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Modified Laplace-type Transform and Its Properties

İnci Ege

Department of Mathematics, Faculty of Science, University of Aydın Adnan Menderes, Aydın, Türkiye

Abstract. In this article, we introduce the Modified Laplace-type transform, develop convergence properties, and obtain fundamental formulas of some elementary functions such as power functions, sine, cosine, hyperbolic sine, hyperbolic cosine, and exponential functions. We derive translation theorems and a scale-preserving theorem and also show the relationship between the modified Laplace type transform and the modified degenerate Gamma function. This integral transform is applied to solve linear ordinary differential equations with constant coefficients and a Volterra integral equation of the second kind.

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1. Introduction

A transformation is a mathematical technique that changes one function into another. An integral transform maps a function from its original function space into another function space by using integration as a tool to solve differential and integral equations. Its motivation comes from some classes of problems that are difficult to solve in their original representations. An integral transform takes a function from its original domain into another, which may make solving the equation much easier than in the original domain. The transformed function can generally be mapped back to the original function space using the inverse transform. An integral transform T is of the form

$$(Tv)(t) = \int_{x_1}^{x_2} v(x)k(x,t)\,dx,$$

where v is the input function, Tv is the output function and $k(x,t)$ is the kernel of the transform.

Numerous useful integral transforms have been defined; see for example [1–20]. Each is specified by a choice of the kernel function k of two variables. Perhaps the most well-known integral transform is the Laplace transform. Besides mathematics, it is utilized

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Email address: iege@adu.edu.tr (İ. Ege)

in other fields of study wildly as engineering, physics, astronomy, etc. For example, it is used to solve differential equations occurring in the analysis of electronic circuits. Laplace transform is defined by

$$F(u) = L\{f(t)\} = \int_0^\infty e^{-ut} f(t)\, dt \tag{1}$$

provided that the integral converges [10].

There are many integral transforms in the Laplace class and most of them have been named after the mathematicians who introduced them. Some of these are the Sumudu transform [21], the Natural transform [22], the Elzaki transform [23], the Aboodh transform [24], the ZZ transform [25] and the Polynomial integral transform [26]. The Sumudu [21] and Elzaki transforms [23] are defined respectively as

$$S(u) = S\{f(t)\} = \frac{1}{u} \int_0^\infty e^{-\frac{t}{u}} f(t)\, dt, \tag{2}$$

and

$$E(u) = E\{f(t)\} = u \int_0^\infty e^{-\frac{t}{u}} f(t)\, dt. \tag{3}$$

Recently, the Laplace-type integral transform is introduced by Kim in [27] as

$$F_\alpha(u) = G_\alpha\{f(t)\} = u^\alpha \int_0^\infty e^{-\frac{t}{u}} f(t)\, dt, \tag{4}$$

where $f(t)$ be an integrable function on $[0.\infty)$, $u > 0$ and $\alpha \in \mathbb{Z}$.

The modified degenerate Gamma function defined in [28] as

$$\Gamma^*_\lambda(x) = \int_0^\infty t^{x-1}(1+\lambda)^{-\frac{t}{\lambda}}\, dt, \quad \lambda \in (0,1) \text{ and } Re(x) > 0, \tag{5}$$

and satisfies the properties that

$$\Gamma^*_\lambda(x+1) = \frac{\lambda x}{\ln(1+\lambda)} \Gamma^*_\lambda(x),$$

$$\Gamma^*_\lambda(n+1) = \frac{\lambda^{n+1} n!}{\ln^{n+1}(1+\lambda)}, \;\; n = 1, 2, \ldots. \tag{6}$$

Degenerate versions of existing integral transforms have also been studied in the last few years. For example, Kim and Kim introduced the degenerate Laplace transform [29] and Upadhyaya gives further results for the degenerate Laplace transform [30–32]. Campos et al. defined degenerate Laplace-type integral transform and gave its properties [3]. Also, Duran defined the degenerate Sumudu transform [33] and Kalavathi et al. defined the degenerate Elzaki transform [34].

Motivated by the above-mentioned research, in this paper we proposed the modified Laplace-type transform. The mentioned transform is indicated by the operator $G^*_{\alpha,\lambda}$

through this research. We define the modified Laplace-type transform and provide some of their properties and relations, and derive the modified Laplace-type transform of some functions such as power functions, sine, cosine, hyperbolic sine, hyperbolic cosine, exponential function, and function derivatives. Moreover, we attain relations between the Laplace-type transform and the modified Laplace-type transform and also give a relation between the modified Laplace-type transform and the modified degenerate Gamma function. Also, we give some operational properties of modified Laplace-type transform.

2. The main results

In this section, we present the modified Laplace-type transform $G^*_{\alpha,\lambda}$, give sufficient conditions for the existence, and calculate the modified Laplace-type transform of some frequently used functions.

Definition 1. *Let $\lambda \in (0, \infty)$, $\alpha \in \mathbb{Z}$, $u > 0$ and $f(t)$ be an integrable function defined for all $t \geq 0$. Then the integral*

$$G^*_{\alpha,\lambda}\{f(t)\} = u^\alpha \int_0^\infty (1+\lambda)^{-\frac{t}{u\lambda}} f(t)\, dt \tag{7}$$

is said to be the modified Laplace-type transform $G^*_{\alpha,\lambda}$ of $f(t)$ provided that the integral in (7) exists.
Since the function $G^*_{\alpha,\lambda}\{f(t)\}$ is depend on the variable u, it can be denoted as $F^*_{\alpha,\lambda}(u)$.

We note that

$$\lim_{\lambda\to 0} G^*_{\alpha,\lambda}\{f(t)\} = G_\alpha\{f(t)\}, \tag{8}$$

$$\lim_{\substack{\lambda\to 0\\ \alpha=1}} G^*_{\alpha,\lambda}\{f(t)\} = E\{f(t)\} \tag{9}$$

and

$$\lim_{\substack{\lambda\to 0\\ \alpha=-1}} G^*_{\alpha,\lambda}\{f(t)\} = S\{f(t)\}. \tag{10}$$

Now we give sufficient conditions for the existence of the new integral transform.

Theorem 1. *(Existence property of $G^*_{\alpha,\lambda}$) Let $f(t)$ be a piecewise-continious function on every finite interval $[0, a]$ and of exponential order as t goes to infinity with $|f(t)| \leq Me^{kt}$ for $t > L$, where and k, L, M are constants and $M > 0$. Then $G^*_{\alpha,\lambda}\{f(t)\}$ exists for $\frac{1}{u} > \frac{k\lambda}{ln(1+\lambda)}$.*

Proof. We can write

$$u^\alpha \int_0^\infty (1+\lambda)^{-\frac{t}{u\lambda}} f(t)\, dt = u^\alpha \int_0^L (1+\lambda)^{-\frac{t}{u\lambda}} f(t)\, dt + u^\alpha \int_L^\infty (1+\lambda)^{-\frac{t}{u\lambda}} f(t)\, dt. \tag{11}$$

Since the function $f(t)$ is continuous on the interval $[0, L]$, there exist an $M > 0$ such that $|f(t)| \leq M$. It gives that

$$u^{\alpha} \int_0^L (1+\lambda)^{-\frac{t}{u\lambda}} f(t)\, dt \leq M u^{\alpha} \int_0^L (1+\lambda)^{-\frac{t}{u\lambda}}\, dt = \frac{M\lambda u^{\alpha+1}}{\ln(1+\lambda)}[1-(1+\lambda)^{-\frac{L}{u\lambda}}] < \infty.$$

Hence the first integral on the right-hand side of equation (11) exists. Also, since

$$|(1+\lambda)^{-\frac{t}{u\lambda}} f(t)| \leq M(1+\lambda)^{-\frac{t}{u\lambda}} e^{kt} = Me^{-\frac{t}{u\lambda}\ln(1+\lambda)} e^{kt} = Me^{t\left(k-\frac{\ln(1+\lambda)}{u\lambda}\right)}$$

we have

$$\begin{aligned} \left| u^{\alpha} \int_L^{\infty} (1+\lambda)^{-\frac{t}{u\lambda}} f(t)\, dt \right| &\leq u^{\alpha} \int_L^{\infty} e^{-\frac{t}{u\lambda}\ln(1+\lambda)} \, |f(t)|\, dt \\ = \quad M u^{\alpha} \lim_{h\to\infty} \int_L^h e^{t\left(k-\frac{\ln(1+\lambda)}{u\lambda}\right)}\, dt &= \frac{M\lambda u^{\alpha+1}}{\ln(1+\lambda)-ku\lambda} e^{L\left(k-\frac{\ln(1+\lambda)}{u\lambda}\right)} < \infty \end{aligned}$$

for $k < \frac{\ln(1+\lambda)}{u\lambda}$. Then we proved that the integral

$$u^{\alpha} \int_0^{\infty} (1+\lambda)^{-\frac{t}{u\lambda}} f(t)\, dt \tag{12}$$

exists, and the result follows.

Theorem 2. *The modified Laplace-type transform of the function $f(t) = t^n$, $n = 1, 2, \ldots$ is given by*

$$G^{*}_{\alpha,\lambda}\{t^n\} = \frac{n!\lambda^{n+1} u^{\alpha+n+1}}{\ln^{n+1}(1+\lambda)}.$$

Proof. By considering the Definition 1 for $f(t) = t^n$, leads to

$$G^{*}_{\alpha,\lambda}\{t^n\} = u^{\alpha} \int_0^{\infty} (1+\lambda)^{-\frac{t}{u\lambda}}\, t^n\, dt = u^{\alpha} \lim_{h\to\infty} \int_0^h (1+\lambda)^{-\frac{t}{u\lambda}}\, t^n\, dt.$$

Using integration by parts, we have

$$G^{*}_{\alpha,\lambda}\{t^n\} = u^{\alpha} \lim_{h\to\infty} \left[\frac{-t^n u\lambda (1+\lambda)^{\frac{-t}{u\lambda}}}{\ln(1+\lambda)} \Big|_0^h + \frac{nu\lambda}{\ln(1+\lambda)} \int_0^h t^{n-1}(1+\lambda)^{\frac{-t}{u\lambda}}\, dt \right].$$

Since for $\frac{1}{u\lambda} > 0$ we have $\lim_{h\to\infty} \dfrac{h^n u\lambda(1+\lambda)^{\frac{-h}{u\lambda}}}{\ln(1+\lambda)} = 0$, then

$$G^{*}_{\alpha,\lambda}\{t^n\} = \frac{nu\lambda}{\ln(1+\lambda)} G^{*}_{\alpha,\lambda}\{t^{n-1}\},$$

and so

$$G^{*}_{\alpha,\lambda}\{t^{n-1}\} = \frac{(n-1)u\lambda}{\ln(1+\lambda)} G^{*}_{\alpha,\lambda}\{t^{n-2}\}.$$

Hence, we get

$$G^*_{\alpha,\lambda}\{t^n\} = \frac{n(n-1)u^2\lambda^2}{\ln^2(1+\lambda)} G^*_{\alpha,\lambda}\{t^{n-2}\}.$$

Continiuting this process we get

$$G^*_{\alpha,\lambda}\{t^n\} = \frac{n(n-1)\dots 2\, u^n\lambda^n}{\ln^n(1+\lambda)} G^*_{\alpha,\lambda}\{1\}.$$

Also by taking $f(t) = 1$ in the Definition 1, we have

$$G^*_{\alpha,\lambda}\{1\} = u^\alpha \int_0^\infty (1+\lambda)^{-\frac{t}{u\lambda}}\, dt = u^\alpha \lim_{h\to\infty} \int_0^h (1+\lambda)^{-\frac{t}{u\lambda}}\, dt = \frac{\lambda\, u^{\alpha+1}}{\ln(1+\lambda)},$$

and the result follows.

Note that, from the Theorem 2 we have

$$\lim_{\lambda\to 0} G^*_{\alpha,\lambda}\{t^n\} = \lim_{\lambda\to 0} \frac{n!\lambda^{n+1}u^{\alpha+n+1}}{\ln^{n+1}(1+\lambda)} = n!u^{\alpha+n+1} = G_\alpha\{t^n\}$$

for $n = 0, 1, 2, \dots$.
Also note that, by using the equation (6) we give the relation between the modified Laplace-type transform and the modified degenerate Gamma function Γ^*_λ for $\lambda \in (0,1)$ as

$$G^*_{\alpha,\lambda}\{t^n\} = \Gamma^*_\lambda(n+1)u^{\alpha+n+1}.$$

Theorem 3. *The modified Laplace-type transform of the function $f(t) = e^{at}$, is given by*

$$G^*_{\alpha,\lambda}\{e^{at}\} = \frac{\lambda u^{\alpha+1}}{\ln(1+\lambda) - a\lambda u} \text{ for } u < \frac{\ln(1+\lambda)}{a\lambda} \text{ and } a \in \mathbb{R}.$$

Proof. Using the equation (7) for $f(t) = e^{at}$ and writing $(1+\lambda)^{-\frac{t}{u\lambda}} = e^{\frac{-t\ln(1+\lambda)}{\lambda u}}$ we obtain

$$\begin{aligned} G^*_{\alpha,\lambda}\{e^{at}\} &= u^\alpha \int_0^\infty (1+\lambda)^{-\frac{t}{u\lambda}} e^{at}\, dt = u^\alpha \lim_{h\to\infty} \int_0^h e^{t\left(a - \frac{\ln(1+\lambda)}{\lambda u}\right)}\, dt \\ &= \lim_{h\to\infty} \frac{\lambda u^{\alpha+1}}{a\lambda u - \ln(1+\lambda)} \left[e^{h\left(a - \frac{\ln(1+\lambda)}{\lambda u}\right)} - 1 \right] = \frac{\lambda u^{\alpha+1}}{\ln(1+\lambda) - a\lambda u} \end{aligned}$$

for $a < \frac{\ln(1+\lambda)}{\lambda u}$, and the proof is completed.

Note that, from the Theorem 3 we have

$$\lim_{\lambda\to 0} G^*_{\alpha,\lambda}\{e^{at}\} = \lim_{\lambda\to 0} \frac{\lambda u^{\alpha+1}}{\ln(1+\lambda) - a\lambda u} = \frac{u^{\alpha+1}}{1 - au} = G_\alpha\{e^{at}\}.$$

Theorem 4. *The modified Laplace-type transform of the function $f(t) = \sin at$ is given by*

$$G^*_{\alpha,\lambda}\{\sin at\} = \frac{a\lambda^2 u^{\alpha+2}}{\ln^2(1+\lambda) + a^2\lambda^2 u^2}. \tag{13}$$

Proof. By writing $\sin at = \frac{e^{iat} - e^{-iat}}{2i}$, we obtain

$$
\begin{aligned}
G^*_{\alpha,\lambda}\{\sin at\} &= u^\alpha \int_0^\infty (1+\lambda)^{-\frac{t}{u\lambda}} \sin at \, dt = u^\alpha \int_0^\infty e^{\frac{-t\ln(1+\lambda)}{\lambda u}} \left(\frac{e^{iat} - e^{-iat}}{2i} \right) dt \\
&= \frac{u^\alpha}{2i} \lim_{h\to\infty} \int_0^h \left[e^{-t\left(\frac{\ln(1+\lambda)}{\lambda u} - ia\right)} - e^{-t\left(ia + \frac{\ln(1+\lambda)}{\lambda u}\right)} \right] dt \\
&= \frac{u^\alpha}{2i} \lim_{h\to\infty} \left[\frac{\lambda u}{ia\lambda u - \ln(1+\lambda)} e^{-h\left(\frac{\ln(1+\lambda)}{\lambda u} - ia\right)} + \frac{\lambda u}{\ln(1+\lambda) - ia\lambda u} \right] \\
&\quad - \frac{u^\alpha}{2i} \lim_{h\to\infty} \left[\frac{-\lambda u}{ia\lambda u + \ln(1+\lambda)} e^{-h\left(ia + \frac{\ln(1+\lambda)}{\lambda u}\right)} + \frac{\lambda u}{\ln(1+\lambda) + ia\lambda u} \right] \\
&= \frac{\lambda u^{\alpha+1}}{2i} \left[\frac{1}{\ln(1+\lambda) - ia\lambda u} - \frac{1}{\ln(1+\lambda) + ia\lambda u} \right] = \frac{a\lambda^2 u^{\alpha+2}}{\ln^2(1+\lambda) + a^2\lambda^2 u^2}
\end{aligned}
$$

for $ia - \frac{\ln(1+\lambda)}{\lambda u} < 0$ and $ia + \frac{\ln(1+\lambda)}{\lambda u} > 0$. Also since $\frac{\ln(1+\lambda)}{\lambda} > 0$, we get the result.

Note that, from the Theorem 4 we have

$$
\lim_{\lambda\to 0} G^*_{\alpha,\lambda}\{\sin at\} = \lim_{\lambda\to 0} = \frac{a\lambda^2 u^{\alpha+2}}{\ln^2(1+\lambda) + a^2\lambda^2 u^2} = \frac{au^{\alpha+2}}{1 + a^2 u^2} = G_\alpha\{\sin at\}.
$$

Now, we give the transform of the mathematical construct Dirac's delta function δ defined as

$$
\delta(t) = \begin{cases} +\infty, & t = 0 \\ 0, & \text{otherwise} \end{cases}, \qquad \int_{-\infty}^\infty \delta(t)\, dt = 1.
$$

Theorem 5. *The modified Laplace-type transform of the Dirac's delta function is*

$$
G^*_{\alpha,\lambda}\{\delta(t-a)\} = u^\alpha (1+\lambda)^{-\frac{a}{\lambda u}}, \ \ a \geq 0.
$$

Proof. Let

$$
f_h(t-a) = \begin{cases} \frac{1}{h}, & a \leq t \leq a+h \\ 0, & \text{otherwise.} \end{cases}
$$

Then we have,

$$
\begin{aligned}
G^*_{\alpha,\lambda}\{f_h(t-a)\} &= u^\alpha \int_0^\infty (1+\lambda)^{-\frac{t}{u\lambda}} f_h(t-a)\, dt = \frac{u^\alpha}{h} \int_a^{a+h} (1+\lambda)^{-\frac{t}{u\lambda}}\, dt \\
&= -\frac{\lambda u^{\alpha+1}}{\ln(1+\lambda)h} (1+\lambda)^{-\frac{a}{u\lambda}} \left[(1+\lambda)^{-\frac{h}{u\lambda}} - 1 \right].
\end{aligned}
$$

Since $\delta(t-a)$ is the limit of f_h as $h \to 0$ we get

$$
G^*_{\alpha,\lambda}\{\delta(t-a)\} = \lim_{h\to 0} G^*_{\alpha,\lambda}\{f_h(t-a)\} = u^\alpha (1+\lambda)^{-\frac{a}{\lambda u}}
$$

by L'hospital rule, and the result follows.

Note that, from the theorem 5 we have

$$
\lim_{\lambda\to 0} G^*_{\alpha,\lambda}\{\delta(t-a)\} = \lim_{\lambda\to 0} u^\alpha (1+\lambda)^{-\frac{a}{\lambda u}} = u^\alpha e^{-\frac{a}{u}} = G_\alpha\{\delta(t-a)\}.
$$

3. Some operational properties

In this section, we present some useful operational properties of the modified Laplace-type transform. Firstly, the modified Laplace-type transform satisfies the following linearity property.

Theorem 6. *(Linearity) Let $i = 1, 2, \dots n$. If $f_i(t)$ is a function whose modified Laplace-type integral transform exists, then for any constant α_i we have*

$$G^*_{\alpha,\lambda}\left\{\sum_{i=1}^{n}\alpha_i f_i(t)\right\} = \sum_{i=1}^{n}\alpha_i G^*_{\alpha,\lambda}\{f_i(t)\}. \tag{14}$$

Proof. Let $f_i(t)$ be any function whose modified Laplace-type integral transform exists for $i = 1, 2, \dots n$. Then

$$\begin{aligned} G^*_{\alpha,\lambda}\left\{\sum_{i=1}^{n}\alpha_i f_i(t)\right\} &= u^\alpha \int_0^\infty \left[(1+\lambda)^{-\frac{t}{u\lambda}}\sum_{i=1}^{n}\alpha_i f_i(t)\right] dt \\ &= \sum_{i=1}^{n}\alpha_i u^\alpha \int_0^\infty (1+\lambda)^{-\frac{t}{u\lambda}} f_i(t)\, dt = \sum_{i=1}^{n}\alpha_i G^*_{\alpha,\lambda}\{f_i(t)\}, \end{aligned}$$

and the result follows.

Theorem 7. *The modified Laplace-type transform of the function $f(t) = \sinh at$ is given by*

$$G^*_{\alpha,\lambda}\{\sinh at\} = \frac{a\lambda^2 u^{\alpha+2}}{\ln^2(1+\lambda) - a^2\lambda^2 u^2} \text{ for } u < \frac{\ln(1+\lambda)}{a\lambda}.$$

Proof. By using the equation $\sinh at = \dfrac{e^{at} - e^{-at}}{2}$ and the linearity properity (14) we get

$$G^*_{\alpha,\lambda}\{\sinh at\} = \frac{1}{2}\left[G^*_{\alpha,\lambda}\{e^{at}\} - G^*_{\alpha,\lambda}\{e^{-at}\}\right].$$

Now, using the Theorem 3 the result follows.

Note that,

$$\lim_{\lambda\to 0} G^*_{\alpha,\lambda}\{\sinh at\} = \lim_{\lambda\to 0}\frac{a\lambda^2 u^{\alpha+2}}{\ln^2(1+\lambda) - a^2\lambda^2 u^2} = \frac{a u^{\alpha+2}}{1 - a^2u^2} = G_\alpha\{\sinh at\}.$$

Theorem 8. *(Transform of Derivatives) If $f(t), f'(t), \dots f^{(n-1)}(t)$ are continious functions on the interval $[0, L]$ and of exponential order as t goes to infinity for $t > L$ while $f^{(n)}(t)$ is piecewise-continious on the interval $[0, L]$, then*

$$\begin{aligned} G^*_{\alpha,\lambda}\{f^{(n)}(t)\} = \frac{\ln^n(1+\lambda)}{\lambda^n u^n} G^*_{\alpha,\lambda}\{f(t)\} - u^\alpha \frac{\ln^{n-1}(1+\lambda)}{\lambda^{n-1}u^{n-1}} f(0) \\ - u^\alpha \frac{\ln^{n-2}(1+\lambda)}{\lambda^{n-2}u^{n-2}} f'(0) - \dots - u^\alpha f^{(n-1)}(0), \end{aligned} \tag{15}$$

where $f^{(n)}(t) = \frac{d^n}{dt^n} f(t)$ and $n = 1, 2, 3, \dots$.

Proof. Using the Definition (1) and the integration by parts, we have

$$G^*_{\alpha,\lambda}\{f'(t)\} = u^\alpha \int_0^\infty (1+\lambda)^{-\frac{t}{u\lambda}} f'(t)\,dt = u^\alpha \lim_{h\to\infty}\int_0^h (1+\lambda)^{-\frac{t}{u\lambda}} f'(t)\,dt$$

$$= u^\alpha \lim_{h\to\infty}\left[(1+\lambda)^{-\frac{t}{u\lambda}} f(t)\Big|_0^h + \frac{\ln(1+\lambda)}{\lambda u}\int_0^h (1+\lambda)^{-\frac{t}{u\lambda}} f(t)\,dt\right] \tag{16}$$

$$= -u^\alpha f(0) + \frac{\ln(1+\lambda)}{\lambda u} G^*_{\alpha,\lambda}\{f(t)\},$$

and the equation (15) follows for $n = 1$.
Now for proving the equation (15), we use induction method on n. The equation (15) is true for $n = 1$. By using the equation (16) we can write

$$G^*_{\alpha,\lambda}\{f^{(k+1)}(t)\} = G^*_{\alpha,\lambda}\{\frac{d}{dt}f^{(k)}(t)\} = \frac{\ln(1+\lambda)}{\lambda u}\,G^*_{\alpha,\lambda}\{f^{(k)}(t)\} - u^\alpha f^n(0)$$

$$= \frac{\ln(1+\lambda)}{\lambda u}\left[\frac{\ln^k(1+\lambda)}{\lambda^k u^k}G^*_{\alpha,\lambda}\{f(t)\} - u^\alpha\,\frac{\ln^{k-1}(1+\lambda)}{\lambda^{k-1}u^{k-1}}f(0) - \ldots - u^\alpha f^{(k-1)}(0)\right] - u^\alpha f^{(k)}(0)$$

$$= \frac{\ln^{k+1}(1+\lambda)}{\lambda^k u^k}G^*_{\alpha,\lambda}\{f(t)\} - u^\alpha\,\frac{\ln^k(1+\lambda)}{\lambda^k u^k}f(0) - \ldots - u^\alpha\,\frac{\ln(1+\lambda)}{\lambda u}f^{(k-1)}(0) - u^\alpha f^{(k)}(0).$$

This proves that the equation (15) is true for $n = k+1$ and the equation (15) follows.

Note that, by using the Theorem 8 we have

$$\begin{aligned}
&\lim_{\lambda\to 0} G^*_{\alpha,\lambda}\{f^{(n)}(t)\}\\
&= \lim_{\lambda\to 0}\left[\frac{\ln^n(1+\lambda)}{\lambda^n u^n}G^*_{\alpha,\lambda}\{f(t)\} - u^\alpha\,\frac{\ln^{n-1}(1+\lambda)}{\lambda^{n-1}u^{n-1}}f(0) - \ldots - u^\alpha f^{(n-1)}(0)\right]\\
&= \frac{1}{u^n}G_\alpha\{f(t)\} - \frac{1}{u^{n-1}}f(0)u^\alpha - \frac{1}{u^{n-2}}f'(0)u^\alpha - \ldots - u^\alpha f^{(n-1)}(0)\\
&= G_\alpha\{f^{(n)}(t)\}.
\end{aligned}$$

for $n = 1, 2, 3, \ldots$.

Corollary 1. *The modified Laplace-type transform of the function $f(t) = \cos at$ is given by*

$$G^*_{\alpha,\lambda}\{\cos at\} = \frac{\lambda\ln(1+\lambda)u^{\alpha+1}}{\ln^2(1+\lambda) + a^2\lambda^2u^2}. \tag{17}$$

Proof. Let $f(t) = \frac{1}{a}\sin at$. Then $f'(t) = \cos at$ and $f(0) = 0$. Now using the linearity property (14) and the equation (16) we get

$$G^*_{\alpha,\lambda}\{\cos at\} = \frac{\ln(1+\lambda)}{a\lambda u} \text{ and } G^*_{\alpha,\lambda}\{\sin at\} = \frac{\lambda\ln(1+\lambda)u^{\alpha+1}}{\ln^2(1+\lambda) + a^2\lambda^2u^2}.$$

Corollary 2. *The modified Laplace-type transform of the function* $f(t) = \cosh at$ *is given by*

$$G^*_{\alpha,\lambda}\{\cosh at\} = \frac{\lambda \ln(1+\lambda) u^{\alpha+1}}{\ln^2(1+\lambda) - a^2\lambda^2 u^2} \text{ for } u < \frac{\ln(1+\lambda)}{a\lambda}.$$

Proof. With the similar proof of the Corollary 1 we get the result.

Note that, from the Corollaries 1 and 2 we have

$$\lim_{\lambda\to 0} G^*_{\alpha,\lambda}\{\cos at\} = \lim_{\lambda\to 0} \frac{\lambda \ln(1+\lambda) u^{\alpha+1}}{\ln^2(1+\lambda) + a^2\lambda^2 u^2} = \frac{u^{\alpha+1}}{1+a^2u^2} = G_\alpha\{\cos at\},$$

$$\lim_{\lambda\to 0} G^*_{\alpha,\lambda}\{\cosh at\} = \lim_{\lambda\to 0} \frac{\lambda \ln(1+\lambda) u^{\alpha+1}}{\ln^2(1+\lambda) - a^2\lambda^2 u^2} = \frac{u^{\alpha+1}}{1-a^2u^2} = G_\alpha\{\cosh at\}.$$

Let $F_{\alpha,\lambda}$ be the modified Laplace-type transform and F_α be the the Laplace-type integral transform of $f(t)$, $G^*_{\alpha,\lambda}\{f(t)\} = F^*_{\alpha,\lambda}(u)$ and $L_\alpha\{f(t)\} = F_\alpha(u)$. Besides the relation $\lim_{\lambda\to 0} G^*_{\alpha,\lambda}\{f(t)\} = G_\alpha\{f(t)\}$ we give the following equation relation between the modified Laplace-type transform and the Laplace-type integral transform. Since

$$\begin{aligned} G^*_{\alpha,\lambda}\{f(t)\} &= u^\alpha \int_0^\infty (1+\lambda)^{-\frac{t}{u\lambda}} f(t)\,dt = u^\alpha \int_0^\infty e^{\frac{-t\ln(1+\lambda)}{\lambda u}} f(t)\,dt \\ &= \left(\frac{\ln(1+\lambda)}{\lambda}\right)^\alpha F_\alpha\left(\frac{\lambda u}{\ln(1+\lambda)}\right), \end{aligned}$$

this implies that

$$F^*_{\alpha,\lambda}(u) = \left(\frac{\ln(1+\lambda)}{\lambda}\right)^\alpha F_\alpha\left(\frac{\lambda u}{\ln(1+\lambda)}\right). \tag{18}$$

Now replace u by $\dfrac{\ln(1+\lambda)}{\lambda}u$ in the equation (18) we have the relation

$$F_\alpha(u) = \left(\frac{\lambda}{\ln(1+\lambda)}\right)^\alpha F^*_{\alpha,\lambda}\left(\frac{\ln(1+\lambda)}{\lambda}u\right).$$

Example 1. *Suppose we want to find* $G^*_{\alpha,\lambda}\{\cos at\}$ *using the relation by the equation (18).*

Let $G^*_{\alpha,\lambda}\{f(t)\} = F^*_{\alpha,\lambda}(u)$ and $L_\alpha\{f(t)\} = F_\alpha(u)$. Since

$$L_\alpha\{cos\, at\} = F_\alpha(u) = \frac{u^{\alpha+1}}{1+a^2u^2},$$

we get

$$\begin{aligned} G^*_{\alpha,\lambda}\{cos\, at\} &= \left(\frac{\ln(1+\lambda)}{\lambda}\right)^\alpha F_\alpha\left(\frac{\lambda u}{\ln(1+\lambda)}\right) \\ &= \left(\frac{\ln(1+\lambda)}{\lambda}\right)^\alpha \left(\frac{\lambda u}{\ln(1+\lambda)}\right)^{\alpha+1} \frac{1}{1+\dfrac{a^2\lambda^2u^2}{\ln^2(1+\lambda)}} = \frac{\lambda\ln(1+\lambda)u^{\alpha+1}}{\ln^2(1+\lambda)+a^2\lambda^2u^2}. \end{aligned}$$

Theorem 9. *(The First Translation) Let* $G_\alpha\{f(t)\} = F_\alpha(u)$. *Then*

$$G^*_{\alpha,\lambda}\{e^{at}f(t)\} = \left(\frac{\ln(1+\lambda)-a\lambda u}{\lambda}\right)^\alpha F_\alpha\left(\frac{\lambda u}{\ln(1+\lambda)-a\lambda u}\right) \text{ for } u < \frac{\ln(1+\lambda)}{a\lambda}.$$

Proof. By using the equation (7) we have

$$G^*_{\alpha,\lambda}\{e^{at}f(t)\} = u^\alpha \int_0^\infty (1+\lambda)^{-\frac{t}{u\lambda}} e^{at} f(t)\,dt = u^\alpha \int_0^\infty e^{-t\left(\frac{\ln(1+\lambda)}{\lambda u}-a\right)} f(t)\,dt$$

$$= u^\alpha \left(\frac{\lambda u}{\ln(1+\lambda)-a\lambda u}\right)^\alpha \left(\frac{\ln(1+\lambda)-a\lambda u}{\lambda u}\right)^\alpha \int_0^\infty e^{-\frac{t}{\frac{\lambda u}{\ln(1+\lambda)-a\lambda u}}} f(t)\,dt$$

$$= \left(\frac{\ln(1+\lambda)-a\lambda u}{\lambda u}\right)^\alpha F_\alpha\left\{\frac{\lambda u}{\ln(1+\lambda)-a\lambda u}\right\}$$

for $\dfrac{\lambda u}{\ln(1+\lambda)-a\lambda u} > 0$, and the result follows.

Note that, by using the Theorem 9 we have

$$\lim_{\lambda\to 0} G^*_{\alpha,\lambda}\{e^{at}f(t)\} = \lim_{\lambda\to 0}\left[\left(\frac{\ln(1+\lambda)-a\lambda u}{\lambda}\right)^\alpha F_\alpha\left(\frac{\lambda u}{\ln(1+\lambda)-a\lambda u}\right)\right]$$

$$= (1-au)^\alpha F_\alpha\left\{\frac{u}{1-au}\right\} = G_\alpha\{e^{at}f(t)\},$$

and since

$$G_\alpha\{\sin bt\} = \frac{bu^{\alpha+2}}{1+b^2u^2} = F_\alpha(u),$$

$$G_\alpha\{\cos bt\} = \frac{u^{\alpha+1}}{1+b^2u^2} = F_\alpha(u)$$

in [27], we get the following results:

$$G^*_{\alpha,\lambda}\{e^{at}\sin bt\} = \frac{b\lambda^2u^{\alpha+2}}{(\ln(1+\lambda)-a\lambda u)^2+b^2\lambda^2u^2},$$

$$G^*_{\alpha,\lambda}\{e^{at}\cos bt\} = \frac{\lambda u^{\alpha+1}(\ln(1+\lambda)-a\lambda u)}{(\ln(1+\lambda)-a\lambda u)^2+b^2\lambda^2u^2}.$$

Theorem 10. *(The Second Translation) Let* $G^*_{\alpha,\lambda}\{f(t)\} = F^*_{\alpha,\lambda}(u)$. *Then for* $a \geq 0$ *we have*

$$G^*_{\alpha,\lambda}\{f(t-a)H(t-a)\} = (1+\lambda)^{\frac{-a}{\lambda u}} F^*_{\alpha,\lambda}(u),$$

where $H(t)$ *is the Heaviside function, which is defined by* $H(t) = 1$ *if* $t \geq 0$ *and* $H(t) = 0$ *if* $t < 0$.

In particular, the modified Laplace-type transform of the Heaviside function is

$$G^*_{\alpha,\lambda}\{H(t-a)\} = \frac{\lambda}{\ln(1+\lambda)}(1+\lambda)^{\frac{-a}{\lambda u}} u^{\alpha+1}.$$

Proof. We have

$$G^*_{\alpha,\lambda}\{f(t-a)H(t-a)\} = u^\alpha \int_0^\infty (1+\lambda)^{-\frac{t}{u\lambda}} f(t-a)H(t-a)\,dt$$
$$= u^\alpha \int_a^\infty (1+\lambda)^{-\frac{t}{u\lambda}} f(t-a)\,dt.$$

Now, using the change of variable $t-a=y$ we get

$$G^*_{\alpha,\lambda}\{f(t-a)H(t-a)\} = u^\alpha \int_0^\infty (1+\lambda)^{-\frac{(a+y)}{u\lambda}} f(y)\,dy$$
$$= u^\alpha (1+\lambda)^{-\frac{a}{u\lambda}} \int_0^\infty (1+\lambda)^{-\frac{y}{u\lambda}} f(y)\,dy = (1+\lambda)^{-\frac{a}{u\lambda}} F^*_{\alpha,\lambda}(u)$$

and

$$G^*_{\alpha,\lambda}\{H(t-a)\} = u^\alpha \int_0^\infty (1+\lambda)^{-\frac{t}{u\lambda}} H(t-a)\,dt$$
$$= u^\alpha \int_a^\infty (1+\lambda)^{-\frac{t}{\lambda u}}\,dt = \frac{\lambda}{\ln(1+\lambda)}(1+\lambda)^{-\frac{a}{\lambda u}} u^{\alpha+1}.$$

Note that by using the Theorem 10 we have

$$\lim_{\lambda\to 0} G^*_{\alpha,\lambda}\{f(t-a)H(t-a)\} = \lim_{\lambda\to 0}(1+\lambda)^{\frac{-a}{\lambda u}} F^*_{\alpha,\lambda}(u)$$
$$= e^{-\frac{a}{u}} F_\alpha(u) = G_\alpha\{f(t-a)H(t-a)\}, \tag{19}$$

and

$$\lim_{\lambda\to 0} G^*_{\alpha,\lambda}\{H(t-a)\} = \lim_{\lambda\to 0}\frac{\lambda}{\ln(1+\lambda)}(1+\lambda)^{\frac{-a}{\lambda u}} u^{\alpha+1} = e^{-\frac{a}{u}} u^{\alpha+1} = G_\alpha\{H(t-a)\}. \tag{20}$$

Theorem 11. *(Transform of an Integral) Let $f(t)$ be a piecewise-continious function for $t \geq 0$ and integrable. Then if $G^*_{\alpha,\lambda}\{f(t)\} = F^*_{\alpha,\lambda}(u)$ we have*

$$G^*_{\alpha,\lambda}\left\{\int_0^t f(s)\,ds\right\} = \frac{\lambda u}{\ln(1+\lambda)} F^*_{\alpha,\lambda}(u). \tag{21}$$

Proof. Let $g(t) = \int_0^t f(s)\,ds$. Then by using the Theorem 8 we get

$$G^*_{\alpha,\lambda}\{f(t)\} = G^*_{\alpha,\lambda}\{g^{'}(t)\} = \frac{\ln(1+\lambda)}{\lambda u} G^*_{\alpha,\lambda}\{g(t)\} - u^\alpha g(0) = \frac{\ln(1+\lambda)}{\lambda u} G^*_{\alpha,\lambda}\{g(t)\},$$

since $g(0)=0$. Hence,

$$F^*_{\alpha,\lambda}(u) = \frac{\ln(1+\lambda)}{\lambda u} G^*_{\alpha,\lambda}\{g(t)\},$$

and the result follows.

Note that by using the Theorem 11 we have

$$\lim_{\lambda\to 0} G^*_{\alpha,\lambda}\left\{\int_0^t f(s)\,ds\right\} = \lim_{\lambda\to 0}\frac{\lambda u}{\ln(1+\lambda)} F^*_{\alpha,\lambda}(u) = uF(u) = G_\alpha\left\{\int_0^t f(s)\,ds\right\}.$$

Theorem 12. *(Change of Scale) Let* $G^*_{\alpha,\lambda}\{f(t)\} = F^*_{\alpha,\lambda}(u)$. *Then*

$$G^*_{\alpha,\lambda}\{f(at)\} = \frac{1}{a^{\alpha+1}} F^*_{\alpha,\lambda}(au) \text{ for } a > 0. \tag{22}$$

Proof. Let $at = w$. Then

$$\begin{aligned} G^*_{\alpha,\lambda}\{f(at)\} &= u^\alpha \int_0^\infty (1+\lambda)^{-\frac{t}{u\lambda}} f(at)\, dt = \frac{u^\alpha}{a} \int_0^\infty (1+\lambda)^{-\frac{w}{\lambda(au)}} f(w)\, dw \\ &= \frac{u^\alpha}{a} \frac{(au)^\alpha}{(au)^\alpha} \int_0^\infty (1+\lambda)^{-\frac{w}{\lambda(au)}} f(w)\, dw = \frac{1}{a^{\alpha+1}} F^*_{\alpha,\lambda}(au) \end{aligned}$$

for $a > 0$, and the result follows.

Note that by using the Theorem 12 we have

$$\lim_{\lambda \to 0} G^*_{\alpha,\lambda}\{f(at)\} = \lim_{\lambda \to 0} \frac{1}{a^{\alpha+1}} F^*_{\alpha,\lambda}(au) = \frac{1}{a^{\alpha+1}} F_\alpha(au).$$

Theorem 13. *Let* $G^*_{\alpha,\lambda}\{f(t)\} = F^*_{\alpha,\lambda}(u)$ *and* $g(t) = f(t-a)$ *for* $t \geq a$ *and* $g(t) = 0$ *for* $0 \leq t < a$, $a > 0$. *Then*

$$G^*_{\alpha,\lambda}\{g(t)\} = (1+\lambda)^{-\frac{a}{u\lambda}} F^*_{\alpha,\lambda}(u).$$

Proof. Let us write

$$G^*_{\alpha,\lambda}\{g(t)\} = u^\alpha \int_0^\infty (1+\lambda)^{-\frac{t}{u\lambda}} g(t)\, dt = u^\alpha \int_a^\infty (1+\lambda)^{-\frac{t}{u\lambda}} f(t-a)\, dt.$$

Now let $t - a = y$. Then we find

$$G^*_{\alpha,\lambda}\{g(t)\} = u^\alpha \int_0^\infty (1+\lambda)^{-\frac{y+a}{u\lambda}} f(y)\, dy = (1+\lambda)^{-\frac{a}{u\lambda}} u^\alpha \int_0^\infty (1+\lambda)^{-\frac{y}{u\lambda}} f(y)\, dy,$$

and the result follows.

Now, we give an example to illustrate the last theorem.

Example 2. *For*

$$g(t) = \begin{cases} \sin t, & t \geq \frac{\pi}{2} \\ 0, & 0 \leq t < \frac{\pi}{2} \end{cases}$$

we want to find $G^*_{\alpha,\lambda}\{g(t)\}$.

Since $\cos\left(t - \frac{\pi}{2}\right) = \sin t$, we have

$$g(t) = \begin{cases} \cos\left(t - \frac{\pi}{2}\right), & t \geq \frac{\pi}{2} \\ 0, & 0 \leq t < \frac{\pi}{2}. \end{cases}$$

Then by the Theorem 13 for $f(t) = \cos t$ we have $G^*_{\alpha,\lambda}\{g(t)\} = (1+\lambda)^{-\frac{\pi}{2u\lambda}} F^*_{\alpha,\lambda}(u)$. Also, since

$$G^*_{\alpha,\lambda}\{\cos t\} = F^*_{\alpha,\lambda}(u) = \frac{\lambda \ln(1+\lambda) u^{\alpha+1}}{\ln^2(1+\lambda) + \lambda^2 u^2},$$

by the equation (17) we have

$$G^*_{\alpha,\lambda}\{g(t)\} = (1+\lambda)^{-\frac{\pi}{2u\lambda}} \frac{\lambda \ln(1+\lambda) u^{\alpha+1}}{\ln^2(1+\lambda) + \lambda^2 u^2}.$$

Theorem 14. *(Multiplication of Powers of the Variable) Let* $G^*_{\alpha,\lambda}\{f(t)\} = F^*_{\alpha,\lambda}(u)$. *Then*

$$\begin{aligned} G^*_{\alpha,\lambda}\{t^n f(t)\} &= \frac{\lambda^n}{\ln^n(1+\lambda)} u^{2n} \frac{d^n}{du^n} F^*_{\alpha,\lambda}(u) - \frac{\lambda^n}{\ln^n(1+\lambda)} \binom{n}{1} (\alpha-(n-1)) u^{2n-1} \frac{d^{n-1}}{du^{n-1}} F^*_{\alpha,\lambda}(u) \\ &+ \frac{\lambda^n}{\ln^n(1+\lambda)} \binom{n}{2} (\alpha-(n-1))(\alpha-(n-2)) u^{2n-2} \frac{d^{n-2}}{du^{n-2}} F^*_{\alpha,\lambda}(u) \\ &- \frac{\lambda^n}{\ln^n(1+\lambda)} \tbinom{n}{3} (\alpha-(n-1))(\alpha-(n-2))(\alpha-(n-3)) u^{2n-3} \frac{d^{n-3}}{du^{n-3}} F^*_{\alpha,\lambda}(u) + \dots \\ &+ (-1)^n \frac{\lambda^n}{\ln^n(1+\lambda)} (\alpha-(n-1))(\alpha-(n-2)) \dots \alpha u^n F^*_{\alpha,\lambda}(u) \end{aligned} \tag{23}$$

for $n = 1, 2, 3, \dots$.

Proof. We use the induction method on n for proving the equation (23). Since

$$\begin{aligned} \frac{d}{du} F^*_{\alpha,\lambda}(u) &= \alpha u^{\alpha-1} \int_0^\infty (1+\lambda)^{-\frac{t}{u\lambda}} f(t)\, dt + \frac{u^{\alpha-2} \ln(1+\lambda)}{\lambda} \int_0^\infty (1+\lambda)^{-\frac{t}{u\lambda}}\, t f(t)\, dt \\ &= \frac{\alpha}{u} F^*_{\alpha,\lambda}(u) + \frac{\ln(1+\lambda)}{\lambda u^2} G^*_{\alpha,\lambda}\{t f(t)\}, \end{aligned}$$

the equation (23) is true for $n = 1$. Now suppose that the equation (23) is valid for n. Since

$$G^*_{\alpha,\lambda}\{t^n f(t)\} = u^\alpha \int_0^\infty (1+\lambda)^{-\frac{t}{u\lambda}}\, t^n f(t)\, dt$$

we have

$$\frac{d}{du} G^*_{\alpha,\lambda}\{t^n f(t)\} = \frac{\alpha}{u} G^*_{\alpha,\lambda}\{t^n f(t)\} + \frac{\ln(1+\lambda)}{\lambda u^2} G^*_{\alpha,\lambda}\{t^{n+1} f(t)\}.$$

Then

$$G^*_{\alpha,\lambda}\{t^{n+1} f(t)\} = \frac{\lambda u^2}{\ln(1+\lambda)} \frac{d}{du} G^*_{\alpha,\lambda}\{t^n f(t)\} - \frac{\lambda \alpha u}{\ln(1+\lambda)} G^*_{\alpha,\lambda}\{t^n f(t)\}. \tag{24}$$

Now, by the inductive hypothesis, we get

$$G^*_{\alpha,\lambda}\{t^{n+1}f(t)\} = \frac{\lambda^{n+1}}{\ln^{n+1}(1+\lambda)}\left[u^{2n+1}\frac{d^{n+1}}{du^{n+1}}F^*_{\alpha,\lambda}(u) - \binom{n+1}{1}(\alpha-n)u^{2n+1}\frac{d^n}{du^n}F^*_{\alpha,\lambda}(u)\right]$$
$$+\frac{\lambda^{n+1}}{\ln^{n+1}(1+\lambda)}\left[\binom{n+1}{2}(\alpha-n)\left(\alpha-(n-1)\right)u^{2n}\frac{d^{n-1}}{du^{n-1}}F^*_{\alpha,\lambda}(u)+\dots\right]$$
$$+\frac{\lambda^{n+1}}{\ln^{n+1}(1+\lambda)}\left[(-1)^{n+1}(\alpha-n)\left(\alpha-(n-1)\right)\dots\alpha u^{n+1}F^*_{\alpha,\lambda}(u)\right].$$

Hence the equation (23) is valid for $n+1$, and the result follows.

Example 3. *We want to find* $G^*_{\alpha,\lambda}\{te^t\}$.

By the Theorem 14 we can write

$$G^*_{\alpha,\lambda}\{te^t\} = \frac{\lambda u^2}{\ln(1+\lambda)}\frac{d}{du}F^*_{\alpha,\lambda}(u) - \frac{\lambda\alpha u}{\ln(1+\lambda)}F^*_{\alpha,\lambda}(u),$$

where $F^*_{\alpha,\lambda}(u) = G^*_{\alpha,\lambda}\{e^t\}$. Since $G^*_{\alpha,\lambda}\{e^t\} = \dfrac{\lambda u^{\alpha+1}}{\ln(1+\lambda)-\lambda u}$ for $u < \dfrac{\ln(1+\lambda)}{\lambda}$, then

$$G^*_{\alpha,\lambda}\{te^t\} = \tfrac{\lambda u^2}{\ln(1+\lambda)}\frac{d}{du}\left(\frac{\lambda u^{\alpha+1}}{\ln(1+\lambda)-\lambda u}\right) - \tfrac{\lambda\alpha u}{\ln(1+\lambda)}\frac{\lambda u^{\alpha+1}}{\ln(1+\lambda)-\lambda u} = \frac{\lambda^2 u^{\alpha+2}}{\left(\ln(1+\lambda)-\lambda u\right)^2}.$$

Note that, we can find the same result for $G^*_{\alpha,\lambda}\{te^t\}$ by using the Theorem 9 as the following: Since

$$G^*_{\alpha,\lambda}\{e^t t\} = \left(\frac{\ln(1+\lambda)-\lambda u}{\lambda}\right)^{\alpha} F_\alpha\left(\frac{\lambda u}{\ln(1+\lambda)-\lambda u}\right)$$

where $F_\alpha(u) = G_\alpha\{t\}$ and $G_\alpha\{t\} = u^{\alpha+2}$ we get

$$G^*_{\alpha,\lambda}\{te^t\} = \frac{\lambda^2 u^{\alpha+2}}{\left(\ln(1+\lambda)-\lambda u\right)^2}.$$

Theorem 15. *(Convolution) Let* $G^*_{\alpha,\lambda}\{f(t)\} = F^*_{\alpha,\lambda}(u)$, *and* $G^*_{\alpha,\lambda}\{g(t)\} = G^*_{\alpha,\lambda}(u)$. *Then the modified Laplace-type transform of the convolution is given as*

$$G^*_{\alpha,\lambda}\{(f*g)(t)\} = \frac{1}{u^\alpha}F^*_{\alpha,\lambda}(u)G^*_{\alpha,\lambda}(u), \tag{25}$$

where $f*g$ *is the convolution of two functions defined by*

$$(f*g)(t) = \int_0^t f(x)g(t-x)\,dx. \tag{26}$$

Proof. By the equations (7) and (26) we have

$$G^*_{\alpha,\lambda}\{(f*g)(t)\} = u^\alpha\int_0^\infty (1+\lambda)^{-\frac{t}{u\lambda}}\left(\int_0^t f(x)g(t-x)\,dx\right)dt$$

$$= u^\alpha \int_0^\infty \int_x^\infty (1+\lambda)^{-\frac{t}{u\lambda}} f(x)g(t-x)\,dt\,dx.$$

Now putting $t - x = w$ we have

$$G^*_{\alpha,\lambda}\{(f*g)(t)\} = u^\alpha \int_0^\infty \int_0^\infty (1+\lambda)^{-\frac{w+x}{u\lambda}} f(x)g(w)\,dw\,dx$$
$$= \frac{1}{u^\alpha}\left(u^\alpha \int_0^\infty (1+\lambda)^{-\frac{w}{u\lambda}} g(w)\right)\left(u^\alpha \int_0^\infty (1+\lambda)^{-\frac{x}{u\lambda}} f(x)\,dx\right),$$

and the result follows.

Note that, the modified Laplace-type transform preserves the associative property concerning the convolution operator:

$$G^*_{\alpha,\lambda}\{((f*g)*h)(t)\} = G^*_{\alpha,\lambda}\{(f*(g*h))(t)\}.$$

Let $G^*_{\alpha,\lambda}\{f(t)\} = F^*_{\alpha,\lambda}(u)$. Then $f(t)$ is called as the inverse Laplace-type transform of $F^*_{\alpha,\lambda}(u)$ and defined by $G^{*^{-1}}_{\alpha,\lambda}\{F^*_{\alpha,\lambda}(u)\} = f(t)$.

Also note that, the inverse modified Laplace-type transform is linear. Namely, let $\alpha_i \in \mathbb{R}$, $G^*_{\alpha,\lambda}\{f_i(t)\} = F^*_{i,\alpha,\lambda}(u)$ for $i = 1, 2, \ldots$. Then

$$G^{*^{-1}}_{\alpha,\lambda}\left\{\sum_{i=1}^n \alpha_i F^*_{i,\alpha,\lambda}(u)\right\} = \sum_{i=1}^n \alpha_i G^{*^{-1}}_{\alpha,\lambda}\left\{F^*_{i,\alpha,\lambda}(u)\right\}.$$

4. Applications

In this section, we give examples to illustrate the use of the mentioned transform in solving certain initial value problems described by ordinary differential equations and a Volterra integral equation of the second kind.

Example 4. *Consider the first order differential equation*

$$\frac{dx}{dt} + x = 0 \quad \textit{with the condition} \quad x(0) = 1. \tag{27}$$

Applying the modified Laplace-type transform to both sides of the equation (27) and using the linearity property we get

$$-u^\alpha x(0) + \frac{\ln(1+\lambda)}{\lambda u} G^*_{\alpha,\lambda}\{x(t)\} + G^*_{\alpha,\lambda}\{x(t)\} = 0.$$

Putting $x(0) = 1$ we have

$$G^*_{\alpha,\lambda}\{x(t)\} = \frac{\lambda u^{\alpha+1}}{\ln(1+\lambda) + \lambda u}.$$

Now, applying the inverse modified Laplace-type transform, and using the Theorem 3 gives the solution $x(t) = e^{-t}$.

Example 5. *Consider the first order differential equation*

$$\frac{dx}{dt} + x = 3t \quad \text{with the condition} \quad x(0) = 1. \tag{28}$$

Applying the modified Laplace-type transform we have

$$-u^{\alpha}x(0) + \frac{\ln(1+\lambda)}{\lambda u} G^*_{\alpha,\lambda}\{x(t)\} + 2G^*_{\alpha,\lambda}\{x(t)\} = \frac{3\lambda^2 u^{\alpha+2}}{\ln^2(1+\lambda)}.$$

Then by using initial condition and partial fraction we get

$$\begin{aligned} G^*_{\alpha,\lambda}\{x(t)\} &= \frac{3\lambda^3 u^{\alpha+3}}{\ln^2(1+\lambda)\,[\ln(1+\lambda)+\lambda u]} + \frac{\lambda u^{\alpha+1}}{\ln(1+\lambda)+\lambda u} \\ &= -\frac{3\lambda u^{\alpha+1}}{\ln(1+\lambda)} + \frac{3\lambda^2 u^{\alpha+2}}{\ln^2(1+\lambda)} + \frac{4\lambda u^{\alpha+1}}{\ln(1+\lambda)+\lambda u}. \end{aligned}$$

Taking the inverse modified Laplace-type transform of the last equation and using the Theorems 2 and 3 leads to the solution $x(t) = -3 + 3t + 4e^{-t}$.

Now, we use the modified Laplace-type transform for solving a Volterra integral equation of the second kind.

Example 6. *Consider the integral equation*

$$x(t) = t^2 + \int_0^t x(v)\sin(t-v)\,dv. \tag{29}$$

By the equation (26) we write the equation (29) as

$$x(t) = t^2 + (x * \sin)(t).$$

Operating the modified Laplace-type transform on both sides to the last equation and using the Convolution Theorem 15 we have

$$G^*_{\alpha,\lambda}\{x(t)\} = G^*_{\alpha,\lambda}\{t^2\} + G^*_{\alpha,\lambda}\{(x*\sin)(t)\} = G^*_{\alpha,\lambda}\{t^2\} + \frac{1}{u^{\alpha}} G^*_{\alpha,\lambda}\{x(t)\}\, G^*_{\alpha,\lambda}\{\sin t\}.$$

Then,

$$G^*_{\alpha,\lambda}\{x(t)\}\left(1 - \frac{G^*_{\alpha,\lambda}\{\sin t\}}{u^{\alpha}}\right) = G^*_{\alpha,\lambda}\{t^2\}.$$

Now using the Theorems 2 and 4 we get

$$G^*_{\alpha,\lambda}\{x(t)\} = \frac{2\lambda^3 u^{\alpha+3}}{\ln^3(1+\lambda)} u^{\alpha} \frac{1}{u^{\alpha} - \dfrac{\lambda^2 u^{\alpha+2}}{\ln^2(1+\lambda)+\lambda^2 u^2}}$$

$$= \frac{2\lambda^3 u^{\alpha+3}}{\ln^3(1+\lambda)} \frac{\ln^2(1+\lambda)+\lambda^2 u^2}{\ln^2(1+\lambda)} = \frac{2\lambda^3 u^{\alpha+3}}{\ln^3(1+\lambda)} \frac{2\lambda^5 u^{\alpha+5}}{\ln^5(1+\lambda)}.$$

Lastly, taking the inverse modified Laplace-type transform of the last equation leads to the solution $x(t) = t^2 + \frac{1}{12}t^4$.

Table 1: The modified Laplace-type transform of some elementary functions.

$f(t) = G^{*^{-1}}_{\alpha,\lambda}\{F^*_{\alpha,\lambda}(u)\}$	$F^*_{\alpha,\lambda}(u) = G^*_{\alpha,\lambda}\{f(t)\}$
1	$\frac{\lambda\, u^{\alpha+1}}{\ln(1+\lambda)}$
t	$\frac{\lambda^2\, u^{\alpha+2}}{\ln^2(1+\lambda)}$
t^n, $(n = 1, 2, \ldots)$	$\frac{n!\lambda^{n+1} u^{\alpha+n+1}}{\ln^{n+1}(1+\lambda)}$
e^{at}	$\frac{\lambda u^{\alpha+1}}{\ln(1+\lambda) - a\lambda u}$
$\sin at$	$\frac{a\lambda^2 u^{\alpha+2}}{\ln^2(1+\lambda) + a^2\lambda^2 u^2}$
$\cos at$	$\frac{\lambda \ln(1+\lambda) u^{\alpha+1}}{\ln^2(1+\lambda) + a^2\lambda^2 u^2}$
$\sinh at$	$\frac{a\lambda^2 u^{\alpha+2}}{\ln^2(1+\lambda) - a^2\lambda^2 u^2}$
$\cosh at$	$\frac{\lambda \ln(1+\lambda) u^{\alpha+1}}{\ln^2(1+\lambda) - a^2\lambda^2 u^2}$
$e^{at}\sin bt$	$\frac{b\lambda^2 u^{\alpha+2}}{(\ln(1+\lambda) - a\lambda u)^2 + b^2\lambda^2 u^2}$
$e^{at}\cos bt$	$\frac{\lambda u^{\alpha+1}(\ln(1+\lambda) - a\lambda u)}{(\ln(1+\lambda) - a\lambda u)^2 + b^2\lambda^2 u^2}$

5. Conclusion

In the presented paper, we have considered the concept of the modified Laplace-type transform and give relations between some integral transforms in the Laplace class such as the Laplace-type, Sumudu, and Elzaki transforms. We have proved some properties and derived the modified Laplace-type transform of power functions, sine, cosine, hyperbolic sine, hyperbolic cosine, exponential function, and function derivatives. We give some operational properties such as linearity, translations, and scale preserving. Besides these, we have also examined the relation between the modified Laplace-type transform and the modified degenerate Gamma function. We have applied the new transform to solve some ordinary differential equations and a Volterra integral equation. In the future, the presented transform can be used in many scopes, such as solving various complicated problems by developing a mathematical model using some differential equations.

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