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A New Method for Generating Continuous Distributions with Applications

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Abstract. In this paper, a new modifying method has been introduced by adding an extra parameter to generate a new family of distributions that has more flexibility and better model fitting. A special case has been considered; the exponential distribution. All the main properties of the new modified exponential distribution are derived, including the CDF, PDF, and quantile function. The maximum likelihood estimation method is used to estimate unknown parameters. The modified exponential distribution has been applied to two-lifetime data sets to illustrate its efficiency.

2020 Mathematics Subject Classifications: 62E10, 62N02, 62N05
Key Words and Phrases: Reliability function, Maximum likelihood estimation, Exponential distribution

1. Introduction

The distribution of random variables is one of the key concepts used in statistics to model and fit data sets. There are several well-known distributions used in the literature. These distributions show strong effectiveness in many situations but fail in other cases. Because of this, more general and flexible distributions have appeared recently by many authors by adding a new parameter to a family of distribution functions to give better fitting for the empirical data. Recently, Ahmad and Asha[1] introduced the exponential reliability method which is a new method for generating a new family of distribution functions. A well-known approach for this purpose is the TX family of distributions, which was suggested by Aljarrah et al. [2]. This work is similar to the beta-normal distribution proposed by Eugene et al. [3]. Other researchers made improvements for classical distributions like the Weibull distribution[4], such as the transmuted modified Weibull distribution [5] and the transmuted inverse Weibull distribution [6]. More details on this context can be found in [7] and [8].

This paper aims to insert a new parameter into a family of distribution functions to produce a new more powerful family that can adjust reliability and has better fit and

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more flexibility as well as simplicity. This new method is called the alpha-R ($\alpha - R$) method. The $\alpha - R$ method can be used easily and effectively for data analysis. First, we present the modified family and discuss its main properties. Then, the technique is applied to the exponential distribution. Two data sets are fitted using the classical distributions and the modified exponential distribution using the alpha-R method, we compare them using Akaike's information criterion [9], the Bayesian information criterion and Kolmogorov-smirnov test. The maximum likelihood estimation method is used to estimate the parameters. All calculations are obtained using R software.

2. The $\alpha - R$ Family

The cumulative distribution function (CDF) of the $\alpha - R$ family is defined as follows

$$G(y) = 1 - R(y)\alpha^{F(y)}, \quad y \in \mathbb{R}, \quad 0 < \alpha \leq e \tag{1}$$

where F(y) and R(y) are the CDF and reliability function of a baseline distribution, respectively. A new parameter $\alpha(0 < \alpha \leq e)$ is utilized to extend the use of a random variable Y. It is clear that if $\alpha = 1$, then $G(y) = F(y)$. The probability density function (PDF) corresponding to (1) is given by:

$$g(y) = f(y)\alpha^{F(y)}[1 - (\log\alpha)R(y)], \quad y \in \mathbb{R}, \quad 0 < \alpha \leq e, \tag{2}$$

where f(y) represents the PDF of a baseline distribution. Further, it can be shown that the hazard rate function of the $\alpha - R$ family is given as

$$z(y) = h(y)[1 - (\log\alpha)R(y)], \quad y \in \mathbb{R}, \quad 0 < \alpha \leq e, \tag{3}$$

where h(y) is the hazard rate function of the original distribution.

3. $\alpha - R$ Exponential Distribution

In this section, we present a special case of the $\alpha - R$ family. This special distribution is considered as a generalization of the exponential distribution.

Definition 1. *A random variable Y is said to have $\alpha - R$ exponential ($\alpha - RE$) distribution if its CDF is given by:*

$$G(y, \alpha, \theta) = 1 - e^{-\theta y} \times \alpha^{1-e^{-\theta y}}, \quad y > 0, \quad \theta > 0, \quad 0 < \alpha \leq e \tag{4}$$

The Corresponding PDF of the $\alpha - RE$ distribution is defined as:

$$g(y, \alpha, \theta) = \theta\alpha^{1-e^{-\theta y}} \left(e^{-\theta y} - \log\alpha \times e^{-2\theta y}\right), \quad y > 0, \quad \theta > 0, \quad 0 < \alpha \leq e \tag{5}$$

Plots for the CDF of the $\alpha - RE$ distribution are provided in Figures 1 and 2. In addition, Figures 3 and 4 display the plots of the corresponding PDFs.

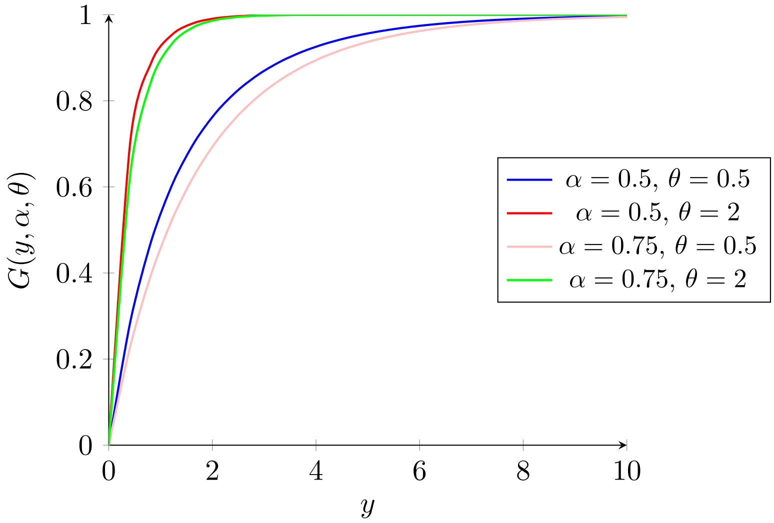

Figure 1: The CDF curves of the $\alpha - R\,E$ distribution for the parameter values $\theta = 0.5$ and $\theta = 2$, and $\alpha = 0.5$ and $\alpha = 0.75$

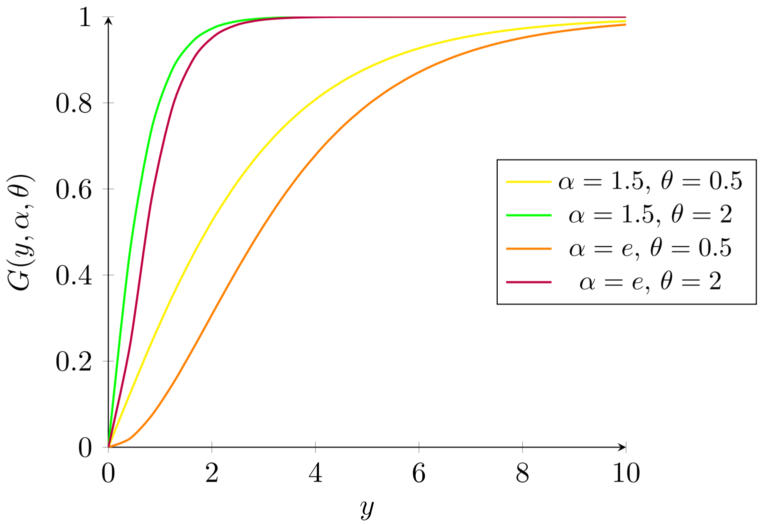

Figure 2: The CDF curves of the $\alpha - R\,E$ distribution for the parameter values $\theta = 0.5$ and $\theta = 2$, and $\alpha = 1.5$ and $\alpha = e$

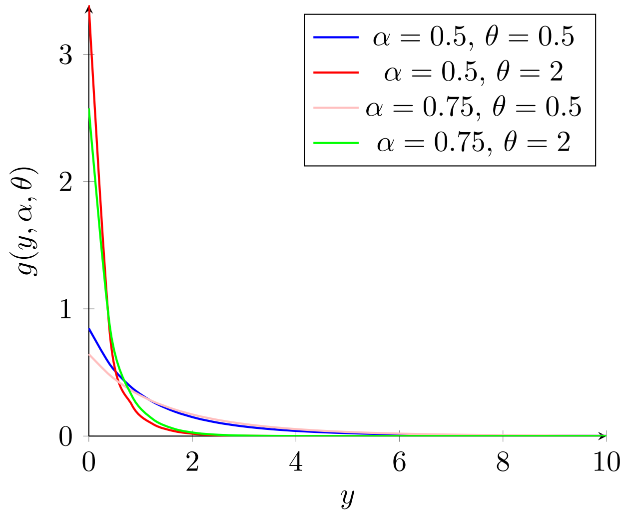

Figure 3: The PDF curves of the $\alpha - RE$ distribution for the parameter values $\theta = 0.5$ and $\theta = 2$, and $\alpha = 0.5$ and $\alpha = 0.75$

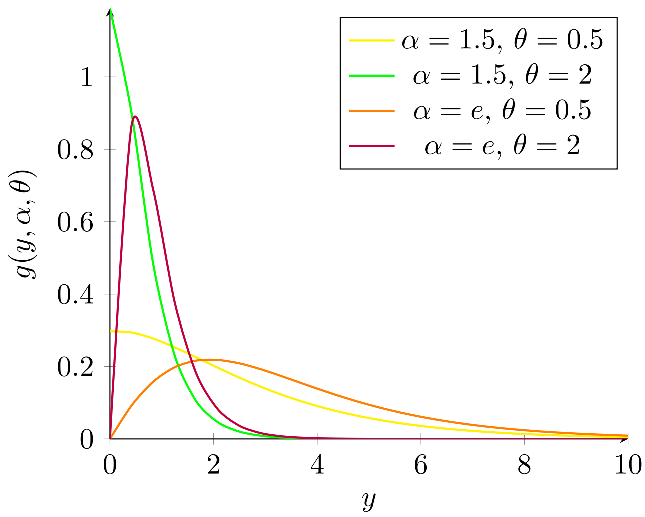

Figure 4: The PDF curves of the $\alpha - RE$ distribution for the parameter values $\theta = 0.5$ and $\theta = 2$, and $\alpha = 1.5$ and $\alpha = e$

4. Statistical Properties of the $\alpha - RE$ distribution

In this section, some statistical properties of the $\alpha - RE$ distribution are discussed, including the moments, the quantile function, and the maximum likelihood estimation.

4.1. Quantile Function

If the CDF of a distribution Y is $H(y)$, then $q = H^{-1}(p),\ 0 < p < 1$ is the corresponding quantile function. Thus, the quantile function of the $\alpha - RE$ distribution can be obtained by solving the equation:

$$-\theta q - \left(1 - e^{-\theta q}\right)\log\alpha = \log\left(1-p\right), \quad 0 < p < 1, \tag{6}$$

Or equivalently:

$$\log\alpha e^{-u} + u - \log\left[\alpha\left(1-p\right)\right] = 0, \tag{7}$$

where $u = -\theta q$. Eq.(7) has an analytical solution in terms of the Lambert W function as follows:

$$q(p) = \frac{1}{\theta}\left[W\left(-\frac{(1-p)}{\alpha}\log\alpha\right) + \log\left[\frac{(1-p)}{\alpha}\right]\right], \quad 0 < p < 1 \tag{8}$$

where $W(.)$ is the Lambert W function. Here, data from the $\alpha - RE$ distribution can be obtained by applying its quantile function to a sample from a uniform distribution.

The median of the $\alpha - R\ E$ distribution is given by:

$$\text{median} = \frac{1}{\theta}\left[W\left(\frac{1}{2\alpha}\log\left(\frac{1}{\alpha}\right) + \log\left(\frac{1}{4\alpha}\right)\right)\right].$$

4.2. Moments

This subsection discusses the computation of the K^{th} moment of the $\alpha - R\ E$ distribution. The K^{th} moment about the origin of the proposed model is derived as follows:

$$\begin{aligned} M_k &= \theta\int_0^\infty y^k\alpha^{1-e^{-\theta y}}\left(e^{-\theta y} - \log\alpha\times e^{-2\theta y}\right)\,dy, \\ &= \theta\left[\int_0^\infty \alpha^{1-e^{-\theta y}}e^{-\theta y}y^k\,dy - \log\alpha\int_0^\infty \alpha^{1-e^{-\theta y}}e^{-2\theta y}y^k\,dy\right]. \end{aligned} \tag{9}$$

It is known that the lower incomplete gamma function is given as:

$$\gamma(x,a) = \int_0^x t^{a-1}e^{-t}dt,$$

Consequently, it can be shown that:

$$\int_0^1 (\log t)^k t^{a-1}e^{-t}\,dt = \frac{\partial^k}{\partial a^k}\,\gamma(1,a). \tag{10}$$

Thus, the first integral (I_1) in Eq.(9) can be handled by letting $t = e^{-\theta y}$, so we get:

$$I_1 = \frac{(-1)^k\alpha}{(\theta)^{k+1}(\log\alpha)^{k+1}}\int_0^1 \left(\log t\right)^k e^{-t}\,dt = \frac{(-1)^k\alpha}{(\theta\log\alpha)^{k+1}}\frac{\partial^k}{\partial a^k}\,\gamma(1,a)\Big|_{a=1} \tag{11}$$

Similarly, the second integral (I_2) in Eq.(9) can be expressed as:

$$\begin{aligned} I_2 &= \frac{(-1)^k \alpha}{(\theta \log \alpha)^{k+1}} \int_0^1 t(\log t)^k e^{-t}\, dt \\ I_2 &= \frac{(-1)^k \alpha}{(\theta \log \alpha)^{k+1}} \frac{\partial^k}{\partial a^k}\, \gamma(1, a)\Big|_{a=2} \end{aligned} \tag{12}$$

Hence, using Eq.s (11) and (12) the k^{th} moment of the $\alpha - RE$ distribution is given by:

$$M_k = \frac{(-1)^k \alpha}{\theta^k (\log \alpha)^{k+1}} \left[\frac{\partial^k}{\partial a^k}\, \gamma(1, a)\Big|_{a=1} - (\log \alpha) \frac{\partial^k}{\partial a^k}\, \gamma(1, a)\Big|_{a=2} \right] \tag{13}$$

Therefore, the mean of the $\alpha - RE$ distribution is given by:

$$E(Y) = \frac{(-1)^k \alpha}{\theta} \left[\frac{1}{(\log \alpha)^2} \frac{\partial}{\partial a}\, \gamma(1, a)\Big|_{a=1} - \frac{1}{\log \alpha} \frac{\partial}{\partial a}\, \gamma(1, a)\Big|_{a=2} \right] \tag{14}$$

4.3. Moment Generating Function

The moment generating function (MGF) of $Y \sim \alpha - RE(\alpha, \theta)$ is obtained as:

$$M(t) = E(e^{tY}) = \theta \int_0^\infty e^{ty} \alpha^{1-e^{-\theta y}} \left(e^{-\theta y} - \log \alpha \times e^{-2\theta y} \right) dy$$

By letting $u = e^{-\theta y}$, we have:

$$M(t) = \alpha \left[\underbrace{\int_0^1 u^{-\frac{t}{\theta}} \alpha^{-u} du}_{I_1} - \log \alpha \underbrace{\int_0^1 u^{1-\frac{t}{\theta}} \alpha^{-u} du}_{I_2} \right]$$

$$I_1 = \frac{1}{(\log \alpha)^{1-\frac{t}{\theta}}}\, \gamma(\log \alpha, 1 - \frac{t}{\theta})$$

$$\begin{aligned} I_2 &= \frac{1}{(\log \alpha)^{1-\frac{t}{\theta}}} \int_0^{\log \alpha} u^{1-\frac{t}{\theta}} e^{-u} du \\ &= \frac{1}{(\log \alpha)^{1-\frac{t}{\theta}}}\, \gamma(\log \alpha, 2 - \frac{t}{\theta}) \end{aligned}$$

Accordingly, the MGF of the $\alpha - R\ E$ distribution is given by:

$$M(t) = \frac{\alpha}{(\log\alpha)^{1-\frac{t}{\theta}}}\ \gamma(\log\alpha, 1-\frac{t}{\theta}) - (\log\alpha)^{\frac{t}{\theta}}\ \gamma(\log\alpha, 2-\frac{t}{\theta}) \tag{15}$$

4.4. Maximum Likelihood Estimation

The foremost method and extensively utilized technique for the purpose of estimating parameters is the maximum likelihood method. In this subsection, we utilize the employment of the maximum likelihood method to estimate the parameters of the $\alpha - RE$ distribution.

Assume that $Y_1, Y_2, ..., Y_n$ is a random sample from $\alpha - RE$ distribution, the corresponding likelihood function is:

$$L(\alpha, \theta, y_i) = \theta^n\ \alpha^{n-\sum_{i=1}^{n} e^{-\theta y_i}}\ \cdot \prod_{i=1}^{n}\left(e^{-\theta y_i} - \log\alpha \times e^{-2\theta y_i}\right) \tag{16}$$

The log-likelihood function is given by:

$$l(\alpha, \theta, y_i) = n\log\theta\ + \left(n - \sum_{i=1}^{n} e^{-\theta y_i}\right)\log\alpha + \sum_{i=1}^{n}\log\left(e^{-\theta y_i} - \log\alpha \times e^{-2\theta y_i}\right) \tag{17}$$

In order to obtain the maximum likelihood estimators (MLEs) of α and θ, the partial derivatives of l in Eq.(17) are provided:

$$\frac{\partial l}{\partial\alpha} = \frac{\left(n - \sum_{i=1}^{n} e^{-\theta y_i}\right)}{\alpha} + \sum_{i=1}^{n}\frac{e^{-2\theta y_i}}{\alpha\left(e^{-\theta y_i} - \log\alpha \times e^{-2\theta y_i}\right)} \tag{18}$$

$$\frac{\partial l}{\partial\theta} = \frac{n}{\theta} + \log\alpha\sum_{i=1}^{n} y_i e^{-\theta y_i} + \sum_{i=1}^{n}\frac{-y_i e^{-\theta y_i} + 2\log\alpha\ y_i e^{-2\theta y_i}}{e^{-\theta y_i} - \log\alpha \times e^{-2\theta y_i}} \tag{19}$$

By equating the non-linear Eq.s (18) and (19) to zero and solving subsequently, we obtain the MLEs of α and θ. In fact, numerical solutions are required to find out such MLEs. Newton-Raphson method via R software can be applied in this context.

5. Simulation Experiment

In this section, a simulation experiment is performed to assess the efficiency and precision of the MLEs of the parameters θ and α of the $\alpha - RE$ distribution. First, we generate a random sample $Y_1, Y_2, \ldots, Y_n$ from $\alpha - RE$ distribution. Using Eq.(18) and Eq.(19) we compute the estimates. The average bias (AB), and the mean squared error (MSE), of the suggested estimators are used to compare the performance of such estimators. The $AB(\hat{\beta})$ of an estimator $\hat{\beta}$ of any parameter β indicates the typical disparity between an outcome and its observed counterpart. It quantifies the precision of a model's predictions, with diminished mean bias implying greater proximity between obtained and actual values.

$$AB(\hat{\beta}) = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\beta}_i - \beta\right),$$

where $\hat{\beta}_i$ is the estimate achieved in iteration $i, i = 1, \ldots, N$. The $MSE(\hat{\beta})$ reflects the average squared difference between an estimated value and the actual value of β. Precisely, it measures how accurate a model's efficiency is. The lower the MSE, the closer the estimated values are to the actual values, and the better the model's performance, formally

$$MSE(\hat{\beta}) = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\beta}_i - \beta\right)^2.$$

A simulation experiment is conducted based on different sample sizes and parameter values. For this, we generate random samples of $\alpha - RE$ distribution by considering the true values of θ and α to be 1.5 and 2, respectively. Samples from $\alpha - RE$ distribution are randomly generated under this setup with 1000 replications of the simulation process. Using these random samples, ABs and $MSEs$ of the estimators are calculated. We propose several sample sizes $n = 50, 100, 200, 500, 1000$. We display the results in Table 1.

Table 1: ABs and $MSEs$ of the parameter estimates when $\theta = 1.5, \alpha = 2$

n	**AB**$(\hat{\theta})$	**MSE**$(\hat{\theta})$	**AB**$(\hat{\alpha})$	**MSE**$(\hat{\alpha})$
50	0.0242	0.0630	0.0417	0.2188
100	0.0191	0.02823	0.0374	0.0904
200	0.0144	0.0133	0.0129	0.0427
500	0.0027	0.0060	0.0053	0.01750
1000	0.0019	0.00297	0.0017	0.0093

It can be observed that the $AB(\hat{\beta})$ and $MSE(\hat{\beta})$, of the estimators decrease as the sample size increases, which means all estimators tend to the true value of the parameter.

6. Real Data Applications

In this section, we examine the adaptability of the new distribution using real-life applications and compare the goodness of fit of the $\alpha - RE$ distribution to other existing distributions.

The goodness of fit of the suggested distribution is compared with the following distributions: Two-parameter exponential distribution (Johnson et al.)[10], with PDF:

$$f_{Exp}(x) = \theta e^{-\theta(x-b)},\ x > b,\ \theta > 0.$$

Gamma distribution (Johnson et al.)[10], with PDF:

$$f_{Gam}(x) = \frac{\beta^{\alpha} x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)},\ x > 0,\ \alpha > 0,\ \beta > 0.$$

Weibull distribution (Johnson et al.)[10], with PDF:

$$f_{Wei}(x) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^{\alpha}},\ x > 0,\ \alpha > 0,\ \beta > 0.$$

The performance of each model is obtained according to many criteria, including $-2\ln L$, Kolmogorov-Smirnov statistic (KSS), and its corresponding p-value, (Chakravarti et al.)[11], Akaike information criterion (AIC), see (Akaike)[12], and Bayesian information criterion (BIC), (Konishi et al.)[13]. We choose the best model so that it has lower values of $-2\ln L,\ KSS,\ AIC,\ and\ BIC$ and higher p-value. Here, L represents the likelihood function. The values of KSS, AIC, BIC are given respectively:

$$KSS = sup\ x|F_n(x) - F_0(x)|$$

$$AIC = -2\ln L + 2m$$

$$BIC = -2\ln L + m\ln(n)$$

where n is the sample size, m represents the number of parameters, and $F_n(x)$ stands for the empirical distribution function. We consider two data sets that are widely used in the literature for the purpose of fitting new distributions.

6.1. Example 1

We have a dataset that contains information about the lifespan of Kevlar 373/epoxy samples exposed to constant pressure at 90% of their stress capacity until each sample reaches the point of failure. This dataset has been utilized by Abdul-Moniem and Seham [14]. The data is recorded as follows:

Example 1

0.0251	0.0886	0.0891	0.2501	0.3113	0.3451	0.4763	0.5650	0.5671
0.6566	0.6748	0.6751	0.6753	0.7696	0.8375	0.8391	0.8425	0.8645
0.8851	0.9113	0.9120	0.9836	1.0483	1.0596	1.0773	1.1733	1.2570
1.2766	1.2985	1.3211	1.3503	1.3551	1.4595	1.4880	1.5728	1.5733
1.7083	1.7263	1.7460	1.7630	1.7746	1.8275	1.8375	1.8503	1.8808
1.8878	1.8881	1.9316	1.9558	2.0048	2.0408	2.0903	2.1093	2.1330
2.2100	2.2460	2.2878	2.3203	2.3470	2.3513	2.4951	2.5260	2.9911
3.0256	3.2678	3.4045	3.4846	3.7433	3.7455	3.9143	4.8073	5.4005
5.4435	5.5295	6.5541	9.0960					

The results of $-2\ln L,\ KSS,\ p-value,\ AIC,\ and\ BIC$ for $\alpha-RE$, gamma, weibull, and exponential distributions are given in Table 2.

Table 2: $-2\ln L,\ KSS,\ p-value,\ AIC,\ and\ BIC$ statistic and the of the fitted distributions

Distribution	$-2\ln L$	KSS	p-value	AIC	BIC
$\alpha-RE$	242.5566	0.08946	0.5472	246.5567	251.2182
Gamma	244.499	0.09614	0.4554	248.4987	253.1602
Weibull	245.049	0.1101	0.2937	249.0494	253.7109
Exponential	185.3168	0.1663	0.0299	256.2289	258.5594

It can be observed that the $\alpha-RE$ distribution has minimum values of: $-2\ln L,\ KSS,\ p-value,\ AIC,\ and\ BIC$ and the largest $p-value$ among all other distribution. Consequently, the $\alpha-RE$ distribution is more adequate to fit this real data.

Table 3: The Parameter Estimates and Standard Errors of the distributions considered.

Distribution	Estimates of the parameters	Std Error
$\alpha-RE$	$\theta=0.805$	0.0846
	$\alpha=2.250$	0.2797
Gamma	$\alpha=1.6406$	0.2439
	$\beta=1.1941$	0.2072
Weibull	$\alpha=1.3258$	0.1138
	$\beta=2.1336$	0.1946
Exponential	$\theta=0.5104$	0.0586

In Table 3, we present the $MLEs$ of the parameters of the $\alpha-RE$ and the other existing distributions with their corresponding standard error.

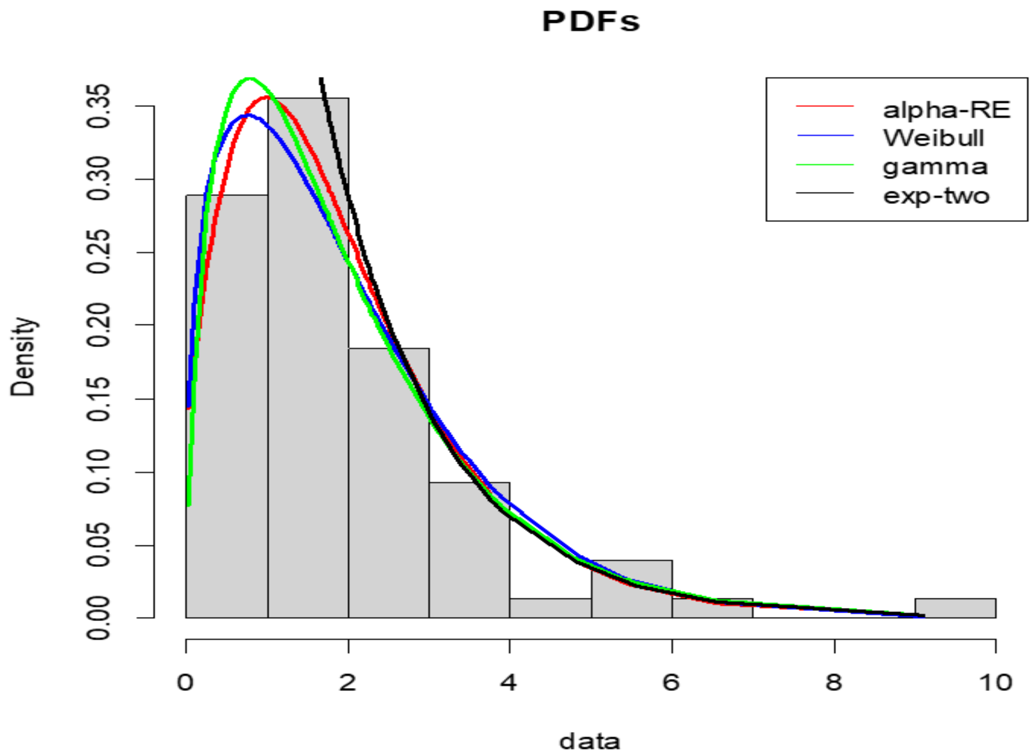

Figure 5: presents plots of the estimated PDFs with the data set.

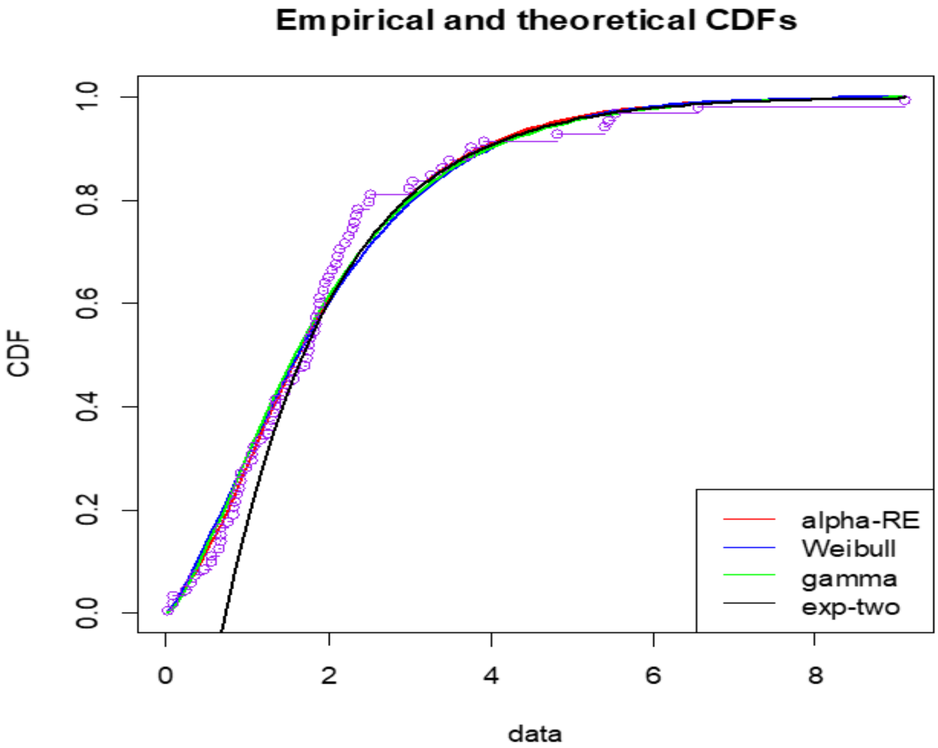

Figure 6: shows the CDFs for the four distribution functions.

Figures 5 and 6 display the PDF and the CDF of the $\alpha - RE$ distribution alongside alternative distributions in the first data set, respectively. Based on the plotted data, it is evident that the $\alpha - RE$ distribution provides a superior match to the data compared to other rival distributions.

6.2. Example 2

The second data set represents the survival times of 121 patients suffering from breast cancer collected from a large hospital in a period from 1929-1938 [15]. The times are reported as follows:

Example 2

0.3	0.3	4.0	5.0	5.6	6.2	6.3	6.6	6.8
7.4	7.5	8.4	8.4	10.3	11.0	11.8	12.2	12.3
13.5	14.4	14.4	14.8	15.5	15.7	16.2	16.3	16.5
16.8	17.2	17.3	17.5	17.9	19.8	20.4	20.9	21.0
21.0	21.1	23.0	23.4	23.6	24.0	24.0	27.9	28.2
29.1	30.0	31.0	31.0	32.0	35.0	35.0	37.0	37.0
37.0	38.0	38.0	38.0	39.0	39.0	40.0	40.0	40.0
41.0	41.0	41.0	42.0	43.0	43.0	43.0	44.0	45.0
45.0	46.0	46.0	47.0	48.0	49.0	51.0	51.0	51.0
52.0	54.0	55.0	56.0	57.0	58.0	59.0	60.0	60.0
60.0	61.0	62.0	65.0	65.0	67.0	67.0	68.0	69.0
78.0	80.0	83.0	88.0	89.0	90.0	93.0	96.0	103.0
105.0	109.0	109.0	111.0	115.0	117.0	125.0	126.0	127.0
129.0	129.0	139.0						

The results of $-2\ln L$, KSS, $p-value$, AIC, $and\ BIC$ for $\alpha-RE$, gamma, weibull, and exponential distributions are given in Table 4.

Table 4: $-2lnL, KSS, p-values, AIC,$ and BIC statistic and the of the fitted distributions

Distribution	$-\mathbf{2}\ln\mathbf{L}$	KSS	$\mathbf{p-value}$	AIC	BIC
$\alpha-RE$	1142.6	0.0577	0.8188	1146.6	1152.2
Gamma	1144.5	0.0805	0.4174	1148.5	1154.1
Weibull	1142.7	0.06762	0.6427	1146.7	1152.3
Exponential	987.6	0.3781	0	982.56	988.15

It can be observed that, the $\alpha-RE$ distribution has minimum values of:$-2lnL$, KSS, AIC, and BIC and the largest $p-value$ among all other distribution. Consequently, the $\alpha-RE$ distribution is more adequate to fit this real data.

Table 5: The Parameter Estimates and Standard Errors of the distributions considered.

Distribution	Estimates of the parameters	Std Error
$\alpha - RE$	$\theta = 0.0329$	0.0031
	$\alpha = 2.134$	0.265
Gamma	$\alpha = 1.515$	0.177
	$\beta = 29.95$	4.15
Weibull	$\alpha = 1.317$	0.0951
	$\beta = 48.77$	3.537
Exponential	$b = 0.3$	0.0586
	$\theta = 22.1$	

In Table 5, we present the MLEs of the parameters of the $\alpha - RE$ and the other existing distributions with their corresponding standard error.

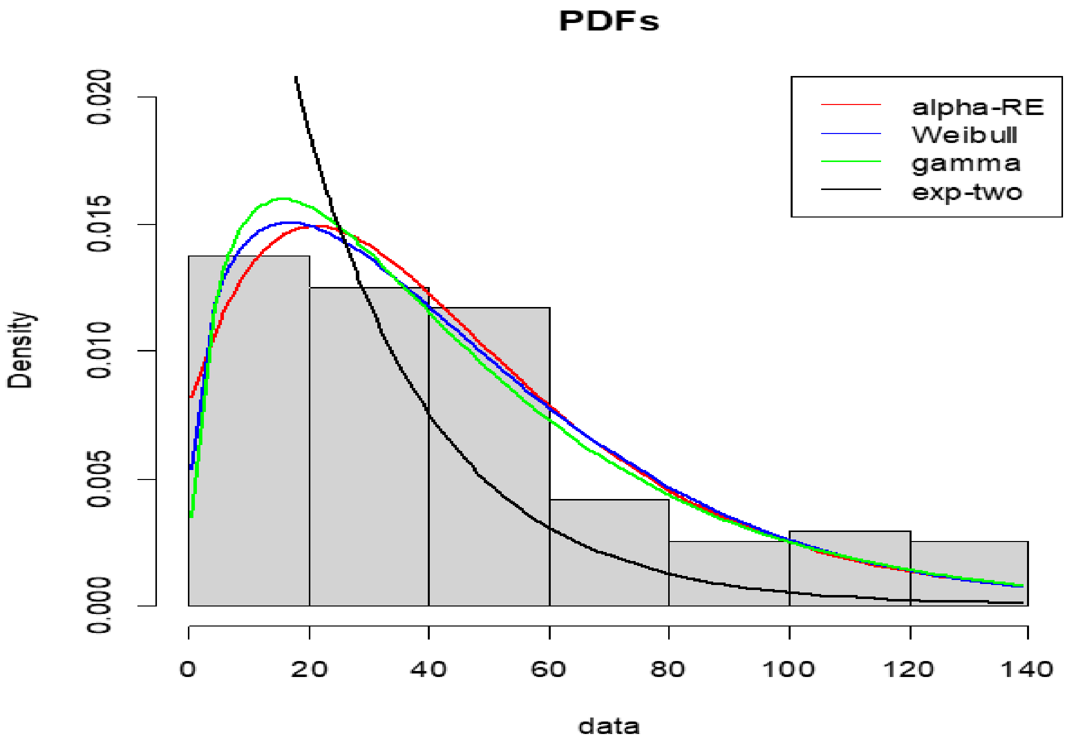

Figure 7: presents plots of the estimated PDFs with the data set.

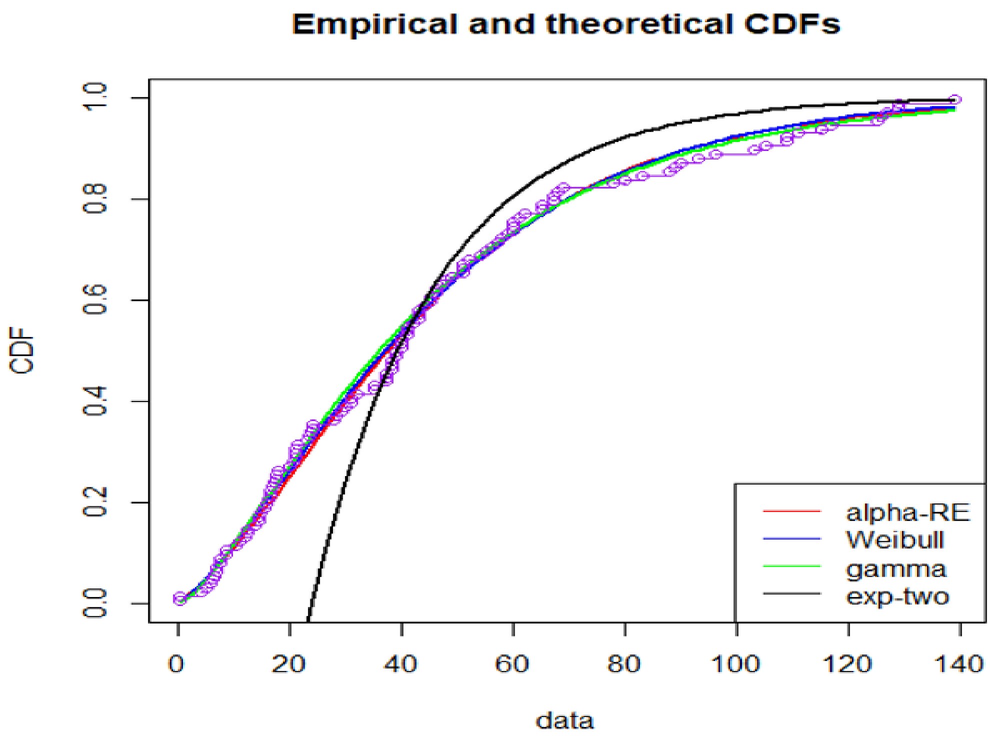

Figure 8: shows the CDFs for the four distribution functions.

Figures 7 and 8 display the PDF and the CDF of the $\alpha - RE$ distribution alongside alternative distributions in the first data set, respectively. Based on the plotted data, it is evident that the $\alpha - RE$ distribution provides a superior match to the data compared to other rival distributions.

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