$

and

$$(a,b) \in Q((a,b)) = Q((u,v)) \subseteq Q(u) \times Q(v).$$

By Lemma 1, we have $(u,v) \in H_a \times H_b$.
(2). Assume that $H_{(a,b)} \subsetneq H_a \times H_b$. There is $(u,v) \in S_1 \times S_2$ such that $(u,v) \in H_a \times H_b$ and $(u,v) \notin H_{(a,b)}$. By (1), we have that

$$\begin{aligned} H_{(u,v)} &\subseteq H_u \times H_v \\ &= H_a \times H_b. \end{aligned}$$

Therefore, $H_{(a,b)}$ and $H_{(u,v)}$ are difference classes contained in $H_a \times H_b$.

Theorem 6. *Let $(a,b) \in S_1 \times S_2$. Then $H_{(a,b)} = H_a \times H_b$ if and only if at least one of the following conditions holds:*

(1) $H_a = \{a\}$ *and* $H_b = \{b\}$;

(2) $a \in aS_1 \cap S_1 a$ *and* $b \in bS_2 \cap S_2 b$.

Proof. Assume that $H_{(a,b)} = H_a \times H_b$. If $H_{(a,b)} = \{(a,b)\}$, then we have

$$\begin{aligned} H_a \times H_b &= H_{(a,b)} \\ &= \{(a,b)\} \\ &= \{a\} \times \{b\}. \end{aligned}$$

Hence, $H_a = \{a\}$ and $H_b = \{b\}$. If $H_{(a,b)} \neq \{(a,b)\}$, then there exists $(u,v) \in S_1 \times S_2$ such that $(u,v) \neq (a,b)$ and

$$Q((u,v)) = Q((a,b)).$$

Since $(u,v) \in H_{(a,b)} = H_a \times H_b$, we have $Q(u) = Q(a)$ and $Q(v) = Q(b)$. Then

$$(uS_1 \cap S_1 u) \times (vS_2 \cap S_2 v) = (aS_1 \cap S_1 a) \times (bS_2 \cap S_2 b).$$

The assumptions $(a,b) \in Q((a,b)) = Q((u,v))$ and $(a,b) \neq (u,v)$ lead to

$$\begin{aligned} (a,b) &\in (uS_1 \cap S_1 u) \times (vS_2 \cap S_2 v) \\ &= (aS_1 \cap S_1 a) \times (bS_2 \cap S_2 b). \end{aligned}$$

Thus, $a \in aS_1 \cap S_1 a$ and $b \in bS_2 \cap S_2 b$.

Conversely, let $(u,v) \in H_a \times H_b$. If (1) holds, then

$$(u,v) = (a,b) \in H_{(a,b)}.$$

Suppose that (2) holds. By Theorem 3, we have $Q((a,b)) = Q(a) \times Q(b)$. Since $(u,v) \in H_a \times H_b$, it follows that

$$Q(a) = Q(u) \text{ and } Q(b) = Q(v).$$

Thus, $Q((u,v)) \subseteq Q(u) \times Q(v) = Q(a) \times Q(b) = Q((a,b))$. On the other hand,

$$\begin{aligned} Q(a,b) &= Q(a) \times Q(b) \\ &= (aS_1 \cap S_1 a) \times (bS_2 \cap S_2 b) \\ &= (Q(a)S_1 \cap S_1 Q(a)) \times (Q(b)S_2 \cap S_2 Q(b)) \\ &= (Q(u)S_1 \cap S_1 Q(u)) \times (Q(v)S_2 \cap S_2 Q(v)) \\ &= (uS_1 \cap S_1 u) \times (vS_2 \cap S_2 v) \\ &= (u,v)(S_1 \times S_2) \cap (S_1 \times S_2)(u,v) \\ &\subseteq Q((u,v)). \end{aligned}$$

Thus, $Q((a,b)) = Q((u,v))$. This means $(u,v) \in H_{(a,b)}$. By these two cases, we conclude that $H_a \times H_b \subseteq H_{(a,b)}$. By Theorem 5, we have $H_{(a,b)} \subseteq H_a \times H_b$. Therefore, $H_a \times H_b = H_{(a,b)}$.

Theorem 7. *Let $(a,b) \in S_1 \times S_2$. If $Q((a,b)) = Q(a) \times Q(b)$, then $H_{(a,b)} = H_a \times H_b$.*

Proof. Assume that $Q((a,b)) = Q(a) \times Q(b)$. By Theorem 3, we have one of the following conditions holds:

(i) $aS_1 \cap S_1 a = \{a\}$;

(ii) $bS_2 \cap S_2 b = \{b\}$;

(iii) $a \in aS_1 \cap S_1 a$ and $b \in bS_2 \cap S_2 b$.

Assume that $aS_1 \cap S_1 a = \{a\}$. Then $H_a = \{a\}$. If $b \in bS_2 \cap S_2 b$, we have that $H_{(a,b)} = H_a \times H_b$ by Theorem 6 (2). If $b \notin bS_2 \cap S_2 b$, then we obtain that $H_b = \{b\}$. Hence, $H_{(a,b)} = H_a \times H_b$ by Theorem 6 (1). The case $bS_2 \cap S_2 b = \{b\}$ can be proved similarly. The case $a \in aS_1 \cap S_1 a$ and $b \in bS_2 \cap S_2 b$ is obtained directly from Theorem 6 (2). Therefore, $H_{(a,b)} = H_a \times H_b$.

However, the converse of Theorem 7 is not true in general. That is, $H_{(a,b)} = H_a \times H_b$ does not implies that $Q((a,b)) = Q(a) \times Q(b)$.

Example 3. *([7]) Let $(S_1, *)$ and $(S_2, \cdot)$ be semigroups where $S_1 = \{a_1, a_2, a_3, a_4\}$ and $S_2 = \{b_1, b_2, b_3, b_4\}$. The binary operations $* : S_1 \times S_1 \longrightarrow S_1$ and $\cdot : S_2 \times S_2 \longrightarrow S_2$ are defined as:*

$*$	a_1	a_2	a_3	a_4
a_1	a_1	a_1	a_1	a_1
a_2	a_1	a_1	a_1	a_1
a_3	a_1	a_1	a_1	a_2
a_4	a_1	a_1	a_2	a_2

$\cdot$	b_1	b_2	b_3	b_4
b_1	b_1	b_1	b_1	b_1
b_2	b_1	b_1	b_1	b_1
b_3	b_1	b_1	b_3	b_3
b_4	b_1	b_1	b_3	b_3

We have $H_{a_3} = \{a_3\}$ and $H_{b_4} = \{b_4\}$. Hence $H_{(a_3,b_4)} = H_{a_3} \times H_{b_4}$ by Theorem 6. Since $(a_3, b_1) \in Q(a_3) \times Q(b_4)$ but $(a_3, b_1) \notin Q((a_3, b_4))$, it follows that $Q((a_3, b_4)) \neq Q(a_3) \times Q(b_4)$.

Theorem 8. *Let $(a, b) \in S_1 \times S_2$. If $\mid H_a \mid > 1$ and $\mid H_b \mid > 1$, then*

(1) $a \in aS_1 \cap S_1a$ *and* $b \in bS_2 \cap S_2b$;

(2) $H_{(a,b)} = H_a \times H_b$.

Proof. (1) Assume that $\mid H_a \mid > 1$ and $\mid H_b \mid > 1$. Then there exist $u \in H_a$ and $v \in H_b$ such that $u \neq a$ and $v \neq b$. It can be observed that

$$(a, b) \in (uS_1 \cap S_1u) \times (vS_2 \cap S_2v) \text{ and } (u, v) \in (aS_1 \cap S_1a) \times (bS_2 \cap S_2b).$$

Thus,

$$\begin{aligned}(a, b) &\in (uS_1 \cap S_1u) \times (vS_2 \cap S_2v)\\ &= (Q(u)S_1 \cap S_1Q(u)) \times (Q(v)S_2 \cap S_2Q(v))\\ &= (Q(a)S_1 \cap S_1Q(a)) \times (Q(b)S_2 \cap S_2Q(b))\\ &= (aS_1 \cap S_1a) \times (bS_2 \cap S_2b).\end{aligned}$$

Therefore, $a \in aS_1 \cap S_1a$ and $b \in bS_2 \cap S_2b$.
(2) It is obtained immediately by Theorem 6.

We obtain that the reverse of Theorem 8 is not true as the following example.

Example 4. *[7] Let $(S_1, *)$ and $(S_2, \cdot)$ be semigroups where $S_1 = \{a_1, a_2, a_3, a_4\}$ and $S_2 = \{b_1, b_2, b_3, b_4\}$. The binary operations $* : S_1 \times S_1 \longrightarrow S_1$ and $\cdot : S_2 \times S_2 \longrightarrow S_2$ are defined as:*

$*$	a_1	a_2	a_3	a_4
a_1	a_1	a_1	a_1	a_1
a_2	a_1	a_2	a_2	a_2
a_3	a_1	a_2	a_2	a_2
a_4	a_1	a_4	a_4	a_4

$\cdot$	b_1	b_2	b_3	b_4
b_1	b_1	b_1	b_1	b_1
b_2	b_1	b_2	b_2	b_2
b_3	b_1	b_2	b_2	b_2
b_4	b_1	b_2	b_2	b_2

We have that $H_{(a_1,b_2)} = \{(a_1, b_2)\} = \{a_1\} \times \{b_2\} = H_{a_1} \times H_{b_2}$ but $\mid H_{a_1} \mid = \mid H_{b_2} \mid = 1$.

Corollary 1. *Let $(a, b) \in S_1 \times S_2$. If $H_a \times H_b$ is a union of at least two $\mathcal{H}$-classes, then*

(1) $\mid H_a \mid > 1$ *and* $H_b = \{b\}$ *or*

(2) $H_a = \{a\}$ *and* $\mid H_b \mid > 1$.

Theorem 9. *Let $(a, b) \in S_1 \times S_2$. Then $H_a \times H_b$ is a union of at least two $\mathcal{H}$-classes if and only if*

(1) $\mid H_a \mid > 1, H_b = \{b\}$ *and* $b \notin bS_2 \cap S_2b$ *or*

(2) $a \notin aS_1 \cap S_1a, H_a = \{a\}$ *and* $\mid H_b \mid > 1$.

Proof. Assume that $H_a \times H_b$ is a union of at least two $\mathcal{H}$-classes. By Corollary 1, we have either $\mid H_a \mid > 1$ and $H_b = \{b\}$ or $H_a = \{a\}$ and $\mid H_b \mid > 1$.
Case 1: $\mid H_a \mid > 1$ and $H_b = \{b\}$. We have that $a \in aS_1 \cap S_1a$. Suppose that $b \in bS_2 \cap S_2b$. Form Theorem 3, we have $H_a \times H_b = H_{(a,b)}$ which contradicts to the assumption. Therefore, $b \notin bS_2 \cap S_2b$.
Case 2: $H_a = \{a\}$ and $\mid H_b \mid > 1$. We can prove similarly Case 1.

Conversely, we assume that (1) or (2) holds. If (1) holds, then $H_{(a,b)} \neq H_a \times H_b$ by Theorem 6. Hence, $H_{(a,b)} \subsetneq H_a \times H_b$. By Theorem 5, we obtain that $H_a \times H_b$ is a union of at least two $\mathcal{H}$-classes. We can proceed analogously in another case.

Let $u \in S$. An $\mathcal{L}$-class (respectively, $\mathcal{R}$-class) of S containing u is denoted by L_u (respectively, R_u). Since relations $\mathcal{L}$ and $\mathcal{R}$ on S are defined in terms of ideals, the inclusion order among these ideals induces a partial order among equivalence classes. For any $a, b \in S$,

$$L_a \leq L_b \Leftrightarrow L(a) \subseteq L(b)$$
$$R_a \leq R_b \Leftrightarrow R(a) \subseteq R(b).$$

These partial orders induce a partial order among $\mathcal{H}$-classes as follows:

$$H_a \leq H_b \Leftrightarrow L_a \leq L_b \text{ and } R_a \leq R_b.$$

We say that H_a is *maximal* if there is no $u \in S$ such that $H_a \leq H_u$ and $H_a \neq H_u$. Equivalently, H_a is maximal if and only if there is no $u \in S$ such that $H_a \leq H_u$ and $Q(a) \neq Q(u)$. A characterization of a maximal $\mathcal{H}$-class is indicated as the following theorem.

Theorem 10. *Let $a \in S$. Then H_a is a maximal if and only if there is no $b \in S$ such that $Q(a) \subsetneq Q(b)$.*

Proof. Suppose that there exists $b \in S$ such that $Q(a) \subsetneq Q(b)$,which implies that

$$H_a \neq H_b.$$

Since

$$a \in Q(a) \subsetneq Q(b) = L(b) \cap R(b),$$

we have $a \in L(b)$ and $a \in R(b)$. Hence, $L(a) \subseteq L(b)$ and $R(a) \subseteq R(b)$. It follows that $L_a \leqslant L_b$ and $R_a \leqslant R_b$. Thus, H_a is not maximal.

Conversely, assume that H_a is not maximal. Then there exists $b \in S$ such that $H_a \leqslant H_b$ and $H_a \neq H_b$. Thus, $L_a \leqslant L_b$ and $R_a \leqslant R_b$. These imply

$$L(a) \subseteq L(b)$$
$$R(a) \subseteq R(b)$$
$$Q(a) \neq Q(b).$$

Hence,

$$Q(a) = L(a) \cap R(a) \subseteq L(b) \cap R(b) = Q(b).$$

Therefore, $Q(a) \subsetneq Q(b)$.

Theorem 11. *Let $(a,b) \in S_1 \times S_2$ such that $(a,b) \in (aS_1 \cap S_1a) \times (bS_2 \cap S_2b)$. Then $H_{(a,b)}$ is maximal if and only if H_a and H_b are maximal.*

Proof. Assume that $H_{(a,b)}$ is not maximal. Then there exists $(u,v) \in S_1 \times S_2$ such that

$$Q((a,b)) \subsetneq Q((u,v)).$$

By Theorem 3, we have

$$\begin{aligned} Q(a) \times Q(b) &= Q((a,b)) \\ &\subsetneq Q((u,v)) \\ &\subseteq Q(u) \times Q(v). \end{aligned}$$

This means that

$$Q(a) \subsetneq Q(u) \text{ or } Q(b) \subsetneq Q(v).$$

Thus, H_a or H_b is not maximal.

Conversely, suppose that H_a is not maximal. Then there exists $u \in S_1$ such that $Q(a) \subsetneq Q(u)$. This implies that $Q(a) \subseteq uS_1 \cap S_1u$ and $u \notin Q(a)$.
Case 1: $u \in uS_1 \cap S_1u$. Then

$$\begin{aligned} Q((a,b)) &= Q(a) \times Q(b) \\ &\subsetneq Q(u) \times Q(b) \\ &= Q((u,b)). \end{aligned}$$

Case 2: $u \notin uS_1 \cap S_1u$. Then

$$\begin{aligned} Q((a,b)) &= Q(a) \times Q(b) \\ &\subsetneq \{(u,b)\} \cup (Q(a) \times Q(b)) \\ &= \{(u,b)\} \cup ((aS_1 \cap S_1a) \times (bS_2 \cap S_2b)) \\ &\subseteq \{(u,b)\} \cup ((uS_1 \cap S_1u) \times (bS_2 \cap S_2b)) \\ &= Q((u,b)). \end{aligned}$$

Therefore, $H_{(a,b)}$ is not maximal. We obtain the same conclusion when H_b is not maximal.

Lemma 2. *Let $(a,b) \in S_1 \times S_2$ such that $(a,b) \notin (aS_1 \cap S_1a) \times (bS_2 \cap S_2b)$. If H_a and H_b are maximal, then $H_{(a,b)}$ is maximal.*

Proof. Assume that $H_{(a,b)}$ is not maximal. Then there exists $(u,v) \in S_1 \times S_2$ such that

$$Q((a,b)) \subsetneq Q((u,v)).$$

It follows that $(a,b) \neq (u,v)$ and $(a,b) \in (uS_1 \cap S_1u) \times (vS_2 \cap S_2v)$. We obtain $Q(a) \subseteq Q(u)$ and $Q(b) \subseteq Q(v)$. If $Q(a) = Q(u)$ and $Q(b) = Q(v)$, then we have

$$\begin{aligned}(a,b) &\in (uS_1 \cap S_1u) \times (vS_2 \cap S_2v)\\ &= (Q(u)S_1 \cap S_1Q(u)) \times (Q(v)S_2 \cap S_2Q(v))\\ &= (Q(a)S_1 \cap S_1Q(a)) \times (Q(b)S_2 \cap S_2Q(b))\\ &= (aS_1 \cap S_1a) \times (bS_2 \cap S_2b).\end{aligned}$$

This contradicts to $(a,b) \notin (aS_1 \cap S_1a) \times (bS_2 \cap S_2b)$. We therefore conclude that $Q(a) \subsetneq Q(u)$ or $Q(b) \subsetneq Q(v)$. Thus, H_a or H_b is not maximal.

Lemma 3. *Let $H_{(a,b)}$ and $H_{(u,v)}$ be difference $\mathcal{H}$-classes contained in $H_a \times H_b$. If $(a,b) \notin (aS_1 \cap S_1a) \times (bS_2 \cap S_2b)$, then $(u,v) \notin (uS_1 \cap S_1u) \times (vS_2 \cap S_2v)$.*

Proof. Assume that

$$(u,v) \in (uS_1 \cap S_1u) \times (vS_2 \cap S_2v).$$

Since $H_{(u,v)} \subseteq H_a \times H_b$, we have $H_u \times H_v = H_a \times H_b$. This implies $Q(u) = Q(a)$ and $Q(v) = Q(b)$. Thus,

$$\begin{aligned}(a,b) &\in Q(a) \times Q(b)\\ &= Q(u) \times Q(v)\\ &= (uS_1 \cap S_1u) \times (vS_2 \cap S_2v)\\ &= (Q(u)S_1 \cap S_1Q(u)) \times (Q(v)S_2 \cap S_2Q(v))\\ &= (Q(a)S_1 \cap S_1Q(a)) \times (Q(b)S_2 \cap S_2Q(b))\\ &= (aS_1 \cap S_1a) \times (bS_2 \cap S_2b).\end{aligned}$$

Lemma 4. *Let H_a and H_b be maximal $\mathcal{H}$-classes of S_1 and S_2, respectively.*

(1) *If $H_a = \{a\}$ and $H_b = \{b\}$, then $H_a \times H_b = H_{(a,b)}$ is maximal.*

(2) *If $\mid H_a \mid > 1$ and $\mid H_b \mid > 1$, then $H_a \times H_b = H_{(a,b)}$ is maximal.*

Proof. (1) Assume that $H_a = \{a\}$ and $H_b = \{b\}$. Let $(u,v) \in S_1 \times S_2$ such that $Q((a,b)) \subseteq Q((u,v))$. If $(u,v) \notin Q((a,b))$, then we have $(u,v) \neq (a,b)$, that is,

$$u \neq a \text{ or } v \neq b.$$

Assume that $u \neq a$. Then we have two cases to consider:

(i) If $u \notin aS_1 \cap S_1a$, then $u \notin Q(a)$. This implies that $Q(a) \subsetneq Q(u)$ which contradicts to H_a is maximal.

(ii) If $u \in aS_1 \cap S_1a$, then $Q(a) = Q(u)$. Thus, $u \in H_a$ which contradicts to $H_a = \{a\}$.

The case $v \neq b$ can be proved similarly. Therefore, $(u,v) \in Q((a,b))$. Thus, $Q((u,v)) = Q((a,b))$. This means that $H_{(a,b)}$ is maximal.
(2) Assume that $\mid H_a \mid > 1$ and $\mid H_b \mid > 1$. Then $a \in aS_1 \cap S_1a$ and $b \in bS_2 \cap S_2b$. These imply that $H_a \times H_b = H_{(a,b)}$. By Theorem 11, the direct product $H_a \times H_b$ is maximal.

Theorem 12. *Let $(a,b) \in S_1 \times S_2$. If H_a and H_b are maximal, then one of the following conditions holds:*

(1) *$H_a \times H_b$ is maximal;*

(2) *$H_a \times H_b$ is the union of at least two maximal $\mathcal{H}$-classes in $S_1 \times S_2$.*

Proof. Assume that H_a and H_b are maximal. By Lemma 4, $H_a \times H_b = H_{(a,b)}$ is maximal when one of the following conditions holds:

(i) $\mid H_a \mid = 1$ and $\mid H_b \mid = 1$ or

(ii) $\mid H_a \mid > 1$ and $\mid H_b \mid > 1$.

Assume that $\mid H_a \mid > 1$ and $\mid H_b \mid = 1$. If $(a,b) \in (aS_1 \cap S_1a) \times (bS_2 \cap S_2b)$, then we obtain that $H_a \times H_b = H_{(a,b)}$ by Theorem 6. By Theorem 11, $H_a \times H_b$ is maximal. On the other hand, assume that

$$(a,b) \notin (aS_1 \cap S_1a) \times (bS_2 \cap S_2b).$$

Then $H_{(a,b)} \subsetneq H_a \times H_b$. By Theorem 5, the direct product $H_a \times H_b$ contains at least two $\mathcal{H}$-classes in $S_1 \times S_2$.

Let $H_{(u,v)}$ be arbitrary $\mathcal{H}$-class of $S_1 \times S_2$ contained in $H_a \times H_b$. By Lemma 3, we have $(u,v) \notin (us_1 \cap S_1u) \times (vS_2 \cap S_2v)$. Suppose that $H_{(u,v)}$ is not maximal. By Lemma 2, $H_a = H_u$ is not maximal or $H_b = H_v$ is not maximal. This contradicts to assumption. Therefore, $H_{(u,v)}$ is maximal.

The case $\mid H_a \mid = 1$ and $\mid H_b \mid > 1$ can be proved similarly.

Definition 3. *Let S be a semigroup. An element $a \in S$ is* decomposable *if there are $u, v \in S$ such that $a = uv$. We say that $b \in S$ is* indecomposable *if b is not decomposable, equivalently, $b \in S \setminus S^2$.*

Theorem 13. *Let $(a,b) \in S_1 \times S_2$. If (a,b) is indecomposable, then $H_{(a,b)}$ is a maximal $\mathcal{H}$-class of $S_1 \times S_2$.*

Proof. Assume that $H_{(a,b)}$ is not a maximal $\mathcal{H}$-class. This means that there exists $(u,v) \in S_1 \times S_2$ such that

$$Q((a,b)) \subsetneq Q((u,v)).$$

Since $(a,b) \neq (u,v)$ and $(a,b) \in Q((u,v))$, we have

$$(a,b) \in (uS_1 \cap S_1u) \times (vS_2 \cap S_2v).$$

Therefore, (a,b) is decomposable.

The following example shows that the reverse of the above theorem is not generally true.

Example 5. *[7] Let $S = \{a,b,c,d\}$ be a semigroup with the binary operation $*$ on S defined by*

$*$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	b	c	c
d	a	b	c	c

Then $H_{(b,b)}$ is maximal, but (b,b) is decomposable.

3. Conclusions

In this paper, we studied quasi-ideals and H-classes in the direct product of two semigroups $S_1 \times S_2$ by focusing on being the principal quasi-ideal of $Q(a) \times Q(b)$ and being the H-class of $H_a \times H_b$ in $S_1 \times S_2$, where $Q(a)$ and $Q(b)$ are the principal quasi-ideal of S_1 generated by $a \in S_1$ and the principal quasi-ideal of S_2 generated by $b \in S_2$, respectively. Similarly, H_a and H_b are an H-class of S_1 containing $a \in S_1$ and an H-class of S_2 containing $b \in S_2$, respectively.

First, we proved that $Q(a) \times Q(b)$ is a quasi-ideal in $S_1 \times S_2$, and we provided an example to indicate that it is not guaranteed to be the principal quasi-ideal generated by (a,b). Furthermore, we ensured that if $Q(a) \times Q(b)$ is not the principal quasi-ideal generated by (a,b), then it is not the principal quasi-ideal generated by any other element in $S_1 \times S_2$. We then characterized when the direct product of two principal quasi-ideals is the principal quasi-ideal by presenting the sufficient and necessary conditions for $Q(a) \times Q(b) = Q((a,b))$. In the context of the H-class, we gave an example to demonstrate that $H_a \times H_b$ is not guaranteed to be an H-class in $S_1 \times S_2$. The relation between the H-class $H_{(a,b)}$ and the direct product $H_a \times H_b$ is explained. After that, we provided the necessary and sufficient conditions for $H_a \times H_b = H_{(a,b)}$. The connection between the two main points is as follows: If $Q(a) \times Q(b)$ is the principal quasi-ideal, we can ensure that $H_a \times H_b$ is an H-class in $S_1 \times S_2$. Finally, we also studied the maximal H-class. We proved under some conditions that the maximality of H_a and H_b implies the maximality of $H_{(a,b)}$. Furthermore, the converse of this statement also holds. In addition, we investigated the properties of the direct product of two maximal H-classes $H_a \times H_b$ and studied the maximal H-class of $H_{(a,b)}$ through the maximal H-class of H_a and H_b, including the element, namely, the indecomposable element.

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