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Convex Independent Neighborhood Polynomial of Some Special Graphs

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Abstract. In this paper, we introduced the notion of convex independent neighborhood polynomial for a graph. We further explored the convex independent neighborhood polynomial for some special graphs such as paths, cycles, complete graphs, and star graphs. We generated these polynomials by counting the number of convex subsets of a graph with corresponding maximum independent set in the neighborhood system.

2020 Mathematics Subject Classifications: 05C31, 05C69

Key Words and Phrases: Convex sets, Convex subgraph, Independent neighborhood system, Independent set, Convex independent neighborhood polynomial

1. Introduction

Graph theory offers a broad foundation for studying various mathematical structures, such as graph polynomials and convexity in graphs. Graph polynomials have been the focus of interest in recent developments in graph theory. The use of polynomials to represent graphs has recently become a significant area of study, driven by its practical applications in various scientific fields [1]. As a result, several graph polynomials have been formulated, and substantial results have been obtained such as neighborhood polynomial which was introduced in 2008 by Brown and Nowakowski [2] and independent neighborhood polynomial which was studied by Abdulcarim et al. in 2021 [3]. On the other

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hand, convexity in a graph is defined based on specific path-based properties. A set of vertices is considered convex if, for any two vertices within the set, all shortest paths between them remain entirely within the set. Further, convexity is gaining its resurgence due to its various current applications. Several authors studied graph convexity in various perspectives and some explored convexity in $\mathbb{R}^n$ [4]. In particular, studies in [5] [6] [7] considered applying the concept of convexity to graphs and to its polynomials. Furthermore, in the studies of R. Artes Jr. et al., they integrated the concept of convexity with graph polynomial which involve counting the number of substructures with corresponding neighborhood system cardinality of some property [8][9].

With this development in the study of convex subsets and polynomials in graph, this paper aims to investigate another type of graph polynomial called the *convex independent neighborhood polynomial* and considers applying it to some special graphs.

2. Terminologies and Notations

A graph G is a finite nonempty set $V(G)$ of objects called *vertices* together with a possibly empty set $E(G)$ of 2-element subsets of $V(G)$ called *edges*. Vertices are sometimes called *points* or *nodes*, while edges are sometimes called *lines* or *links*. Given a graph $G = \langle V(G), E(G) \rangle$, where $V(G)$ is the *vertex-set* of G and $E(G)$ is the *edge-set* of G, the number of vertices in a graph G is the *order* of G and the number of edges is the *size* of G. A subset S of $V(G)$ is said to be *independent* in G if no two vertices in the set are adjacent in G. In other words, for any two vertices $u, v \in S$, $uv \notin E(G)$. The cardinality of a maximum independent set is called *independence number* of G. A *path* consists of a sequence of edges, one following another in which no vertex appears more than once. A path of order n with vertices $v_1, v_2, \ldots, v_n$ (in order) is denoted by P_n. A graph G is *connected* if every pair of vertices in the vertex-set of G is connected by a path. A *cycle* is a closed path of order n and is denoted by C_n. A graph G is called *bipartite* if $V(G)$ can be partitioned into two nonempty independent subsets A and B of $V(G)$ (called partite sets). A *complete bipartite graph* is a bipartite graph in which each vertex in A is joined to each vertex in B and is denoted by $K_{m,n}$ where m and n are the order of A and B, respectively. The complete bipartite graph $K_{1,n}$ of order $n + 1$ is called a *star graph*. A graph in which each pair of distinct vertices are adjacent is called a *complete graph* and denoted by K_n where n is the number of vertices. If $v \in V(G)$, the *open neighborhood* or simply *neighborhood* of v in G is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. For the subset S of $V(G)$, the *neighborhood system* of S in G is the set $N_G(S) = \left(\bigcup_{s\in S} N_G(s)\right)\backslash S$. The *independent neighborhood system* of a subset S of $V(G)$ is a subset of the neighborhood system of S in $V(G)$ which is independent. If the independent neighborhood system of a subset S of $V(G)$ is maximum, then we say that it is the *maximum independent neighborhood system* of S, denoted by Γ_{in}-sets. Given a connected graph G and vertices $u, v \in V(G)$, the *distance* $d_G(u, v)$ from a vertex u to vertex v is the smallest length of a u–v path in G. A $u - v$ path of length $d_G(u, v)$ is called a $u - v$ *geodesic*. The *geodesic closure* of $\{u, v\}$ is the set consists of all vertices lies in any $u - v$ geodesic including u and v and is denoted by $I_G[u,v]$. In other words, $I_G[u,v] = \{u,v\} \cup \{x : x \text{ lies in } u-v \text{ geodesic in } G\}$. The *geodesic closure* of a subset S of $V(G)$ is the set $I_G[S] = \bigcup_{u,v\in S} I_G[u,v]$. A subset S of $V(G)$ is *convex* if for every $u,v \in S$, the vertex-set of every $u-v$ geodesic is contained entirely in S. A convex subset of cardinality i is called $i-$*convex*. A subgraph H of G induced by a convex subset of $V(G)$ is called a *convex subgraph.* The *convex independent neighborhood polynomial* of a graph G of order n, denoted by $\Gamma_{cin}(G;x,y)$ in x and y indeterminates, is given by

$$\Gamma_{cin}(G;x,y) = \sum_{j=0}^{n-i}\sum_{i=1}^{n} c_{ij}(G)x^i y^j,$$

where $c_{ij}(G)$ is the number of i-convex subsets of G with maximum independent neighborhood system of cardinality equal to j. The *degree* of an algebraic polynomial is equal to the degree of the term that has the highest exponent. If, in additon, the polynomial is defined by several variables then two or more variables are in a term as factors, wherein the degree of the term is the sum of the exponents of the variables; and the degree of the polynomial will be determined by the highest degree in their terms [10].

3. Preliminary Results

This section illustrates the definition of convex independent neighborhood polynomial of a graph. This illustration form the basis for the more comprehensive analyses and conclusions for the following sections. Consider the graph G in Figure 1.

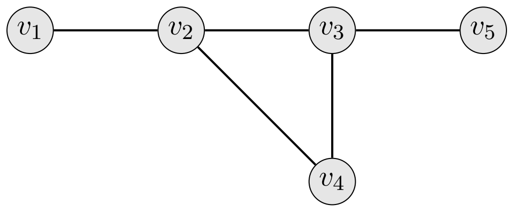

Figure 1: A graph G of order 5 and size 5

The convex subsets of $V(G)$ with corresponding maximum independent neighborhood systems (Γ_{in}-sets) are enumerated in the following Tables 1, 2 and 3:

1-convex	Γ_{in}-sets
$\{v_1\}$	$\{v_2\}$
$\{v_2\}$	$\{v_1,v_3\}$ and $\{v_1,v_4\}$
$\{v_3\}$	$\{v_2,v_5\}$ and $\{v_4,v_5\}$
$\{v_4\}$	$\{v_2\}$ and $\{v_3\}$
$\{v_5\}$	$\{v_3\}$

Table 1: 1-convex subsets of $V(G)$ and its corresponding Γ_{in}-sets.

From Table 1, observe that there are 3 1-convex subset of $V(G)$ with maximum independent neighborhood systems (Γ_{in}-sets) cardinality equal to 1, and 2 1-convex subsets of $V(G)$ with maximum independent neighborhood systems (Γ_{in}-sets) cardinality equal to 2. This contributes to the convex independent neighborhood polynomial of G as $3xy + 2xy^2$.

2-convex	Γ_{in}-sets
$\{v_1, v_2\}$	$\{v_3\}, \{v_4\}$
$\{v_2, v_3\}$	$\{v_1, v_4, v_5\}$
$\{v_2, v_4\}$	$\{v_1, v_3\}$
$\{v_3, v_4\}$	$\{v_2, v_5\}$
$\{v_3, v_5\}$	$\{v_2\}, \{v_4\}$

Table 2: 2-convex subsets of $V(G)$ and its corresponding Γ_{in}-sets.

From Table 2, observe that there are 2 2-convex subsets of $V(G)$ with maximum independent neighborhood systems (Γ_{in}-sets) cardinality equal to 1, 2 2-convex subsets of $V(G)$ with maximum independent neighborhood systems (Γ_{in}-sets) cardinality equal to 2, and 1 2-convex subset of $V(G)$ with maximum independent neighborhood systems (Γ_{in}-sets) cardinality equal to 3. This contributes to the convex independent neighborhood polynomial of G as $2x^2y + 2x^2y^2 + x^2y^3$.

3-convex	Γ_{in}-sets
$\{v_1, v_2, v_3\}$	$\{v_4, v_5\}$
$\{v_1, v_2, v_4\}$	$\{v_3\}$
$\{v_2, v_3, v_4\}$	$\{v_1, v_5\}$
$\{v_2, v_3, v_5\}$	$\{v_1, v_4\}$
$\{v_4, v_3, v_5\}$	$\{v_2\}$

Table 3: 3-convex subsets of $V(G)$ and its corresponding Γ_{in}-sets.

From Table 3, observe that there are 2 3-convex subsets of $V(G)$ that has maximum independent neighborhood systems (Γ_{in}-sets) cardinality equal to 1, and 3 3-convex subsets of $V(G)$ that has maximum independent neighborhood systems (Γ_{in}-sets) cardinality equal to 2. This contributes to the convex independent neighborhood polynomial of G as $2x^3y + 3x^3y^2$.

For 4-convex subsets of $V(G)$, it can be verified that there are 3 4-convex subset with maximum independent neighborhood systems (Γ_{in})-sets cardinality equal to 1. This contributes to convex independent neighborhood polynomial of G as $3x^4y$.

For 5-convex subsets of $V(G)$, it can be verified that there is only 1 5-convex subset with empty (zero cardinality) maximum independent neighborhood systems (Γ_{in})-set. This contributes to convex independent neighborhood polynomial of G as x^5.

Therefore, by combining all the terms, we have

$$\Gamma_{cin}(G;x,y) = x^5 + 3x^4y + 2x^3y + 3x^3y^2 + 2x^2y + 2x^2y^2 + x^2y^3 + 3xy + 2xy^2.$$

Remark 3.1. *The following properties of the convex independent neighborhood polynomial of a graph G are noted:*

i) The degree of the convex independent neighborhood polynomial of a connected graph is n. Moreover, it is a monic polynomial.

ii) The convex independent neighborhood polynomial of a disconnected graph with isolated n vertices is nx.

4. Paths and Cycles

This section discusses the convex independent neighborhood polynomial of paths (P_n) and cycles (C_n).

Theorem 4.1. *Let P_n be a path of order n. Then, for $n \geq 2$, the convex independent neighborhood polynomial of P_n is given by*

$$\Gamma_{cin}(P_n;x,y) = x^n + \sum_{i=1}^{n-1} 2x^iy + \sum_{i=1}^{n-2}(n-1-i)x^iy^2.$$

Proof. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of P_n. Note that for n-convex subset of $V(P_n)$ there is only one set that is $\{v_1, v_2, \ldots, v_n\}$ with empty (zero cardinality) Γ_{in}-set. This contributes to the term x^n of the polynomial. Next, for i-convex subset of $V(P_n)$, $i = 1, 2, ..., n-1$. We consider two cases in choosing the i-convex subset of $V(P_n)$, the first case is choosing set of vertices which include exactly one of the end vertices and for the second case choosing set of vertices which does not include the end vertices.
For the first case, for 1-convex subset of subset of $V(P_n)$, consider the vertex sets $\{v_1\}$ and $\{v_n\}$, both of these convex vertex sets contains only one Γ_{in}-sets namely $\{v_2\}$ and $\{v_{n-1}\}$, respectively. For 2-convex subset of $V(P_n)$, consider the vertex set $\{v_1, v_2\}$ and $\{v_{n-1}, v_n\}$, both of these convex vertex sets contains only one Γ_{in}-sets namely $\{v_3\}$ and $\{v_{n-2}\}$, respectively. Since $V(P_n)$ is finite, it follows that i-convex subsets of $V(P_n)$ is also finite. This means that we may continue this process up to $(n-1)$-convex subsets of P_n. All of these i-convex subsets has Γ_{in}-sets cardinality equal to one. Thus, by combining all of these i-convex subsets with Γ_{in}-sets cardinality equal to one. We have

$$2xy + 2x^2y + 2x^3y + \cdots + 2x^{n-1}y = \sum_{i=1}^{n-1} 2x^iy.$$

For the second case, choosing i-convex subset of $V(P_n)$ which does not include the end vertices. For 1-convex subsets of $V(P_n)$, consider the vertices $\{v_2\}, \{v_3\}, \ldots, \{v_{n-1}\}$. Then there are $(n-2)$ 1-convex subsets of $V(P_n)$ whose Γ_{in}-sets contains two elements. Next, for 2-convex subsets of $V(P_n)$, consider the sets $\{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}, \ldots, \{v_{n-2}, v_{n-1}\}$. This means that there are $(n-3)$ 2-convex subsets of $V(P_n)$ with Γ_{in}-sets equal to two. Since $V(P_n)$ is finite, similarly we can continue this process till we choose the set $\{v_2, v_3, ..., v_{n-1}\}$. All of these i-convex subsets have Γ_{in}-sets with cardinality equal to two. Thus, by combining all of these i-convex subsets of $V(P_n)$ with Γ_{in}-sets cardinality equal to two, we have

$$(n-2)x^1y^2 + (n-3)x^2y^2 + (n-4)x^3y^2 + \cdots + x^{n-2}y^2 = \sum_{i=1}^{n-2}(n-1-i)x^iy^2.$$

Therefore, by combining all the above scenarios, the convex independent neighborhood polynomial of P_n is

$$\Gamma_{cin}(P_n; x, y) = x^n + \sum_{i=1}^{n-1} 2x^iy + \sum_{i=1}^{n-2}(n-1-i)x^iy^2. \qquad \blacksquare$$

Let $V(P_n) = \{v_1, \ldots, v_n\}$ be vertex set of P_n. Note that when $i = n$, there is only 1 n-convex subset of $V(P_n)$ with empty (zero cardinality) maximum independent neighborhood system (Γ_{in}). Now, for $i = 1, 2, ..., (n-1)$, there are $(n-1)$ i-convex subsets of $V(P_n)$ with maximum independent neighborhood system (Γ_{in})-set cardinality equal to 1. Moreover, there are $(n-2)$ i-convex subsets of $V(P_n)$ with maximum independent neighborhood system (Γ_{in})-sets. Thus, by combining all the vertices, we have

$$1 + (n-1) + (n-2) = 2n - 2$$

number of terms of the convex independent neighborhood polynomial of P_n. Thus, we have this corollary

Corollary 4.2. *For $n \geq 2$, the number of terms for the convex independent neighborhood polynomial of P_n is given by $2n-2$.*

Illustration 4.3. Consider P_5 be a path of order 5.

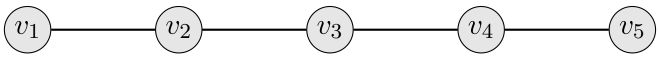

Figure 2: A path P_5 of order 5

Then, by using Theorem 4.1,

$$\begin{aligned}\Gamma_{cin}(P_5; x, y) &= x^5 + \sum_{i=1}^{5-1} 2x^iy + \sum_{i=1}^{5-2}(5-1-i)x^iy^2 \\ &= x^5 + \sum_{i=1}^{4} 2x^iy + \sum_{i=1}^{3}(4-i)x^iy^2\end{aligned}$$

$$= x^5 + (2xy + 2x^2y + 2x^3y + 2x^4y) + (3xy^2 + 2x^2y^2 + x^3y^2)$$
$$= x^5 + 2x^4y + 2x^3y + x^3y^2 + 2x^2y + 2x^2y^2 + 2xy + 3xy^2.$$

And by Corollary 4.2, there are $[2(5) - 2] = 8$ terms in convex independent neighborhood polynomial of P_5.

To see the convex subsets of $V(P_5)$ and its corresponding Γ_{in}-sets of P_5, refer to the Tables 4, 5, and 6:

1-convex	Γ_{in}-sets
$\{v_1\}$	$\{v_2\}$
$\{v_2\}$	$\{v_1, v_3\}$
$\{v_3\}$	$\{v_2, v_4\}$
$\{v_4\}$	$\{v_3, v_5\}$
$\{v_5\}$	$\{v_4\}$

Table 4: 1-convex subsets of $V(P_5)$ and its corresponding Γ_{in}-sets.

Table 4 shows that there are 2 1-convex subsets of $V(P_5)$ with Γ_{in}-sets cardinality equal to 1, and 3 1-convex subsets of $V(P_5)$ with Γ_{in}-sets cardinality equal to 2. This contributes to the convex independent neighborhood polynomial of P_5 as $2xy + 3xy^2$.

2-convex	Γ_{in}-sets
$\{v_1, v_2\}$	$\{v_3\}$
$\{v_2, v_3\}$	$\{v_1, v_4\}$
$\{v_3, v_4\}$	$\{v_2, v_5\}$
$\{v_4, v_5\}$	$\{v_3\}$

Table 5: 2-convex subsets of $V(P_5)$ and its corresponding Γ_{in}-sets.

Table 5 shows that there are 2 2-convex subsets of $V(P_5)$ with Γ_{in}-sets cardinality equal to 1, and 2 2-convex subsets of $V(P_5)$ with Γ_{in}-sets cardinality equal to 2. This contributes to the convex independent neighborhood polynomial of P_5 as $2x^2y + 2x^2y^2$.

3-convex	Γ_{in}-sets
$\{v_1, v_2, v_3\}$	$\{v_4\}$
$\{v_2, v_3, v_4\}$	$\{v_1, v_5\}$
$\{v_3, v_4, v_5\}$	$\{v_2\}$

Table 6: 3-convex subsets of $V(P_5)$ and its corresponding Γ_{in}-sets.

Table 6 shows that there are 2 3-convex subsets of $V(P_5)$ with Γ_{in}-sets cardinality equal to 1, and 1 3-convex subsets of $V(P_5)$ with Γ_{in}-sets cardinality equal to 2. This contributes to the convex independent neighborhood polynomial of P_5 as $2x^3y + x^3y^2$.

For 4-convex subsets of $V(P_5)$, it can be verified that there are 2 4-convex subset with Γ_{in}-sets cardinality equal to one. This contributes to the convex independent neighborhood polynomial of P_5 as $2x^4y$.

For 5-convex subsets of $V(P_5)$, it can be verified that there is only 1 5-convex subset with empty (zero cardinality) Γ_{in}-set. This contributes to the convex independent neighborhood polynomial of P_5 as x^5.

Theorem 4.4. *Let C_n be a cycle of order n. Then, for $n \geq 6$, the convex independent neighborhood polynomial of C_n is*

$$\Gamma_{cin}(C_n; x, y) = \begin{cases} x^n + n\sum_{i=1}^{\frac{n+1}{2}} x^i y^2, & \text{if } n \text{ is odd} \\ x^n + n\sum_{i=1}^{\frac{n}{2}} x^i y^2, & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of C_n. We consider the following cases:

Case 1:
Let n be odd. First, there is only one n-convex subset of $V(C_n)$ with empty (zero cardinality) Γ_{in}-sets and this contributes to the term x^n of the polynomial. Next, we consider the i-convex subsets of $V(C_n)$ for $i = 1, 2, ..., \frac{n+1}{2}$. We only consider subsets less than or equal to $\frac{n+1}{2}$ because these are the only convex subsets of $V(C_n)$. Subsets more than $\frac{n+1}{2}$ are no longer convex subsets. Now, for i-convex subsets of C_n such that $i = 1, 2, ..., \frac{n+1}{2}$, all of these contains Γ_{in}-sets equal to two and each of these i-convex subsets has n choices. Thus, we have the following polynomial

$$nxy^2 + nx^2y^2 + \cdots + nx^{\frac{n+1}{2}}y^2 = n\sum_{i=1}^{\frac{n+1}{2}} x^i y^2.$$

Hence, the convex independent neighborhood polynomial of C_n if n is odd is given by

$$\Gamma_{cin}(C_n; x, y) = x^n + n\sum_{i=1}^{\frac{n+1}{2}} x^i y^2.$$

Case 2:
Let n be even. By similar argument as case 1 and integer $i = 1, 2, ..., \frac{n}{2}$, we will obtain the convex neighborhood polynomial of C_n, i.e.,

$$\Gamma_{cin}(C_n; x, y) = x^n + n\sum_{i=1}^{\frac{n}{2}} x^i y^2.$$

This complete the proof. ■

Let $V(C_n) = \{v_1, v_2, ..., v_n\}$ be vertex set of C_n. Note that the vertices $\{v_1, v_2, ..., v_n\}$ convex subset of $V(C_n)$ is the only subset that has empty (zero cardinality) Γ_{in}-set which is the leading term of the convex independent neighborhood polynomial of C_n.
Now, consider the following cases, if n is odd, then there are $\frac{n+1}{2}$ terms with Γ_{in}-set cardinality equal to two. This means that, for $n \geq 6$ when n is odd, there are

$$1 + \frac{n+1}{2} = \frac{n+3}{2}$$

terms for the convex independent neighborhood polynomial of C_n. On the other hand, if n is even, then there are $\frac{n}{2}$ terms with Γ_{in}-set cardinality equal to two. This means that, for $n \geq 6$ when n is even, there are

$$1 + \frac{n}{2} = \frac{n+2}{2}$$

terms for the convex independent neighborhood polynomial of C_n. Thus, we have the following corollary

Corollary 4.5. *For $n \geq 6$, the number of terms of the convex independent neighborhood polynomial of C_n in terms of n is given by,*

$$\begin{cases} \frac{n+3}{2}, & \text{if } n \text{ is odd} \\ \frac{n+2}{2}, & \text{if } n \text{ is even.} \end{cases}$$

Illustration 4.6. Consider the cycle C_6.

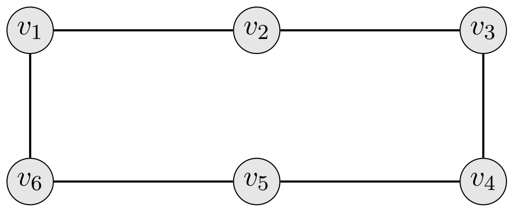

Figure 3: A cycle C_6 of order 6

Then, by using Theorem 4.4 when n is even,

$$\begin{aligned}\Gamma_{cin}(C_6; x, y) &= x^6 + 6\sum_{i=1}^{\frac{6}{2}} x^i y^2 \\ &= x^6 + 6\sum_{i=1}^{3} x^i y^2 \\ &= x^6 + 6(xy^2 + x^2y^2 + x^3y^2)\end{aligned}$$

$$= x^6 + 6x^3y^2 + 6x^2y^2 + 6xy^2.$$

And by Corollary 4.5, there are $\frac{6+2}{2} = 4$ terms in the convex independent neighborhood polynomial of C_6.

To see the convex subsets of $V(C_6)$ and its corresponding Γ_{in}-sets, refer to the Tables 7, 8, and 9:

1-convex	Γ_{in}-sets
$\{v_1\}$	$\{v_6, v_2\}$
$\{v_2\}$	$\{v_1, v_3\}$
$\{v_3\}$	$\{v_2, v_4\}$
$\{v_4\}$	$\{v_3, v_5\}$
$\{v_5\}$	$\{v_4, v_6\}$
$\{v_6\}$	$\{v_5, v_1\}$

Table 7: 1-convex subsets of $V(C_6)$ and its corresponding Γ_{in}-sets.

Table 7 shows that there are 6 1-convex subsets of $V(C_6)$ with Γ_{in}-sets cardinality equal to 2. This contributes to the convex independent neighborhood polynomial of C_6 as $6xy^2$.

2-convex	Γ_{in}-sets
$\{v_1, v_2\}$	$\{v_6, v_3\}$
$\{v_2, v_3\}$	$\{v_1, v_4\}$
$\{v_3, v_4\}$	$\{v_2, v_5\}$
$\{v_4, v_5\}$	$\{v_3, v_6\}$
$\{v_5, v_6\}$	$\{v_4, v_1\}$
$\{v_6, v_1\}$	$\{v_5, v_2\}$

Table 8: 2-convex subsets of $V(C_6)$ and its corresponding Γ_{in}-sets.

Table 8 shows that there are 6 2-convex subsets of $V(C_6)$ with Γ_{in}-sets cardinality equal to 2. This contributes to the convex independent neighborhood polynomial of C_6 as $6x^2y^2$.

3-convex	Γ_{in}-sets
$\{v_1, v_2, v_3\}$	$\{v_6, v_4\}$
$\{v_2, v_3, v_4\}$	$\{v_1, v_5\}$
$\{v_3, v_4, v_5\}$	$\{v_2, v_6\}$
$\{v_4, v_5, v_6\}$	$\{v_3, v_1\}$
$\{v_5, v_6, v_1\}$	$\{v_4, v_2\}$
$\{v_6, v_1, v_2\}$	$\{v_5, v_3\}$

Table 9: 3-convex subsets of $V(C_6)$ and its corresponding Γ_{in}-sets.

Table 9 shows that there are 6 3-convex subsets of $V(C_6)$ with Γ_{in}-sets cardinality equal to 2. This contributes to the convex independent neighborhood polynomial of C_6 as $6x^3y^2$.

For both 4-convex and 5-convex subsets of $V(C_6)$, it can be verified that there are no 4-convex and 5-convex subsets of $V(C_6)$ since any subsets of C_6 that contains four or more elements is no longer convex sets.

For 6-convex subsets of $V(C_6)$, it can be verified that there is only 1 6-convex subset of $V(C_6)$ with empty (zero cardinality) Γ_{in}-set. This contributes to the convex independent neighborhood polynomial of C_6 as x^6.

5. Complete Graph and Star Graph

This section discusses the convex independent neighborhood polynomial of complete graph (K_n) and star graph ($K_{1,n}$).

Theorem 5.1. *Let K_n be a complete graph of order n. Then, the convex independent neighborhood polynomial of K_n is*

$$\Gamma_{cin}(K_n; x, y) = x^n + \sum_{i=1}^{n-1} \binom{n}{i} x^i y$$

where $n \geq 3$, $\binom{n}{i}$ is the number of i-convex subsets of $V(K_n)$ with maximum independent neighborhood system cardinality equal to 1 which is the combination of n vertices taken i at a time.

Proof. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ be vertex set of K_n. First, note that there is only one n-convex subset of $V(K_n)$ which has empty (zero cardinality) Γ_{in}-sets. This contributes to x^n. Now, for integer $i = 1, 2, ..., n-1$, there always exist an i-convex subsets of $V(K_n)$. This means that the Γ_{in}-sets of i-convex subsets of $V(K_n)$ contains exactly one element, for all $i = 1, 2, ..., n-1$ since every vertices are connected to each other. Moreover, there are $\binom{n}{i}$ i-convex subsets of $V(K_n)$ whose Γ_{in}-sets contain exactly one element.

Therefore, the convex independent neighborhood polynomial of K_n is

$$\Gamma_{cin}(K_n; x, y) = x^n + \sum_{i=1}^{n-1} \binom{n}{i} x^i y. \qquad \blacksquare$$

Let $V(K_n) = \{v_1, v_2, ..., v_n\}$ be vertex set of complete graph K_n. Note that the vertices $\{v_1, v_2, ..., v_n\}$ is the n-convex subset of $V(K_n)$ which contributes to the leading term of the convex independent neighborhood polynomial of K_n. Now, for $i = 1, ..., n-1$, the i-convex subsets of $V(K_n)$ has Γ_{in}-sets cardinality equal to one. This means that for $n \geq 3$, there are

$$1 + (n-1) = n$$

terms for the convex independent neighborhood polynomial of K_n. Thus, we have the following corollary

Corollary 5.2. *For $n \geq 3$, the number of terms of the convex independent neighborhood polynomial of K_n is n.*

Illustration 5.3. Consider K_4 be a complete graph of order 4.

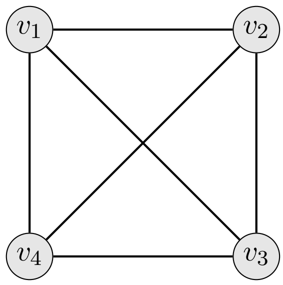

Figure 4: A complete graph K_4 of order 4

Then, by using Theorem 5.1,

$$\begin{aligned}\Gamma_{cin}(K_4; x, y) &= x^4 + \sum_{i=1}^{4-1} \binom{4}{i} x^i y \\ &= x^4 + \sum_{i=1}^{3} \binom{4}{i} x^i y \\ &= x^4 + \left[\binom{4}{1} xy + \binom{4}{2} x^2 y + \binom{4}{3} x^3 y\right] \\ &= x^4 + 4x3y + 6x^2 y + 4x^3 y.\end{aligned}$$

And by Corollary 5.2, there are $n = 4$ terms in the convex independent neighborhood polynomial of K_4.

To see the convex subsets of $V(K_4)$ and its corresponding Γ_{in}-sets, refer to the Tables 10, 11, and 12:

1-convex	Γ_{in}-sets
$\{v_1\}$	$\{v_2\}, \{v_3\}, \{v_4\}$
$\{v_2\}$	$\{v_1\}, \{v_3\}, \{v_4\}$
$\{v_3\}$	$\{v_1\}, \{v_2\}, \{v_4\}$
$\{v_4\}$	$\{v_1\}, \{v_2\}, \{v_3\}$

Table 10: 1-convex subsets of $V(K_4)$ and its corresponding Γ_{in}-sets.

Table 10 shows that there are 4 1-convex subsets of $V(K_4)$ with Γ_{in}-sets cardinality equal to 1. This contributes to the convex independent neighborhood polynomial of K_4 as $4xy$.

2-convex	Γ_{in}-sets
$\{v_1, v_2\}$	$\{v_3\}, \{v_4\}$
$\{v_2, v_3\}$	$\{v_1\}, \{v_4\}$
$\{v_3, v_4\}$	$\{v_1\}, \{v_2\}$
$\{v_4, v_1\}$	$\{v_2\}, \{v_3\}$
$\{v_1, v_3\}$	$\{v_2\}, \{v_4\}$
$\{v_2, v_4\}$	$\{v_1\}, \{v_3\}$

Table 11: 2-convex subsets of $V(K_4)$ and its corresponding Γ_{in}-sets.

Table 11 shows that there are 6 2-convex subsets of $V(K_4)$ with Γ_{in}-sets cardinality equal to 1. This contributes to the convex independent neighborhood polynomial of K_4 as $6x^2y$.

3-convex	Γ_{in}-sets
$\{v_1, v_2, v_3\}$	$\{v_4\}$
$\{v_2, v_3, v_4\}$	$\{v_1\}$
$\{v_1, v_2, v_4\}$	$\{v_3\}$
$\{v_1, v_3, v_4\}$	$\{v_2\}$

Table 12: 3-convex subsets of $V(K_4)$ and its corresponding Γ_{in}-sets.

Table 12 shows that there are 4 3-convex subsets of $V(K_4)$ with Γ_{in}-sets cardinality equal to 1. This contributes to the convex independent neighborhood polynomial of K_4 as $4x^3y$.

For the 4-convex subsets of $V(K_4)$, it can be verified that there is only 1 4-convex subset of $V(K_4)$ with empty (zero cardinality) Γ_{in}-set. This contributes to the convex independent neighborhood polynomial of K_4 as x^4.

Theorem 5.4. *Let $K_{1,n}$ be a star graph of order $n+1$. Then, the convex independent neighborhood polynomial of $K_{1,n}$ is*

$$\Gamma_{cin}(K_{1,n};x,y) = x^{n+1} + \sum_{i=2}^{n} \binom{n}{i-1} x^i y^{[n-(i-1)]} + xy^n + nxy,$$

where $n \geq 3$, $\binom{n}{i-1}$ is the number of i-convex subsets of $V(K_{1,n})$ with maximum independent neighborhood system cardinality equal to $[n-(i-1)]$ which is the combination of n vertices taken $(i-1)$ at a time.

Proof. Let $V(K_{1,n}) = \{u_1, v_1, v_2, ..., v_n\}$ be a vertex set of star graph of order $K_{1,n}$ and let u_1 be the center vertex. First, note that every n vertices are all adjacent only to the center vertex, namely, u_1. This means that there are exactly n 1-convex subset of $V(K_{1,n})$ with Γ_{in}-set contain only one element, namely u_1. This contribute to the convex independent neighborhood polynomial of $K_{1,n}$ as nxy. Also, there is only 1 1-convex subset of $V(K_{1,n})$ with Γ_{in}-set contain n elements, namely u_1, the center vertex. This contribute to the polynomial as xy^n. Now, for $(n+1)$-convex subset of $V(K_{1,n})$, there is exactly one $(n+1)$-convex with empty (zero cardinality) Γ_{in}-set which contributes as x^{n+1} to the polynomial. Now, for i-convex subsets of $V(K_{1,n})$ where $i = 2, ..., n$, there are $\binom{n}{i-1}$ i-convex subsets of $V(K_{1,n})$ with Γ_{in}-sets contain $[n-(i-1)]$ elements. Thus, by combining all i-convex with Γ_{in}-sets cardinality equal to $[n-(i-1)]$ we have

$$\binom{n}{1} x^2 y^{n-1} + \binom{n}{2} x^3 y^{n-2} + \cdots + \binom{n}{n-1} x^n y = \sum_{i=2}^{n} \binom{n}{i-1} x^i y^{n-(i-1)}$$

Therefore, the convex independent neighborhood polynomial of $K_{1,n}$ is

$$\Gamma_{cin}(K_{1,n};x,y) = x^{n+1} + \sum_{i=2}^{n} \binom{n}{i-1} x^i y^{[n-(i-1)]} + xy^n + nxy. \qquad \blacksquare$$

Let $V(K_{1,n}) = \{u_1, v_1, v_2, ..., v_n\}$ be a vertex-set of star graph of order $K_{1,n}$ and let u_1 be the center vertex. Note that every n vertices are all adjacent only to the center vertex, namely, u_1. This means that the n 1-convex subsets of $V(K_{1,n})$ with Γ_{in}-sets cardinality equal to one contribute as the nxy term in the polynomial. Moreover, there is only 1 1-convex subset of $V(K_{1,n})$ with Γ_{in}-sets cardinality equal to n, i.e., u_1 which contribute to the polynomial as xy^n. Now, the vertices $\{u_1, v_1, ..., v_n\}$ is the $(n+1)$-convex subset of $V(K_{1,n})$ which contributes to the leading term of the convex independent neighborhood polynomial of $K_{1,n}$, i.e., x^{n+1}. Now, for $i = 2, ..., n$, the i-convex subsets of $V(K_{1,n})$ contains Γ_{in}-sets $[n-(i-1)]$ elements. This means that for $n \geq 3$, there are

$$1 + 1 + 1 + (n-1) = n + 2$$

terms for the convex independent neighborhood polynomial of $K_{1,n}$. Thus, we have the following corollary

Corollary 5.5. *For $n \geq 3$, the number of terms of the convex independent neighborhood polynomial of $K_{1,n}$ is $(n+2)$.*

Illustration 5.6. Consider $K_{1,4}$ be a star graph of order 5.

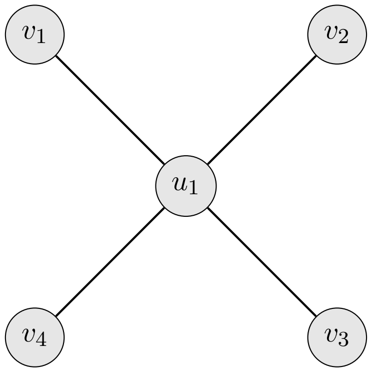

Figure 5: A star graph $K_{1,4}$ of order 5

Then, by using Theorem 5.4,

$$\begin{aligned}
\Gamma_{cin}(K_{1,4};x,y) &= x^{4+1} + \sum_{i=2}^{4} \binom{4}{i-1} x^i y^{[4-(i-1)]} + xy^4 + 4xy \\
&= x^5 + \left[\binom{4}{1} x^2y^3 + \binom{4}{2} x^3y^2 + \binom{4}{3} x^4y \right] + xy^4 + 4xy \\
&= x^5 + 4x^2y^3 + 6x^3y^2 + 4x^4y + xy^4 + 4xy \\
&= x^5 + 4x^4y + 6x^3y^2 + 4x^2y^3 + xy^4 + 4xy.
\end{aligned}$$

And by Corollary 5.5, there are $4+2=6$ terms in the convex independent neighborhood polynomial of $K_{1,4}$.

To see the convex subsets of $V(K_{1,4})$and its corresponding Γ_{in}-sets, refer to the following Tables 13, 14, and 15:

1-convex	Γ_{in}-sets
$\{u_1\}$	$\{v_1, v_2, v_3, v_4\}$
$\{v_1\}$	$\{u_1\}$
$\{v_2\}$	$\{u_1\}$
$\{v_3\}$	$\{u_1\}$
$\{v_4\}$	$\{u_1\}$

Table 13: 1-convex subsets of $V(K_{1,4})$ and its corresponding Γ_{in}-sets.

Table 13 shows that there are 4 1-convex subsets of $V(K_{1,4})$ with Γ_{in}-set cardinality equal to 1 and exactly 1 1-convex subset of $V(K_{1,4})$ with Γ_{in}-set cardinality equal to 4. This contributes to the convex independent neighborhood polynomial of $K_{1,4}$ as $xy^4 + 4xy$.

2-convex	Γ_{in}-sets
$\{v_1, u_1\}$	$\{v_2, v_3, v_4\}$
$\{v_2, u_1\}$	$\{v_1, v_3, v_4\}$
$\{v_3, u_1\}$	$\{v_1, v_2, v_4\}$
$\{v_4, u_1\}$	$\{v_1, v_2, v_3\}$

Table 14: 2-convex subsets of $V(K_{1,4})$ and its corresponding Γ_{in}-sets.

Table 14 shows that there are 4 2-convex subsets of $V(K_{1,4})$ with Γ_{in}-set cardinality equal to 3. This contributes to the convex independent neighborhood polynomial of $K_{1,4}$ as $4x^2y^3$.

3-convex	Γ_{in}-sets
$\{v_1, v_2, u_1\}$	$\{v_3, v_4\}$
$\{v_1, v_3, u_1\}$	$\{v_2, v_4\}$
$\{v_1, v_4, u_1\}$	$\{v_2, v_3\}$
$\{v_2, v_3, u_1\}$	$\{v_1, v_4\}$
$\{v_2, v_4, u_1\}$	$\{v_1, v_3\}$
$\{v_3, v_4, u_1\}$	$\{v_1, v_2\}$

Table 15: 3-convex subsets of $V(K_{1,4})$ and its corresponding Γ_{in}-sets.

Table 15 shows that there are 6 3-convex subsets of $V(K_{1,4})$ with Γ_{in}-set cardinality equal to 2. This contributes to the convex independent neighborhood polynomial of $K_{1,4}$ as $6x^3y^2$.

For 4-convex subsets of $V(K_{1,4})$, it can be verified that there are 4 4-convex subsets of $V(K_{1,4})$ with Γ_{in}-set cardinality equal to 1. This contributes to the convex independent neighborhood polynomial of $K_{1,4}$ as $4x^4y$.

For 5-convex subset of $V(K_{1,4})$, there is only 1 5-convex subset of $V(K_{1,4})$ with empty (zero cardinality) Γ_{in}-set. This contributes to the convex independent neighborhood polynomial of $K_{1,4}$ as x^5.

6. Conclusion

The study of graph polynomials and convexity in graphs continues to be a significant area of research in graph theory, offering valuable insights and applications across various scientific fields. Researchers have investigated new approaches to count and describe substructures based on their neighborhood system by combining the ideas of convexity with graph polynomials. In this paper, we have extended this line of study by considering independent neighborhood systems of convex subgraphs, which leads to the introduction of the convex independent neighborhood polynomial for paths, cycles, complete and star graph. This new polynomial provides a new way to look at how convexity and neighborhood structures work together, creating a foundation for future graph theory researches.

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