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Common Fixed Point Theorems for Mappings Satisfying implicit Relation in Bipolar metric Space

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Abstract. In this article, we introduce the concept of compatible mappings of type (A) and weaken the same in the setting of Bipolar metric spaces and established fixed point results in the setting of Bipolar metric spaces, using implicit relation function. The derived results have been supplemented with suitable non-trivial examples. Our results have extended and generalized some results proven in the past and some open problems for future research has been given.

2020 Mathematics Subject Classifications: 47H10, 54H25

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1. Introduction

It would be fair to say that the concept of metric fixed point theory started with the famous contraction mapping theorem of S. Banach [1]. This theory has seen rapid development in the past nineteenth and twentieth centuries. In the overlaps made in these centuries, while metric spaces and normed spaces developed, the domains were only taken as value regions with single variables and real positive numbers. In other words, new metric spaces are produced by taking the domains X, X^2, and X^3. However, bipolar metric space is defined as a new space by going beyond the conventional definition of metric spaces that have been defined for years. At the same time, this theory has been applied to real life and various fields of science, namely engineering, economics, medical sciences,

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and computer, etc.

In the sequel of generalisation of contractive conditions, Jungck proposed a very different type of generalization of the contraction condition introduced by Banach for a pair of compatible maps in metric spaces. (see [2]). The concept of compatible mappings of type (A) was described in [3] on complete metric spaces. The common fixed point theorems have been proved for the compatible mappings of type (A). This type (A) was shown to be equivalent to the context of compatible mappings defined by Jungck, with some restrictions (see [3]). Valerie Popa demonstrated fixed point theorems for compatible mappings satisfying an implicit relation in [4].

In [5], the (E.A.) property in metric space was defined, which generalizes the concept of non-compatible mappings, and some common fixed point theorems were proved.

Later, in 2016 Mutlu and Gürdal [6] introduced the concept of a bipolar metric space which is the generalization of a metric space. They have proved some generalizations of Banach Fixed Point Theorem [1] in bipolar metric spaces. Given the theorem proved herein, it is highly demanded to recall the most basic definitions and properties in bipolar metric spaces.

Subsequently, in the recent past, various authors have reported fixed point results in the setting of bipolar metric spaces using various contractive conditions. For more details, [7–16].

Inspired, in this article we aim to prove common fixed point theorems with the concepts of maps of type (A) and property $(E.A)$, and implicit relations in bipolar metric spaces, which generalizes some famous well-known results, namely Kannan [17], Reich [18] and Gaba [16].

The rest of the paper is organized as follows: In Section-2, we review some basic preliminaries and monograph. We present our main results in Section-3, establishing the fixed point results and generalising various fixed point results proven in the past. The derived results have been supported with suitable examples. Finally we conclude the manuscript presenting the scope for further research by presenting some open problems for future research.

2. Preliminaries

The following are required in the sequel.

Definition 1. *[6] Let X and Y be two non nonempty sets and $d : X \times Y \to [0, +\infty)$ be a function. Then the triplet (X, Y, d) is called bipolar metric space and d is called bipolar*

metric on (X, Y), if the following conditions holds:
(BP1) $d(x, y) = 0$ *if and only if* $x = y$ *where* $(x, y) \in X \times Y$,
(BP2) If $x, y \in X \cap Y$ *then* $d(x, y) = d(y, x)$,
(BP3) $d(x_1, y_2) \leq d(x_1, y_1) + d(x_2, y_1) + d(x_2, y_2)$ *for all* $x_1, x_2 \in X$ *and* $y_1, y_2 \in Y$.

Definition 2. *[6] Let (X, Y, d) be a bipolar metric space. A sequence $\{u_n\}$ is said to be convergent to a point t if and only if $\{u_n\}$ is a sequence in X, t is a point in Y and $\lim_{n \to +\infty} d(u_n, t) = 0$; or $\{u_n\}$ is a sequence in Y, t is a point in X and $\lim_{n \to +\infty} d(t, u_n) = 0$.*
A sequence $\{(x_n, y_n)\}$ in $X \times Y$ is called a bisequence on (X, Y). This sequence is simply denoted by (x_n, y_n). If both the sequences $\{x_n\}$ and $\{y_n\}$ converge, then the bisequence (x_n, y_n) is said to be convergent. If both the sequences $\{x_n\}$ and $\{y_n\}$ converge to a same point $u \in X \cap Y$ then (x_n, y_n) is called biconvergent. If $\lim_{n,m \to +\infty} d(x_n, y_m) = 0$ then the bisequence (x_n, y_n) is called a Cauchy bisequence.
In a bipolar metric space, every convergent Cauchy bisequence is biconvergent. A bipolar metric space is called complete, if every Cauchy bisequence is convergent, hence biconvergent.

Remark 1. *Limit of a convergent sequence in a bipolar metric space need not be unique, but if a limit is in $X \cap Y$, then it is the unique limit of the sequence.*

Definition 3. *[6] Let X_1, Y_1, X_2 and Y_2 be four sets. A function $f : X_1 \cup Y_1 \to X_2 \cup Y_2$ is said to be a covariant map if $f(X_1) \subseteq X_2$ and $f(Y_1) \subseteq Y_2$ and is denoted as $f : (X_1, Y_1) \rightrightarrows (X_2, Y_2)$. In particular, if (X_1, Y_1, d_1) and (X_2, Y_2, d_2) are two bipolar metric space then we use the notaion $f : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ for covariant map f.*
A function $g : X_1 \cup Y_1 \to X_2 \cup Y_2$ is said to be a contravariant map if $g(X_1) \subseteq Y_2$ and $g(Y_1) \subseteq X_2$ and is denoted as $g : (X_1, Y_1) \rightleftarrows (X_2, Y_2)$.

Definition 4. *Let (X_1, Y_1, d_1) and (X_2, Y_2, d_2) be two bipolar metric spaces. A covariant map $f : (X_1, Y_1) \rightrightarrows (X_2, Y_2)$ is continuous at v if and only if $\{u_n\}$ converges to v on (X_1, Y_1, d_1) implies $\{f(u_n)\}$ converges to $f(v)$ on (X_2, Y_2, d_2).*
A contravariant map $g : (X_1, Y_1, d_1) \rightleftarrows (X_2, Y_2, d_2)$ is continuous if and only if it is continuous as a covariant map $g : (X_1, Y_1, d_1) \rightrightarrows (Y_2, X_2, \bar{d_2})$, where $\bar{d_2}$ is defined as $\bar{d_2}(y, x) = d_2(x, y)$, for all $(y, x) \in Y_2 \times X_2$.

Definition 5. *[7] If S and T are covariant or contravariant maps on $X \cup Y$, then*

(i) $u \in X \cup Y$ *is called fixed point of* T *if and only if* $Tu = u$.

(ii) $u \in X \cup Y$ *is called common fixed point of* S *and* T *if and only if* $Tu = Su = u$.

(iii) $u \in X \cup Y$ *is called coincidence point of* S *and* T *if and only if* $Tu = Su$.

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Popa [4] studied a new type of contraction condition by employing the implicit function to obtain fixed points. In the sequel, we are also going to prove a few fixed-point theorems by employing implicit relations in a bipolar metric space.

3. Main Results

The concept of maps of type (A) was introduced initially by Jungck, Murthy, and Cho [3] in a metric space. Now we are ready to introduce the same concept in a bipolar metric space. The definition follows:

Definition 6. *Let (X, Y, d) be a bipolar metric space. Also, let $T : (X, Y, d) \rightrightarrows (X, Y, d)$ be a covariant map and $S : (X, Y, d) \rightleftarrows (X, Y, d)$ be a contravariant map. Then*

(i) S and T are said to be compatible mappings of type (A) with respect to X if and only if $\lim_{n\to+\infty} d(SSu_n, TSu_n) = 0$ or $\lim_{n\to+\infty} d(TTu_n, STu_n) = 0$, whenever $\{u_n\}$ be a sequence in X such that

$$\lim_{n\to+\infty} Su_n = \lim_{n\to+\infty} Tu_n = t$$

for some $t \in X \cap Y$.

(ii) S and T are said to be compatible mappings of type (A) with respect to Y if and only if $\lim_{n\to+\infty} d(TSu_n, SSu_n) = 0$ or $\lim_{n\to+\infty} d(STu_n, TTu_n) = 0$, whenever $\{u_n\}$ is a sequence in Y such that

$$\lim_{n\to+\infty} Su_n = \lim_{n\to+\infty} Tu_n = t$$

for some $t \in X \cap Y$.

(iii) S and T are said to be weak compatible mappings of type (A) if and only if $Tu = Su$ for some $u \in X \cap Y$, then $TSu = SSu$ (or equivalently, $STu = TTu$.)

Example 1. *Let $X = \mathbb{N} \cup \{0\}$, $Y = [0, 1]$ and the metric d is defined by $d(x, y) = |x - y|$, where N is the set of positive integers. Then (X, Y, d) is a bipolar metric space. Let T (covariant) and S (contravariant map) are defined as $S(n) = \frac{1}{n}$, for all $n \in \mathbb{N}$, $S(y) = 1$, for all $y \in Y$, $T(n) = 2n$, for all $n \in X - \{1\}, T1 = 0$*

$$Ty = \begin{cases} 2y, & \text{if } 0 \le y \le \frac{1}{2} \\ \frac{1}{2}, & \text{if } \frac{1}{2} < y < 1 \end{cases}$$

then the maps S and T are compatible of type (A) with respect to X vacuously as there is no sequence $\{x_n\}$ in X such that $\lim_{n\to+\infty} Sx_n = \lim_{n\to+\infty} Tx_n = 1 \in X \cap Y$ or $\lim_{n\to+\infty} Sx_n = \lim_{n\to+\infty} Tx_n = 0 \in X \cap Y$, but it is not compatible of type (A) with respect to Y as the sequence $\{\frac{1}{2} - \frac{1}{2n}\}$ in Y has the following property

$$\lim_{n\to+\infty} Sy_n = \lim_{n\to+\infty} Ty_n = 1$$

but $\lim_{n\to+\infty} d(TSy_n, SSy_n) = d(0, 1) \neq 0$ and $\lim_{n\to+\infty} d(STy_n, TTy_n) = d(1, \frac{1}{2}) \neq 0$.

Now we extend the definition of property (E.A.) [5] to bipolar metric space.

Definition 7. *Let (X, Y, d) be a bipolar metric space and let $T_1, T_2 : (X, Y, d) \rightrightarrows (X, Y, d)$ be covariant maps and $S_1, S_2 : (X, Y, d) \rightleftarrows (X, Y, d)$ be contravariant maps. We say that*

(i) T_1 and S_1 satisfy the property (E.A.) if there exists a sequence $\{x_n\}$ in X and a sequence $\{y_n\}$ in Y such that $\lim_{n\to+\infty} S_1x_n = \lim\limits_{n\to+\infty} T_1x_n = \lim\limits_{n\to+\infty} S_1y_n = \lim\limits_{n\to+\infty} T_1y_n = t$ for some $t \in X \cap Y$

(ii) T_1 and S_1 satisfy the weak form of property (E.A.) if there exists a sequence $\{u_n\}$ in X or Y such that $\lim\limits_{n\to+\infty} S_1u_n = \lim\limits_{n\to+\infty} T_1u_n = t$ for some $t \in X \cap Y$

(iii) the quadruple (S_1, T_1, S_2, T_2) satisfies the property (E.A.) if there exists a sequence $\{x_n\}$ in X and a sequence $\{y_n\}$ in Y such that $\lim\limits_{n\to+\infty} S_1x_n = \lim\limits_{n\to+\infty} T_1x_n = \lim\limits_{n\to+\infty} S_2y_n = \lim\limits_{n\to+\infty} T_2y_n = t$ for some $t \in X \cap Y$

The following proposition gives the connection between compatible mappings of type (A) and a weak compatible mappings of type (A).

Proposition 1. *If S and T are mappings of type (A) with respect to X or Y, then they are weak compatible mappings of type (A).*

Proof. Let S and T are compatible mappings of type (A) with respect to X or Y with $Su = Tu$ for some $u \in X \cap Y$, then the proposition can be proved easily by taking $u_n = u$ in the definitions of compatible mappings of type (A) with respect to X and Y.

We now introduce the following class of implicit functions:
Let Ψ be the collection of all real-valued function $\psi : [0, +\infty)^4 \to \mathbb{R}$ satisfying the following conditions:

(ψ_a): If $\psi(a, b, a, b) \leq 0$ or $\psi(a, b, b, a) \leq 0$ then there exists $k \in [0, 1)$ such that $a \leq kb$.

(ψ_b): If $\psi(a, a, 0, 0) > 0$ for all $a > 0$.

Remark 2. *by definition of ψ, it is clear that the following implications hold:*

- $\psi(a, a, 0, 0) \leq 0$ *implies* $a = 0$.
- $\psi(a, 0, 0, a) \leq 0$ *implies* $a = 0$.
- $\psi(a, a, a, a) \leq 0$ *implies* $a = 0$.

Example 2. *The following functions are members of Ψ.*

- $\psi_1(a, b, c, d) = a - k_1b - k_2c - k_3d, \quad k_1, k_2, k_3 \geq 0, k_1 + k_2 + k_3 < 1$
  If $\psi_1(a, b, a, b) = a - k_1b - k_2a - k_3b \leq 0$ *then* $a \leq \left(\dfrac{k_1 + k_3}{1 - k_2}\right) b$ *with* $0 < \dfrac{k_1 + k_3}{1 - k_2} < 1$.
  If $\psi_1(a, b, b, a) = a - k_1b - k_2b - k_3a \leq 0$ *then* $a \leq \left(\dfrac{k_1 + k_2}{1 - k_3}\right) b$ *with* $0 < \dfrac{k_1 + k_2}{1 - k_3} < 1$.
  take $k = \max\left\{\dfrac{k_1 + k_3}{1 - k_2}, \dfrac{k_1 + k_2}{1 - k_3}\right\}$

- $\psi_2(a,b,c,d) = a - k \max\{b,c,d\},\ k \in [0,1)$
- $\psi_3(a,b,c,d) = a - k_1 b - k_2 \max\{c,d\} \quad k_1,k_2 \geq 0, k_1+k_2<1$
- $\psi_4(a,b,c,d) = a - k_1 \max\{b,c\} - k_2 d \quad k_1,k_2 \geq 0, k_1+k_2<1$
- $\psi(a,b,c,d) = a - kF(max\{b,c,d\})$ *where* $F : [0,+\infty) \to [0,+\infty)$ *is a function satisfying the condition:* $F(t) \leq t$, *for each* $t \in (0,+\infty)$ *and* $k \in [0,1)$.

Now we consider a super class of Ψ which will be denoted by Φ and it is the collection of all real valued functions $\phi : [0,+\infty)^4 \to \mathbb{R}$ satisfying the following condition:
$\phi(a,a,0,0) > 0$, for all $a > 0$.
Now we prove a lemma that will be used in proving our theorems.

Lemma 1. *Let* (X,Y,d) *be a bipolar metric space and* (x_n,y_n) *is a bisequence in* $X \times Y$ *satisfying the following condition:*
There exists $k \in [0,1)$ *such that*

$$d(x_{n+1},y_{n+1}) \leq kd(x_{n+1},y_n) \ \ \textit{and} \\ d(x_{n+1},y_n) \leq kd(x_n,y_n)$$

for all $n \in \mathbb{N} \cup \{0\}$.
Then the bisequence (x_n,y_n) *is Cauchy bisequence.*

Proof. First, we observe that the given condition implies the following condition

$$d(x_{n+1},y_{n+1}) \leq k^2 d(x_n,y_n) \ \text{ for all } \ n \in \mathbb{N}\cup\{0\} \\ d(x_{n+1},y_{n+1}) \leq k^{2(n+1)} d(x_0,y_0),$$

taking limit as $n \to +\infty$, we get

$$\lim_{n\to+\infty} d(x_n,y_n) = 0. \tag{1}$$

Let $n,p \in \mathbb{N}$, then by (BP3) and given condition, we have

$$\begin{aligned}
d(x_n,y_{n+p}) \leq & d(x_n,y_n) + d(x_{n+1},y_n) + d(x_{n+1},y_{n+p}) \\
\leq & d(x_n,y_n) + kd(x_n,y_n) + d(x_{n+1},y_{n+p}) \\
= & (1+k)d(x_n,y_n) + d(x_{n+1},y_{n+p}) \\
\leq & (1+k)k^{2n} d(x_0,y_0) + d(x_{n+1},y_{n+p}) \\
\leq & (1+k)(k^{2n} + k^{2(n+1)} + k^{2(n+2)} + \cdots + k^{2(n+p-1)}) d(x_0,y_0) \\
& \quad + d(x_{n+p},y_{n+p}) \\
\leq & (1+k)(k^{2n} + k^{2(n+1)} + k^{2(n+2)} + \cdots) d(x_0,y_0) + d(x_{n+p},y_{n+p}) \\
\leq & (1+k)k^{2n}(1 + k^2 + k^4 + \cdots) d(x_0,y_0) + k^{2(n+p)} d(x_0,y_0) \\
= & \frac{(1+k)k^{2n}}{1-k^2} d(x_0,y_0) + k^{(2n+2p)} d(x_0,y_0).
\end{aligned}$$

This implies

$$\lim_{n\to+\infty} d(x_n, y_{n+p}) = 0. \tag{2}$$

Now to prove that $\lim_{n\to+\infty} d(x_{n+p}, y_n) = 0$, consider the inequality (by property $(BP3)$)

$$d(x_{n+p}, y_n) \leq d(x_{n+p}, y_{n+p}) + d(x_n, y_{n+p}) + d(x_n, y_n)$$

and take the limit as $n \to +\infty$ and use (1) and (2). Hence (x_n, y_n) is a Cauchy sequence.

Our first main result is the following.

Theorem 1. *Let (X, Y, d) be a complete bipolar metric space and let $T_1, T_2 : (X, Y, d) \rightrightarrows (X, Y, d)$ be two covariant maps and $S_1, S_2 : (X, Y, d) \rightleftarrows (X, Y, d)$ be two contravariant maps satisfying the following conditions:*

(i) The mappings S_2, T_1 are compatible of type (A) with respect to Y.

(ii) The mappings S_1, T_2 are compatible of type (A) with respect to X.

(iii) $S_1(X \cup Y) \subseteq T_1(X \cup Y)$ and $S_2(X \cup Y) \subseteq T_2(X \cup Y)$.

(iv) All the four mappings S_1, S_2, T_1 and T_2 are continuous.

(v) There exists $\psi \in \Psi$ such that

$$\psi(d(S_2y, S_1x), d(T_2x, T_1y), d(T_2x, S_1x), d(S_2y, T_1y)) \leq 0, \tag{3}$$

for all $(x, y) \in X \times Y$.
Then the functions S_1, S_2, T_1 and T_2 have a unique common fixed point.

Proof. Let $x_0 \in X$ and choose $x_1 \in X$ and $y_1 \in Y$ such that $S_1x_0 = T_1y_1 = v_0$ and $S_2y_1 = T_2x_1 = u_1$. This can be done since $S_1(X \cup Y) \subseteq T_1(X \cup Y)$ and $S_2(X \cup Y) \subseteq T_2(X \cup Y)$. In general we can choose $(x_n, y_n) \in X \times Y$ such that $S_1x_n = T_1y_{n+1} = v_n$ and $S_2y_{n+1} = T_2x_{n+1} = u_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Now putting $x = x_{n+1}$ and $y = y_{n+1}$ in (3), we get

$$\psi(d(S_2y_{n+1}, S_1x_{n+1}), d(T_2x_{n+1}, T_1y_{n+1}), d(T_2x_{n+1}, S_1x_{n+1}), d(S_2y_{n+1}, T_1y_{n+1})) \leq 0$$
$$\psi(d(u_{n+1}, v_{n+1}), d(u_{n+1}, v_n), d(u_{n+1}, v_{n+1}), d(u_{n+1}, v_n)) \leq 0.$$

So by property of ψ, there exists $k \in [0, 1)$ such that

$$d(u_{n+1}, v_{n+1}) \leq kd(u_{n+1}, v_n). \tag{4}$$

Again putting $x = x_n$ and $y = y_{n+1}$ in (3), we get

$$\psi((d(S_2y_{n+1}, S_1x_n), d(T_2x_n, T_1y_{n+1}), d(T_2x_n, S_1x_n), d(S_2y_{n+1}, T_1y_{n+1})) \leq 0$$
$$\psi(d(u_{n+1}, v_n), d(u_n, v_n), d(u_n, v_n), d(u_{n+1}, v_n)) \leq 0.$$

So by property of ψ, we have

$$d(u_{n+1}, v_n) \leq kd(u_n, v_n). \tag{5}$$

By (4), (5) and Lemma 1, the sequence (u_n, v_n) is Cauchy bisequence and since given bipolar metric space is complete, hence the sequence (u_n, v_n) biconverges to a point $t \in X \cap Y$. So

$$\lim_{n\to+\infty} S_1x_n = \lim_{n\to+\infty} T_1y_n = \lim_{n\to+\infty} S_2y_n = \lim_{n\to+\infty} T_2x_n = t. \tag{6}$$

By using compatibility of type (A) with respect to Y of mappings S_2 and T_1 and (6), we get

$$\lim_{n\to+\infty} d(T_1S_2y_n, S_2S_2y_n) = 0 \text{ or } \lim_{n\to+\infty} d(S_2T_1y_n, T_1T_1y_n) = 0.$$

By continuity of mappings S_2 and T_1, we have

$$\begin{aligned} d(T_1t, S_2t) &= 0 \\ T_1t &= S_2t. \end{aligned} \tag{7}$$

Similarly by using Compatibility of type (A) with respect to X and Continuity of mappings S_1 and T_2 and (6), we get

$$T_2t = S_1t. \tag{8}$$

Now Putting $x = y = t$ in (3) and using (7) and (8), we get

$$\begin{aligned} \psi(d(S_2t, S_1t), d(T_2t, T_1t), d(T_2t, S_1t), d(S_2t, T_1t)) \leq 0 \\ \psi(d(S_2t, S_1t), d(S_1t, S_2t), 0, 0) \leq 0. \end{aligned}$$

So by property of ψ, this implies that $S_2t = S_1t$. Hence we get

$$S_2t = S_1t = T_2t = T_1t = u \quad (say)$$

that is t is a coincidence point of S_1, S_2, T_1 and T_2.
By Proposition 1 the pairs (S_2, T_1) and (S_1, T_2) are weak compatibility of type (A). So we get $T_1S_2t = S_2S_2t$ and $T_2S_1t = S_1S_1t$. This implies

$$T_1u = S_2u \text{ and } T_2u = S_1u. \tag{9}$$

Now putting $x = y = u$ in (3), we get

$$\begin{aligned} \psi(d(S_2u, S_1u), d(T_2u, T_1u), d(T_2u, S_1u), d(S_2u, T_1u)) \leq 0 \\ \psi(d(S_2u, S_1u), d(S_1u, S_2u), 0, 0) \leq 0 \\ d(S_2u, S_1u) = 0 \text{ (by property of } \psi) \\ S_2u = S_1u. \end{aligned} \tag{10}$$

By (9) and (10), we get

$$T_1u = T_2u = S_1u = S_2u.$$

So u is also a coincidence point of S_1, S_2, T_1 and T_2.
Now Putting $y = u$ and $x = t$ in (3), we get

$$\begin{aligned}\psi(d(S_2u, S_1t), d(T_2t, T_1u), d(T_2t, S_1t), d(S_2u, T_1u)) &\leq 0\\ \psi(d(S_2u, u), d(u, S_2u), 0, 0) &\leq 0\\ \psi(d(S_2u, u), d(S_2u, u), 0, 0) &\leq 0\\ d(S_2u, u) &= 0\\ S_2u &= u.\end{aligned}$$

So u is a common fixed point of all the given four mappings. Now we prove that the fixed point is unique. For this let us assume that u_1 is another common fixed point then putting $y = u$ and $x = u_1$ in (3), we get

$$\begin{aligned}\psi(d(S_2u, S_1u_1), d(T_2u_1, T_1u), d(T_2u_1, S_1u_1), d(S_2u, T_1u)) &\leq 0\\ \psi(d(u, u_1), d(u_1, u), 0, 0) &\leq 0.\end{aligned}$$

By Remark 2, this implies

$$\begin{aligned}d(u, u_1) &= 0\\ u &= u_1.\end{aligned}$$

So u is the unique common fixed point.

Now we prove some corollaries derived from Theorem 1

Corollary 1. *Let (X, Y, d) be a complete bipolar metric space and let $T : (X, Y, d) \rightrightarrows (X, Y, d)$ be a covariant map and $S_1, S_2 : (X, Y, d) \rightleftarrows (X, Y, d)$ be two contravariant maps satisfying the following conditions*

(i) S_2 and T are compatible of type (A) with respect to Y.

(ii) S_1 and T are compatible of type (A) with respect to X.

(iii) $S_1(X \cup Y) \subseteq T(X \cup Y)$ and $S_2(X \cup Y) \subseteq T(X \cup Y)$.

(iv) All the three mappings S_1, S_2 and T are continuous.

(v) There exists $\psi \in \Psi$ such that

$$\psi(d(S_2y, S_1x), d(Tx, Ty), d(Tx, S_1x), d(S_2y, Ty)) \leq 0, \tag{11}$$

for all $(x, y) \in X \times Y$.
Then the functions S_1, S_2, and T have a unique common fixed point.

Proof. Take $T_1 = T_2 = T$ in Theorem 1.

Corollary 2. *Let (X, Y, d) be a complete bipolar metric space and let $T : (X, Y, d) \rightrightarrows (X, Y, d)$ be a covariant map and $S : (X, Y, d) \rightleftarrows (X, Y, d)$ be a contravariant map satisfying the following conditions:*

(i) S and T are compatible of type (A) with respect to X or Y.

(ii) $S(X \cup Y) \subseteq T(X \cup Y)$.

(iii) S and T are continuous.

(iv) There exists $\psi \in \Psi$ such that

$$\psi(d(Sy, Sx), d(Tx, Ty), d(Tx, Sx), d(Sy, Ty)) \leq 0, \tag{12}$$

for all $(x, y) \in X \times Y$.
Then the functions S and T have a unique common fixed point.

Proof. Take $T_1 = T_2 = T$ and $S_1 = S_2 = S$ in Theorem 1.

Corollary 3. *Let (X, Y, d) be a complete bipolar metric space and let $S : (X, Y, d) \rightleftarrows (X, Y, d)$ be a contravariant map satisfying the following conditions:*

(i) S is continuous.

(ii) There exists $\psi \in \Psi$ such that

$$\psi(d(Sy, Sx), d(x, y), d(x, Sx), d(Sy, y)) \leq 0, \tag{13}$$

for all $(x, y) \in X \times Y$.
Then the function S has a unique fixed point.

Proof. Take $T_1 = T_2 = I$ and $S_1 = S_2 = S$ in Theorem 1, where I is the identity mapping.

In the following corollary, we take ψ as a continuous function and S need not be continuous.

Corollary 4. *Let (X, Y, d) be a complete bipolar metric space and let $S : (X, Y, d) \rightleftarrows (X, Y, d)$ be a contravariant map satisfying the following condition:*

$$\psi(d(Sy, Sx), d(x, y), d(x, Sx), d(Sy, y)) \leq 0,$$

for some continuous function $\psi \in \Psi$ and for all $(x, y) \in X \times Y$. Then the function S has a unique fixed point.

Proof. As in the proof of the previous corollary, we obtain a bisequence (Sy_n, Sx_n) biconverging to a point $t \in X \cap Y$, where $Sy_n = x_n$ and $Sx_n = y_{n+1}$. In the given condition, if we take $y = y_n$ and $x = t$, then we get,

$$\psi(d(Sy_n, St), d(t, y_n), d(t, St), d(Sy_n, y_n)) \leq 0,$$

taking limit as $n \to +\infty$, we get

$$\psi(d(t, St), 0, d(t, St), 0) \leq 0$$

So by property of ψ, we get $d(t, St) = 0$. This implies that t is a fixed point of S. Uniqueness can be proved as given in the previous corollary.

The contraction used in the following corollary is Reich-type contraction (see [18], [16]).

Corollary 5. *Let (X, Y, d) be a complete bipolar metric space and let $S : (X, Y, d) \rightleftarrows (X, Y, d)$ be a continuous contravariant map satisfying the following condition:*

$$d(Sy, Sx) \leq k_1 d(x, y) + k_2 d(x, Sx) + k_3 d(Sy, y)),$$

for all $(x, y) \in X \times Y$, where $k_1 + k_2 + k_3 < 1$. Then the function S has a unique fixed point.

Proof. In Corollary 4, take $\psi(a, b, c, d) = a - (k_1 b + k_2 c + k_3 d)$.

Corollary 6. *Let (X, Y, d) be a complete bipolar metric space and let $S : (X, Y, d) \rightleftarrows (X, Y, d)$ be a contravariant map satisfying the following condition:*

$$d(Sy, Sx) \leq k(d(x, y) + d(x, Sx) + d(Sy, y)),$$

for all $(x, y) \in X \times Y$, where $k < \frac{1}{3}$. Then the function S has a unique fixed point.

Proof. In Corollary 5, take $k_1 = k_2 = k_3 = k$.

In the following corollary, Kannan-type contraction (see [16, 17]) is used.

Corollary 7. *Let (X, Y, d) be a complete bipolar metric space and let $S : (X, Y, d) \rightleftarrows (X, Y, d)$ be a contravariant map satisfying the following condition:*

$$d(Sy, Sx) \leq k(d(x, Sx) + d(Sy, y)),$$

for all $(x, y) \in X \times Y$, where $k < \frac{1}{2}$. Then the function S has a unique fixed point.

Proof. In Corollary 5, take $k_1 = 0, k_2 = k_3 = k$.

In our next theorem, we do not require the continuity of maps.

Theorem 2. *Let (X, Y, d) be a bipolar metric space and let $T_1, T_2 : (X, Y, d) \rightrightarrows (X, Y, d)$ be two covariant maps and $S_1, S_2 : (X, Y, d) \rightleftarrows (X, Y, d)$ be two contravariant maps satisfying the following conditions:*

(i) The pairs (S_2, T_1) *and* (S_1, T_2) *are weak compatible of type*(A).

(ii) $S_1(X) \subseteq T_1(Y)$ *or* $S_2(Y) \subseteq T_2(X)$.

(iii) $(T_2(X), T_1(Y), d)$ *or* $(S_2(Y), S_1(X), d)$ *is complete.*

(iv) There exists continuous function $\psi \in \Psi$ *such that*

$$\psi(d(S_2y, S_1x), d(T_2x, T_1y), d(T_2x, S_1x), d(S_2y, T_1y)) \leq 0, \tag{14}$$

for all $(x, y) \in X \times Y$.
Then the functions S_1, S_2, T_1 *and* T_2 *have a unique common fixed point.*

Proof. Let the bisequence (x_n, y_n) and (u_n, v_n) be defined as in Theorem 1. By the same argument as given in the same theorem, (u_n, v_n) is Cauchy bisequence in $(T_2(X), T_1(Y), d)$ and $(S_2(Y), S_1(X), d)$.
The following two cases arise

Case - I : If $(T_2(X), T_1(Y), d)$ is complete, then the sequence (u_n, v_n) biconverges to some point in $T_2(X) \cap T_1(Y)$.

Case - II: If $(S_2(Y), S_1(X), d)$ is complete, then the sequence (u_n, v_n) biconverges to a point in $S_2(Y) \cap S_1(X)$. This implies that (u_n, v_n) biconverges to a point in $T_2(X) \cap T_1(Y)$ as $S_2(Y) \cap S_1(X) \subset T_2(X) \cap T_1(Y)$.

So in both the cases, it converges to a point t (say) in $T_2(X) \cap T_1(Y)$. Hence, there exist $p \in B$ and $q \in A$ such that

$$t = T_1p = T_2q. \tag{15}$$

Now putting $y = y_n$ and $x = q$ in (14), we get

$$\psi(d(S_2y_n, S_1q), d(T_2q, T_1y_n), d(T_2q, S_1q), d(S_2y_n, T_1y_n)) \leq 0,$$

taking limit as $n \to +\infty$, we get

$$\begin{aligned} \psi(d(t, S_1q), d(t, t), d(t, S_1q), 0) &\leq 0 \\ \psi(d(t, S_1q), 0, d(t, S_1q), 0) &\leq 0 \\ S_1q &= t. \end{aligned} \tag{16}$$

Again putting $y = p$ and $x = x_n$ in (14), we get

$$\psi(d(S_2p, S_1x_n), d(T_2x_n, T_1p), d(T_2x_n, S_1x_n), d(S_2p, T_1p)) \leq 0,$$

taking limit as $n \to +\infty$, we get

$$\begin{aligned} \psi(d(S_2p, t), d(t, t), d(t, t), d(S_2p, t)) &\leq 0 \\ \psi(d(S_2p, t), 0, 0, d(S_2p, t)) &\leq 0 \end{aligned}$$

$$S_2p = t. \tag{17}$$

From (15), (16) and (17), we get

$$T_1p = T_2q = S_1q = S_2p = t. \tag{18}$$

Since the pairs (S_2, T_1) and (S_1, T_2) are weak compatible of type(A), equations (18) imply that $T_1S_2p = S_2S_2p$ or $S_2T_1p = T_1T_1p$; and $T_2S_1q = S_1S_1q$ or $S_1T_2q = T_2T_2q$. So $T_1t = S_2t, T_2t = S_1t$.
Now putting $x = y = t$ in (14), we get

$$\begin{aligned}
\psi(d(S_2t, S_1t), d(T_2t, T_1t), d(T_2t, S_1t), d(S_2t, T_1t)) \leq 0 \\
\psi(d(S_2t, S_1t), d(S_1t, S_2t), 0, 0) \leq 0 \\
d(S_2t, S_1t) = 0 \\
S_2t = S_1t.
\end{aligned}$$

So we get

$$T_1t = S_2t = T_2t = S_1t. \tag{19}$$

That is, t is a coincidence point of T_1, S_2, T_2 and S_1. Now we show that t is a common fixed point of these four mappings. For this, substituting $x = t$ and $y = p$ in (14) and using (18) and (19), we get

$$\begin{aligned}
\psi(d(S_2p, S_1t), d(T_2t, T_1p), d(T_2t, S_1t), d(S_2p, T_1p)) \leq 0 \\
\psi(d(t, S_1t), d(S_1t, t), 0, 0) \leq 0 \\
S_1t = t.
\end{aligned}$$

So t is a common fixed point of given four mappings. The uniqueness of a common fixed point can be proved as in Theorem 1.

Our next theorem is about the common fixed point of four mappings and is a generalization of the Theorem 1.

Theorem 3. *Let (X, Y, d) be a complete bipolar metric space and let $T_1, T_2 : (X, Y, d) \rightrightarrows (X, Y, d)$ be two covariant maps and $S_1, S_2 : (X, Y, d) \rightleftarrows (X, Y, d)$ be two contravariant maps satisfying the following conditions:*

(i) S_2 and T_1 are compatible of type (A) with respect to Y.

(ii) S_1 and T_2 are compatible of type (A) with respect to X.

(iii) The quadruple (S_1, T_2, S_2, T_1) satisfies the property (E.A.).

(iv) All the four mappings S_1, S_2, T_1 and T_2 are continuous.

(v) There exists $\phi \in \Phi$ such that

$$\phi(d(S_2y, S_1x), d(T_2x, T_1y), d(T_2x, S_1x), d(S_2y, T_1y)) \leq 0, \tag{20}$$

for all $(x, y) \in X \times Y$.
Then the functions S_1, S_2, T_1 *and* T_2 *have a unique common fixed point.*

Proof. Since the quadruple (S_1, T_2, S_2, T_1) satisfies the property (E.A.), so there exists a sequence $\{(x_n, y_n)\}$ in $X \times Y$ such that

$$\lim_{n\to+\infty} S_1x_n = \lim_{n\to+\infty} T_2x_n = \lim_{n\to+\infty} S_2y_n = \lim_{n\to+\infty} T_1y_n = t. \tag{21}$$

This is the equation (6) in Theorem 1. The remaining proof of the theorem is the same as the proof of the Theorem 1 with ψ replaced by ϕ.

Like Theorem 1, many corollaries can be derived here also. One of the corollaries is given below:

Corollary 8. *Let* (X, Y, d) *be a complete bipolar metric space and let* $T : (X, Y, d) \rightrightarrows (X, Y, d)$ *be a covariant map and* $S : (X, Y, d) \rightleftarrows (X, Y, d)$ *be a contravariant map satisfying the following conditions:*

(i) S *and* T *are compatible of type* (A) *with respect to* X *or* Y.

(ii) T *and* S *satisfy the weak form of property (E.A.).*

(iii) S *and* T *are continuous.*

(iv) There exists $\phi \in \Phi$ *such that*

$$\phi(d(Sy, Sx), d(Tx, Ty), d(Tx, Sx), d(Sy, Ty)) \leq 0$$

for all $(x, y) \in X \times Y$.
Then the functions S *and* T *have a unique common fixed point.*

Example 3. *Let* $a, d \in \mathbb{R}, 0 \neq b \in \mathbb{C}$ *with* $b\bar{b} - ad > 0$ *, where* $\mathbb{C}$ *is the set of complex numbers. Let us define two sets* $C(a, b, d) = \{z \in \mathbb{C} : az\bar{z} + b\bar{z} + \bar{b}z + d = 0\}$ *and* $L(b, d) = \{z \in \mathbb{C} : b\bar{z} + \bar{b}z + d = 0\}$. *It is clear that* $C(a.b, d)$ *and* $L(b, d)$ *represent a circle(if* $a \neq 0$*) and a straight line respectively in a complex plane. Let* $X = \{C(a, b, d) : a, d \in \mathbb{R}\}$ *and* $Y = \{L(b, d) : d \in \mathbb{R}\}$. *Hence* $Y \subseteq X$. *Let* $\rho : X \times Y \to [0, +\infty)$ *is defined as* $\rho(C(a, b, d), L(b, d_1)) = |a| + |d - d_1|$ *for all* $C(a, b, d) \in X, L(b, d_1) \in Y$. *Then* (X, Y, ρ) *is a complete bipolar metric space. Let* $T : (X, Y, \rho) \rightrightarrows (X, Y, \rho)$ *be a covariant map and* $S : (X, Y, \rho) \rightleftarrows (X, Y, \rho)$ *be a contravariant map defined as follows:*
$S(C(a, b, d)) = L(b, \frac{d}{8}), S(L(b, d)) = L(b, \dfrac{d}{8}), T(C(a, b, d)) = C\left(\dfrac{a}{2}, b, \dfrac{d}{2}\right)$,
$T(L(b, d)) = L\left(b, \dfrac{d}{2}\right)$

Then S and T are continuous mappings, $S(X \cup Y) = Y \subseteq X \cup Y = T(X \cup Y)$. The mappings S and T are compatible of type (A) with respect to Y, for let the sequence $\{L(b, d_n)\}$ in Y satisfies the condition:

$$\lim_{n \to +\infty} S(L(b, d_n)) = \lim_{n \to +\infty} T(L(b, d_n)) = L(b, d_0) \in X \cap Y,$$

then $\frac{d_n}{8} \to d_0$ and $\frac{d_n}{2} \to d_0$, so $d_0 = 0$, this implies that $\rho(TS(L(b, d_n)), SS(L(b, d_n))) \to 0$ and $\rho(ST(L(b, d_n)), TT(L(b, d_n))) \to 0$. S and T satisfy the following condition

$$\psi(d(Sy, Sx), d(Tx, Ty), d(Tx, Sx), d(Sy, Ty)) \leq 0,$$

for all $(x, y) \in X \times Y$, where $\psi(a, b, c, d) = a - \frac{1}{4}(b + c + d)$. So all the conditions of Corollary 2 are satisfied, so S and T have unique common fixed point. In fact , $L(b, 0), (that is b\bar{z} + \bar{b}z = 0)$ is the unique common fixed point of S and T.

Example 4. *Let $X = (-\infty, 0]$ and $Y = [0, +\infty)$, then (X, Y, d) is a complete bipolar metric space where d is defined as $d(x, y) = |x - y|$. Let maps T_1, T_2, S_1 and S_2 be defined as $S_1(x) = \frac{-x}{12}$, $S_2(x) = \frac{-x}{6}$, $T_1(x) = \frac{x}{2}$ and $T_2(x) = \frac{x}{4}$, for all $x \in X \cup Y$. Then S_1, S_2 are two continuous contravariant maps and T_1, T_2 are continuous covariant maps. All these maps satisfy the condition*

$$\psi(d(S_2y, S_1x), d(T_2x, T_1y), d(T_2x, S_1x), d(S_2y, T_1y)) \leq 0,$$

for all $(x, y) \in X \times Y$ where $\psi(a, b, c, d) = a - k_1b - k_2c - k_3d$ with $k_1 = \frac{2}{3}, k_2 = \frac{1}{12}, k_3 = \frac{1}{12}$. All the other conditions of Theorem 1 are also satisfied, so S_1, S_2, T_1 and T_2 have a unique common fixed point.

Example 5. *Let $X = (-\infty, 0]$ and $Y = [0, +\infty)$, then (X, Y, d) is a complete bipolar metric space where d is defined as $d(x, y) = |x - y|$. Let T (covariant) and S(contravariant map) are defined as $S(x) = \frac{-x}{3}$ and $T(x) = \frac{x}{2}$, for all $x \in X \cup Y$. Then S and T are continuous functions, compatible of type (A) with respect to X and Y both, $S(X \cup Y) \subseteq T(X \cup Y)$, and satisfy the condition*

$$\psi(d(Sy, Sx), d(Tx, Ty), d(Tx, Sx), d(Sy, Ty)) \leq 0,$$

for all $(x, y) \in X \times Y$ where $\psi(a, b, c, d) = a - k_1b - k_2c - k_3d$ with $k_1 = \frac{2}{3}, k_2 = \frac{1}{12}, k_3 = \frac{1}{12}$. So all the conditions of Corollary 2 are satisfied, so S and T have a unique common fixed point.

4. Conclusion

In the article, fixed point results in the setting of Bipolar metric space generalising some famous results of Kannan [17], Reich [18] and Gaba [16] have been proven. Suitable non-trivial examples have been provided in support of the derived results. It will be an open problem to find some applications to examine the existence and uniqueness of solutions to Differential equations, integral equations and also extend the proven results using generalised contractive conditions.

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Conflict of interest

The authors declare no conflict of interest.

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