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Finite Groups with Certain $\mathcal{SSH}$-subgroups

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Abstract. Let G be a finite group. A subgroup H of G is S-permutable in G if H permutes with every Sylow subgroup of G. A subgroup H of G is called an $\mathcal{SSH}$-subgroup in G if G has an S-permutable subgroup K such that $H^{SG} = HK$ and $H^g \cap N_K(H) \leqslant H$, for all $g \in G$, where H^{SG} is the intersection of all S-permutable subgroups of G containing H. In this paper, we investigate the structure of a finite group G under the assumption that certain subgroups of prime power orders are $\mathcal{SSH}$-subgroups of G.

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1. Introduction

Throughout this paper, we assume that all groups in this paper are finite and G always denotes a group. Recall that a subgroup H of G is called permutable in G if H permutes with every subgroup of G, that is, $HK \leqslant G$, for all $K \leqslant G$; and a subgroup H is said to be S-permutable in G if H permutes with every Sylow subgroup of G. The concept of S-permutability as generalization of normality and permutability was defined by Kegel [1] in 1962.

Another generalization of normality was given by Wang [2] in 1996 as follows: A subgroup H of G is said to be c-normal in G if G has a normal subgroup K such that $G = HK$ and $H \cap K \leqslant H_G$, where $H_G = Core_G(H) = \cap_{g \in G} H^g$ is the largest normal subgroup of G contained in H. In 2000, the concept of $\mathcal{H}$-subgroup was introduced by Bianchi et al. in [3] as follows: A subgroup H of G is called an $\mathcal{H}$-subgroup in G if $H^g \cap N_G(H) \leqslant H$, for all $g \in G$.

Wei and Guo [4], in 2012, defined a new concept, named $\mathcal{HC}$-subgroup, which is a generalization of c-normality and $\mathcal{H}$-subgroup as follows: A subgroup H of G is said to be an $\mathcal{HC}$-subgroup of G if there exists a normal subgroup K of G such that $G = HK$

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and $H^g \cap N_K(H) \leq H$, for all $g \in G$. Clearly, every c-normal subgroup of G is an $\mathcal{H}C$-subgroup of G; to see that, if H is a c-normal subgroup of G, then there exists a normal subgroup K of G such that $G = HK$ and $H \cap K \leq Core_G(H)$. Thus $H^g \cap N_K(H) = (H \cap K)^g \cap N_G(H) \leq H$, for all $g \in G$ and so H is an $\mathcal{H}C$-subgroup of G. However, the converse is not true in general (see [4, Example 1]). Moreover, it is easy to see that every $\mathcal{H}$-subgroup of G is an $\mathcal{H}C$-subgroup of G, but the converse is not true in general (see [4, Example 2]).

In 2016, Asaad and Ramadan [5] introduced the concept of weakly $\mathcal{H}C$-embedded subgroup as a generalization of $\mathcal{H}C$-subgroup as follows: A subgroup H of G is said to be weakly $\mathcal{H}C$-embedded in G if there exists a normal subgroup K of G such that $H^G = HK$ and $H^g \cap N_K(H) \leq H$, for all $g \in G$, where $H^G = \cap\{N : N \trianglelefteq G \text{ and } H \leq N\}$ is the normal closure of H in G.

In 2018, AL-Gafri and Nauman [6] introduced the concept of $\mathcal{SSH}$-subgroup which is a generalization of weakly $\mathcal{H}C$-embedded subgroup as follows: A subgroup H of G is said to be an $\mathcal{SSH}$-subgroup in G if G has an S-permutable subgroup K such that $H^{SG} = HK$ and $H^g \cap N_K(H) \leqslant H$, for all $g \in G$, where H^{SG} is the intersection of all S-permutable subgroups of G containing H, that is, $H^{SG} = \cap\{L \leq G : H \leqslant L \text{ and } L \text{ is an } S\text{-permutable subgroup in } G\}$. Clearly, every weakly $\mathcal{H}C$-embedded in G is an $\mathcal{SSH}$-subgroup in G; to see that, assume that H is weakly $\mathcal{H}C$-embedded in G. Then there exists a normal subgroup T of G such that $H^G = HT$ and $H^g \cap N_T(H) \leqslant H$, for all $g \in G$. Note that H^{SG} is S-permutable in G and $H^{SG} \leqslant H^G$ by Lemma 6. So, $H^{SG} = H^{SG} \cap HT = H\left(H^{SG} \cap T\right) = HK$, where $K = H^{SG} \cap T$. Moreover, K is S-permutable in G by [1, Satz 2]. Clearly, $H^g \cap N_K(H) = H^g \cap N_G(H) \cap T \cap H^{SG} = H^g \cap N_T(H) \cap H^{SG} \leqslant H \cap H^{SG} = H$, for all $g \in G$. Thus H is an $\mathcal{SSH}$-subgroup in G. But the converse is not true in general (see [6, Example 1.5]).

Several researchers have studied the structure of finite groups by using the above mentioned concepts. For example, in 1980, Srinivasan [7] proved that if all maximal subgroups of every Sylow subgroup of a group G are normal in G, then G is supersolvable. Wang [2] got the supersolvability of the group G when all maximal subgroups of every Sylow subgroup of G are c-normal in G. Moreover, Asaad in [8] proved that if all maximal subgroups of every Sylow subgroup of G are $\mathcal{H}$-subgroups in G, then G is supersolvable. In addition, in [4], Wei and Guo obtained the same previous result by replacing $\mathcal{H}$-subgroup with $\mathcal{H}C$-subgroup. Asaad and Ramadan [5], studied extensively the structure of a finite group by using the weakly $\mathcal{H}C$-embedded subgroup concept and proved that: Let G be a group and P a Sylow p-subgroup of G. Then G is p-nilpotent if and only if $N_G(P)$ is p-nilpotent and every maximal subgroup of P is weakly $\mathcal{H}C$-embedded in G. In the same line of these studies, AL-Gafri and Nauman [6] used the $\mathcal{SSH}$-subgroup concept to get a new structure of the group G. In fact, they proved that let P be a Sylow p-subgroup of a group G, for some prime p. Then G is p-nilpotent if and only if $N_G(P)$ is p-nilpotent and every maximal subgroup of P is an $\mathcal{SSH}$-subgroup in G. Also, they proved that a group

G is supersolvable if and only if the maximal subgroups of the non-cyclic Sylow subgroups of G' are $\mathcal{SSH}$-subgroup in G. For more results along these same lines; see [9–12].

The main aim of this paper is to continue the above mentioned investigations. More precisely, we study the structure of a finite group G under the assumption that certain subgroups of prime power orders are $\mathcal{SSH}$-subgroups in G itself.

Recall that a class of group $\mathfrak{F}$ is said to be a formation if $\mathfrak{F}$ is closed under taking epimorphic images and every group G has a unique smallest normal subgroup with quotient in $\mathfrak{F}$. A formation $\mathfrak{F}$ is called saturated if it is closed under taking Frattini extensions. $\mathfrak{U}$ denotes the class of all supersolvable groups. Clearly, $\mathfrak{U}$ is a saturated formation (see [13, p. 713, Satz 8.6]).

Most of the notation is standard and can be found in [14] and [15]. In particular, $|G|$ denotes the order of G. Moreover, $\Phi(G)$, $F(G)$ and $F^\star(G)$ denote the Frattini subgroup, the Fitting subgroup and the generalized Fitting subgroup of G.

2. Preliminaries

In this section, we state some known results from the literature which will be used in proving our results.

Lemma 1. *Let H and L be normal subgroups of G and let $p \in \pi(G)$. Then, the following hold:*

(i) $\Phi(H) \leqslant \Phi(G)$.

(ii) If $L \leqslant \Phi(G)$, then $F(G/L) = F(G)/L$.

(iii) If $L \leqslant H \cap \Phi(G)$, then $F(H/L) = F(H)/L$.

Proof. For (i), see [13, III, Hilfssatz 3.3]. For (ii), and (iii), see [16, Lemma 2.7].

Lemma 2. *Let H, M and L be subgroups of a group G such that H is an $\mathcal{SSH}$- subgroup in G and $L \lhd G$. Then the following statements hold:*

(i) If $H \leqslant M$, then H is an $\mathcal{SSH}$-subgroup in M.

(ii) Assume that $L \leqslant M$. Then M is an $\mathcal{SSH}$-subgroup in G if and only if M/L is an $\mathcal{SSH}$-subgroup in G/L.

(iii) Assume that H is a p-subgroup of G and L is a p'-subgroup of G, for some prime p. Then HL and HL/L are $\mathcal{SSH}$-subgroups in G and G/L, respectively.

Proof. See [6, Lemma 2.4].

Lemma 3. *Let G be a group and let N be a nontrivial normal subgroup of G. If $N \cap \Phi(G) = 1$, then $F(N)$, the Fitting subgroup of N, is the direct product of the minimal normal subgroups of G which are contained in $F(N)$.*

Proof. See [17, Lemma 2.6].

Lemma 4. *Let G be a group and let H be an $\mathcal{H}$-subgroup in G. If H is subnormal in G, then H is normal in G.*

Proof. See [3, Theorem 6.2].

Lemma 5. *Let G be a solvable group. Suppose that $F(G)$ possesses a normal series*

$$\Phi(G) = K_0 \leqslant K_i \leqslant K_2 \leqslant \ldots \leqslant K_n = F(G),$$

such that $K_{i's}$ are normal subgroups of G and $|K_i/K_{i-1}| = prime$ $(1 \leq i \leq n)$. Then G is supersolvable.

Proof. See [13, p. 720, Satz 9.9].

Lemma 6. *Let G be a group and $H \leqslant K \leqslant G$. Then H^{SG} is S-permutable in G and $H^{SG} \leqslant H^G$.*

Proof. See [18, Lemma 2.5(1)].

Lemma 7. *Let P be an elementary abelian p-subgroup of G such that P is not cyclic. Then the following statements are equivalent:*

(i) The subgroups of order p in P are normal in G.

(ii) The maximal subgroups of P are normal in G.

Proof. See [19, Lemma 2.6].

Lemma 8. *Let G be a group and let L be a subgroup of G:*

(i) If $L \trianglelefteq G$, then $F^*(L) \leqslant F^*(G)$.

(ii) $F^*(F^*(G)) = F^*(G) \geq F(G)$. If $F^*(G)$ is solvable, then $F^*(G) = F(G)$

(iii) Suppose that P is a normal p-subgroup, then $F^*(G/\Phi(P)) = F^*(G)/\Phi(P)$

Proof. See [20, p. 123, X. 13].

Lemma 9. *Let G be a group and let H be a normal cyclic subgroup with G/H supersolvable. Then G is supersolvable.*

Proof. See [21, Theorm 1.2].

3. Main results

In the present section, we will prove some theorems, also we will give some illustrative examples and counterexamples.

We will begin our study with the following theorem:

Theorem 1. *Assume that G is a solvable group and all maximal subgroups of the non-cyclic Sylow subgroups of $F(G)$ are $\mathcal{SSH}$-subgroups of G. Then G is supersolvable.*

Proof. Assume that the result is false and let G be a counterexample of minimal order. We distinguish the following two cases:

Case 1: $\Phi(G) \neq 1$.

Then there exists a prime p such that $p||\Phi(G)|$. Since $\Phi(G) \leqslant F(G)$, it follows that $p||F(G)|$. Let P_1 be a non-cyclic Sylow p-subgroup of $\Phi(G)$. Since P_1 is characteristic in $\Phi(G) \trianglelefteq G$, we have that $P_1 \trianglelefteq G$. By Lemma 1(ii), we have that $F(G/P_1) = F(G)/P_1$. Let P_2/P_1 be a maximal subgroup of the non-cyclic Sylow p-subgroup of $F(G)/P_1$. Then P_2/P_1 is an $\mathcal{SSH}$-subgroup in G/P_1. By Lemma 2(ii), P_2 is an $\mathcal{SSH}$-subgroup in G.

Also, if Q is the non-cyclic Sylow q-subgroup of $F(G)/P_1$, then $Q = F_qP_1/P_1$, where F_q is the non-cyclic Sylow q-subgroup of $F(G)(q \neq p)$. Let M/P_1 be a maximal subgroup of F_qP_1/P_1. Then $M = (M \cap F_q)P_1$, where $M \cap F_q$ is a maximal subgroup of F_q. By hypothesis of the theorem, $M \cap F_q$ an $\mathcal{SSH}$-subgroup in G, which implies that M/P_1 is an $\mathcal{SSH}$-subgroup in G/P_1 by Lemma 2(iii). Therefore, every maximal subgroups of the non-cyclic Sylow subgroup of $F(G)/P_1$ are $\mathcal{SSH}$-subgroups in G/P_1. Then, by minimality choice of $|G|$, G/P_1 is supersolvable. Since $(G/P_1)/(\Phi(G)/P_1) \cong G/\Phi(G)$, we have $G/\Phi(G)$ is supersolvable. By a well-known Theorem of Huppert [13, p. 713, Satz 8.6], G is supersolvable, a contradiction.

Case 2: $\Phi(G) = 1$.

Let P be a non-cyclic Sylow p-subgroup of $F(G)$. Since P is characteristic in $F(G) \trianglelefteq G$, it follows that $P \trianglelefteq G$. By Lemma 1(i), $\Phi(P) \leqslant \Phi(G)$ and since $\Phi(G) = 1$, then $\Phi(P) = 1$, for every Sylow subgroup of $F(G)$. Since G is solvable and $\Phi(G) = 1$, then by [13, p. 279, Staz 4.5], we have $F(G) = R_1 \times R_2 \times R_3 \times \cdots \times R_m$, where $R_i (i = 1, \ldots, m)$ is a minimal normal subgroup of G. Clearly, $R_1 \leq P$. If $R_1 = P$, then P is a minimal normal subgroup of G. If $|R_1| = p^e$, $e > 1$, let P_1 be a maximal subgroup of $R_1 = P$. By hypothesis, P_1 is an $\mathcal{SSH}$-subgroup of G. Then G has an S-permutable subgroup T such that $P^{SG} = P_1T$ and $P_1^g \cap N_T(P_1) \leq P_1$, for all $g \in G$. Since P is a minimal S-permutable subgroup of G, we have that $P^{SG} = P = P_1T$ and $P_1^g \cap N_T(P_1) \leq P_1$, for all $g \in G$. Since $P_1 < P$, we see that $T \neq 1$ and so $P = T$ as P is abelian and so $P_1^g \cap N_T(P_1) = P_1^g \leq P_1$, for all $g \in G$. This means that P_1 is normal in G, a contradiction. Thus we may assume that R_1 is a proper subgroup of P, where P is a non-cyclic Sylow p-subgroup of $F(G)$. Since $\Phi(P) = 1$, then there exists a maximal subgroup P_1 of P such that $R_1 \not\leqslant P_1$. By hypothesis, P_1 is an $\mathcal{SSH}$-subgroup in G, then there exists an S-permutable subgroup K of G such that $(P_1)^{SG} = P_1K$ and $(P_1)^g \cap N_K(P_1) \leqslant P_1$, for all $g \in G$. Assuming that $K = P$, then we have $(P_1)^g \cap N_G(P_1) = (P_1)^g \cap K \cap N_G(P_1) = (P_1)^g \cap N_K(P_1) \leqslant P_1$. Then we get P_1 is an $\mathcal{H}$-subgroup in G and $P_1 \trianglelefteq P$. By Lemma 4, we get $P_1 \trianglelefteq G$. Now, $P_1 \cap R_1 \trianglelefteq G$ and R_1 is a minimal normal subgroup of G, means that $P_1 \cap R_1 = 1$ and since $P = P_1R_1$ we have:

$$p = |P : P_1| = |R_1 : P_1 \cap R_1| = |R_1|.$$

Thus, R_1 is a cyclic subgroup of prime order.

Set $K_i = R_1 \times R_2 \times R_3 \times \cdots \times R_i$, where $i = 1, \ldots, m$ and consider the chain

$$1 = \Phi(G) = K_0 \leqslant K_1 \leqslant K_2 \leqslant \ldots \leqslant K_m = F(G).$$

Clearly, K_i are normal subgroups of G and $|K_i/K_{i-1}| =$ prime $(1 \leq i \leq m)$. Applying Lemma 5, G is supersolvable, a contradiction.

The following example shows that the solvability of G in Theorem 1 can not be omitted.

Example 1. *Consider the group $G = N \times M$, where N is nilpotent and M is a non-abelian simple group. Clearly, G is not solvable. We notice that $F(G) = F(N) = N$ and every maximal subgroup of the non-cyclic Sylow subgroups of $F(G)$ is $\mathcal{SSH}$-subgroup of G.*

The converse of Theorem 1 is not necessary true as the following example

Example 2. *Let $G = Z/3Z \times S_3$. Then G is supersolvable, but there exists a maximal subgroup of the non-cyclic Sylow subgroups of $F(G)$ which is not $\mathcal{SSH}$-subgroup of G.*

Theorem 2. *Let G be a group with a normal solvable subgroup H such that G/H is supersolvable. If all maximal subgroups of the non-cyclic Sylow subgroup of $F(H)$ are $\mathcal{SSH}$-subgroups of G, then G is supersolvable.*

Proof. Assume that the claim is false and choose G to be a counterexample of minimal order. We distinguish the following two cases:

Case 1: $\Phi(G) \cap H \neq 1$.

Then there exists a prime p such that $p||\Phi(G) \cap H|$. Let P_1 be a non-cyclic Sylow p-subgroup of $(\Phi(G) \cap H)$. Since $(\Phi(G) \cap H)$ is a nilpotent, then $P_1 \trianglelefteq (\Phi(G) \cap H)$. Now, P_1 is a normal Hall subgroup of $(\Phi(G) \cap H)$ implies that P_1 is characteristic in $(\Phi(G) \cap H) \trianglelefteq G$, hence $P_1 \trianglelefteq G$. So, $(G/P_1)/(H/P_1) \cong G/H$ is supersolvable.

Now, we show that $F(H/P_1) = F(H)/P_1$. It is clear $F(H)/P_1$ is a normal nilpotent subgroup of H/P_1 and $F(H/P_1)$ is a largest normal nilpotent subgroup of H/P_1 so, $F(H)/P_1 \leqslant F(H/P_1)$. Set $F(H/P_1) = L/P_1$. Since L/P_1 is characteristic in $H/P_1 \trianglelefteq G/P_1$, then $L/P_1 \trianglelefteq G/P_1$. But L is a normal nilpotent subgroup of H holds by the fact that $P_1 \leqslant \Phi(G)$, then $L \leqslant F(H)$, and so $L/P_1 = F(H/P_1) \leqslant F(H)/P_1$. Therefore, $F(H)/P_1 = F(H/P_1)$. Let P_2/P_1 be a maximal subgroup of the non-cyclic Sylow

p-subgroup of $F(H)/P_1$. Then, by hypothesis, P_2/P_1 is an $\mathcal{SSH}$-subgroup in G/P_1. Thus, by minimalty choice of $|G|$, G/P_1 is supersolvable and since $P_1 \leqslant \Phi(G)$, we get $G/\Phi(G)$ is supersolvable. By Huppert's Theorem [13, p. 713, Satz 8.6], G is supersolvable, a contradiction.

Case 2: $\Phi(G) \cap H = 1$

If $H = 1$, nothing need to prove. So, assume that $H \neq 1$. By Lemma 3, $F(H)$ is a direct product of minimal normal subgroups of G which are contained in $F(H)$. Let P be a non-cyclic Sylow p-subgroup of $F(H)$. Then $P = R_1 \times R_2 \times R_3 \times \cdots \times R_t$, where $R_i (i = 1, \ldots, t)$ is a minimal normal subgroup of G. Then there exists a maximal subgroup P_1 of P, and by hypothesis, P_1 is an $\mathcal{SSH}$-subgroup in G. Then there exists an S-permutable subgroup K of G such that $(P_1)^{SG} = P_1K$ and $(P_1)^g \cap N_K(P_1) \leqslant P_1$, for all $g \in G$. Assuming that $K = P$, then we have $(P_1)^g \cap N_G(P_1) = (P_1)^g \cap K \cap N_G(P_1) = (P_1)^g \cap N_K(P_1) \leqslant P_1$. Then we get P_1 is an $\mathcal{H}$-subgroup in G and $P_1 \trianglelefteq P$. Applying Lemma 4, we get $P_1 \trianglelefteq G$. Let Q be a non-cyclic Sylow q-subgroup of $F(H)$ such that $(p, |Q|) = 1$. Now, $P_1Q \leqslant G$ and since P_1 is a normal Hall subgroup of P_1Q, it follows that P_1 is a characteristic subgroup of P_1Q. In particular, P_1 is a normal subgroup of P_1Q. Hence $Q \leqslant N_G(P_1)$ for all Sylow q-subgroup Q of $F(H)$, where $(p, |Q|) = 1$. Since P_1 is a normal subgroup of P and P_1 is a normal subgroup of P_1Q, we get P_1 is a normal subgroup of PQ. Thus we have that every maximal subgroup of P is a normal subgroup of PQ. Since P is an elementary abelian p-group and P is a non-cyclic Sylow p-subgroup, so every subgroup of order p is a normal subgroup in PQ, where $(p, |Q|) = 1$, by Lemma 7(i). On the other hand, we know that $R_i \cap Z(P) \neq 1$, where $(i = 1, ..., t)$. Let L_i be subgroup of $R_i \cap Z(P)$ of order p, where $(i = 1, ..., t)$. Then L_i is normal in P and we have L_i is subnormal in G. Now, if $L_i = P_1$, then L_i is normal in G. Also, if L_i is a proper subgroup of P_1, then L_i is an $\mathcal{H}$-subgroup in G. Applying Lemma 4, we get $L_i \trianglelefteq G$. Since R_i is a minimal normal subgroup of G, it follows that $L_i = R_i$ is a cyclic group of order p, for any i. Therefore, we can write $F(H) = R_1 \times R_2 \times R_3 \times \cdots \times R_m$, where $R_i (i = 1, \ldots, m)$ is a normal subgroup of G of prime order. We have $G/C_G(R_i)$ is isomorphic to a subgroup of $Aut(R_i)$, $G/C_G(R_i)$ is cyclic, in particular $G/C_G(R_i)$ is supersolvable. Hence, $G/\cap_{i=1}^{m} C_G(R_i)$ is supersolvable. Notice that $C_G(F(H)) = \cap_{i=1}^{m} C_G(R_i)$, so $G/C_G(F(H))$ is supersolvable. The supersolvability of G/H and $G/C_G(F(H))$ implies that $G/(H \cap C_G(F(H)) = G/C_H(F(H))$ is supersolvable. Since H is solvable, $C_H(F(H)) \leqslant F(H)$. Moreover, $F(H) \leqslant C_H(F(H))$ as $F(H)$ is abelian. Hence, $F(H) = C_H(F(H))$, and so $G/F(H)$ is supersolvable. Then there exists a chief series:

$$\bar{1} = G_m/F(H) \trianglelefteq G_{m-1}/F(H) \trianglelefteq G_{m-2}/F(H) \trianglelefteq \cdots \trianglelefteq G_0/F(H) = G/F(H),$$

where $((G_{i-1}/F(H)/(G_i/F(H))(1 \leqslant i \leqslant m)$ are cyclic groups of prime order. Then

$$F(H) = G_m \trianglelefteq G_{m-1} \trianglelefteq G_{m-2} \trianglelefteq \cdots \trianglelefteq G_0 = G, \dot{(*)}$$

where $G_{i-1}/G_i \cong ((G_{i-1}/F(H)/(G_i/F(H))(1 \leqslant i \leqslant m)$ are cyclic groups of prime order and $G_i \trianglelefteq G$. Also, we have:

$$1 = G_n \trianglelefteq G_{n-1} \trianglelefteq G_{n-2} \trianglelefteq \cdots G_{m+1} \trianglelefteq G_m = F(H), (**)$$

where $(G_{i-1}/G_i)(m+1 \leqslant i \leqslant n)$ are cyclic groups of prime order and $G_i \trianglelefteq G$. Then, we have from $(*)$ and $(**)$:

$$1 = G_n \trianglelefteq \cdots \trianglelefteq G_m = F(H) \trianglelefteq \cdots \trianglelefteq G_0 = G,$$

where $(G_{i-1}/G_i)(1 \leqslant i \leqslant n)$ are cyclic groups of prime order and $G_i \trianglelefteq G$. Hence, G is supersolvable, a contradiction.

Now, we generalize Theorem 2 to the class of saturated formation as follows:

Theorem 3. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Suppose that G is a group with a solvable normal subgroup H such that $G/H \in \mathfrak{F}$. If all maximal subgroups of the non-cyclic Sylow subgroups of $F(H)$ are $\mathcal{SSH}$-subgroups of G, then $G \in \mathfrak{F}$.*

Proof. Assume that the claim is false and choose G to be a counterexample of minimal order. We aim to obtain that there is no such counterexample of G by the following steps:

(1) $\Phi(G) \cap H = 1$

If not, $\Phi(G) \cap H \neq 1$ and then there exists a prime p such that $p||\Phi(G) \cap H|$. Let P_1 be a non-cyclic Sylow p-subgroup of $(\Phi(G) \cap H)$. Clearly, $P_1 \trianglelefteq G$ and $(G/P_1)/(H/P_1) \cong G/H \in \mathfrak{F}$. By using similar arguments as in the second paragraph of (1) in Theorem 2, we can see that $G/P_1 \in \mathfrak{F}$. But $P_1 \leqslant \Phi(G)$, then $G/\Phi(G) \in \mathfrak{F}$ and, since $\mathfrak{F}$ is saturated, we have $G \in \mathfrak{F}$, a contradiction. Thus $\Phi(G) \cap H = 1$.

(2) Let P be a non-cyclic Sylow p-subgroup of $F(H)$. Then $P = R_1 \times R_2 \times R_3 \times \cdots \times R_t$, where $R_i (i = 1, \ldots, t)$ are normal subgroups of G of order p.

By (1) and Lemma 3, we have $P = R_1 \times R_2 \times R_3 \times \cdots \times R_t$, where $R_i (i = 1, \ldots, t)$ is a minimal normal subgroup of G. It is easily follows, by a similar argument to (2) in Theorem 2, that $|R_i| = p$.

(3) $G/F(H) \in \mathfrak{F}$.

From (2), Denote $F(H) = R_1 \times R_2 \times R_3 \times \cdots \times R_r$, where $R_i (i = 1, \ldots, r)$ are minimal normal subgroups of G. We have $G/C_G(R_i)$ is isomorphic to a subgroup of $Aut(R_i)$, which implies that $G/C_G(R_i)$ is cyclic, and $G/C_G(R_i) \in \mathfrak{U}$. So $G/(\cap_{i=1}^{r} C_G(R_i)) \in \mathfrak{U}$. Notice that $C_G(F(H)) = \cap_{i=1}^{r} C_G(R_i)$, so $G/C_G(F(H)) \in \mathfrak{U} \subseteq \mathfrak{F}$. Since $G/H \in \mathfrak{F}$ and $G/C_G(F(H)) \in \mathfrak{F}$, it follows that $G/(H \cap C_G(F(H)) = G/C_H(F(H)) \in \mathfrak{F}$. As H is solvable, $C_H(F(H)) \leqslant F(H)$. Moreover, $F(H) \leqslant C_H(F(H))$ as $F(H)$ is abelian. Hence, $F(H) = C_H(F(H))$, and so $G/F(H) \in \mathfrak{F}$.

(4) If N is a minimal normal subgroup of G contained in H, then $G/N \in \mathfrak{F}$.

Let N be an arbitrary minimal normal subgroup of G contained in H. Since H is solvable, we may assume that N is an elementary abelian p-group for some prime p and $N \leqslant F(H)$. Now, we show G/N and $F(H)/N$ satisfy the hypothesis of the theorem. Consider the solvable normal subgroup $F(H)/N$. Then

$$(G/N)/(F(H)/N) \cong G/F(H) \in \mathfrak{F}.$$

To prove $G/N \in \mathfrak{F}$, we need only show that all maximal subgroups of the non-cyclic Sylow subgroups of $F(H)/N = F(F(H)/N)$ are $\mathcal{SSH}$-subgroups in G/N. Now P/N is the non-cyclic Sylow p-subgroup of $F(H)/N$, where P is the non-cyclic Sylow p-subgroup of $F(H)$. Thus if P_1/N is maximal in P/N, P_1 is maximal in P, so P_1 is an $\mathcal{SSH}$-subgroup in G by hypothesis, and P_1/N is $\mathcal{SSH}$-subgroup in G/N by Lemma 2(ii). Now suppose q is a prime different from p, so QN/N is the Sylow q-subgroup of $F(H)/N$, where Q is the Sylow q-subgroup of $F(H)$. Then any maximal subgroup of QN/N is of the form Q_1N/N, where Q_1 is a maximal subgroup of Q. Thus Q_1 is an $\mathcal{SSH}$-subgroup in G by hypothesis, so Q_1N/N is an $\mathcal{SSH}$-subgroup in G/N by Lemma 2 (ii). So G/N and $F(H)/N$ satisfy the hypotheses of the theorem. It follows that $G/N \in \mathfrak{F}$.

(5) The final contradiction

By (2) and (4), $F(H) = \langle x_1 \rangle$, is the unique minimal normal subgroup of G contained in H, so $F(H)$ is cyclic of prime order. Let $N = F(H)$, we show that N is the only minimal normal subgroup of G. Suppose that $L \neq N$ is another minimal normal subgroup of G, and consider NL/L normal subgroup of G/L. Since

$$(G/L)/(NL/L) \cong G/NL \cong (G/N)/(NL/N),$$

and $G/N \in \mathfrak{F}$, we have $(G/L)/(NL/L) \in \mathfrak{F}$. Notice that $N \cap L = 1$, hence $(NL/L) \cong N$. And so, the only maximal subgroup of the non-cyclic Sylow subgroup of $F(NL/L) = NL/L$ is trivial subgroup, which is an $\mathcal{SSH}$-subgroup in G/L. By the minimal choice of G, $G/L \in \mathfrak{F}$. So, $G \in \mathfrak{F}$, a contradiction. Thus, $N = F(H) = \langle x_1 \rangle$ is unique minimal normal in G. By (1), $\Phi(G) = \langle x_1 \rangle \cap \Phi(G) = 1$. Let M be maximal subgroup of G such that $\langle x_1 \rangle \nsubseteq M$. Then $G = \langle x_1 \rangle M$ and $\langle x_1 \rangle \cap M = 1$. If $\langle x_1 \rangle < C_G(\langle x_1 \rangle)$, then $1 < C_G(\langle x_1 \rangle) \cap M \leqslant \langle x_1 \rangle M = G$. By the unique minimal normality of $\langle x_1 \rangle$, $\langle x_1 \rangle \leqslant C_G(\langle x_1 \rangle) \cap M \leqslant M$, then $G = \langle x_1 \rangle M = M$, a contradiction. Thus, $\langle x_1 \rangle = C_G(\langle x_1 \rangle)$. It follows that $G/\langle x_1 \rangle = G/C_G(\langle x_1 \rangle) \subseteq Aut(\langle x_1 \rangle)$ is cyclic of order dividing $p-1$ and so $G/\langle x_1 \rangle) \in \mathfrak{U}$. Hence, $G \in \mathfrak{U} \subseteq \mathfrak{F}$, the final contradiction.

In the following remark, we will mention some cases.

Remark 1. *(i) Theorem 3 is not true if we omit the solvability of H. Set $G = N \times M$, where $N = SL(2,5)$, the special linear group of degree 2 and $M \in \mathfrak{U}$. Then $F(N) = Z(N) \cong Z_2$ and $G/N \cong M \in \mathfrak{U}$, but G does not belong to $\mathfrak{U}$.*

(ii) Theorem 3 is not true for saturated formations $\mathfrak{F}$ which do not contain $\mathfrak{U}$. For example, if $\mathfrak{F}$ is the saturated formation of all nilpotent groups, then the symmetric group S_3 of degree three is a counterexample.

Theorem 4. *Let G be a group with a normal subgroup H such that G/H is supersolvable. If all maximal subgroups of the non-cyclic Sylow subgroups of $F^*(H)$ are $\mathcal{SSH}$-subgroups of G, then G is supersolvable.*

Proof. Suppose that the theorem is false and assume that G is a counterexample of minimal order. Then we have:

(1) Every proper normal subgroup of G containing $F^*(H)$ is supersolvable.
If N is a proper normal subgroup of G containing $F^*(H)$, we have $N/N\cap H \cong NH/H$ is supersolvable as $NH/H \leqslant G/H$ which is supersolvable. By Lemma 8((i) and (ii)),

$$F^*(H) = F^*(F^*(H)) \leqslant F^*(H\cap N) \leqslant F^*(H),$$

so, $F^*(H) = F^*(H\cap N)$. Then all maximal subgroups of the non-cyclic Sylow subgroups of $F^*(H\cap N)$ (i. e. of $F^*(H)$) are $\mathcal{SSH}$-subgroups in G. Thus, all maximal subgroups of the non-cyclic Sylow subgroups of $F^*(H\cap N)$ (i. e. of $F^*(H)$) are $\mathcal{SSH}$-subgroups in N by Lemma 2(ii). So, we have N, $H\cap N$ satisfy the hypothesis of the theorem. The minimality of G implies that N is supersolvable.

(2) $H = G$, and $F^*(H) = F(H) < G$
If $H < G$, then H is supersolvable by (1). In particular, H is solvable, so by Lemma 9, G is solvable and $F^*(H) = F(H)$ by Lemma 8(ii), then G is supersolvable by Theorem 2, a contradiction.
If $F^*(H) = G$, then G is supersolvable by Theorem 3, a contradiction. Then $F^*(H) < G$ and it is supersolvable by (1). So, $F^*(H) = F(H)$.

(3) For any Sylow p-subgroup P of $F(G)$, $\Phi(P) = 1$, i.e. P is elementary abelian.
If there exists a Sylow p-subgroup P of $F(G)$ with $\Phi(P) \neq 1$, then consider the factor group $G/\Phi(P)$. By Lemma 8(iii), $F^*(G/\Phi(P)) = F^*(G)/\Phi(P) = F(G)/\Phi(P)$. If $P_1/\Phi(P)$ is a maximal subgroup of the non-cyclic Sylow p-subgroup $P/\Phi(P)$ of $F^*(G)/\Phi(P)$, then P_1 is a maximal subgroup of the non-cyclic Sylow p-subgroup P of $F^*(G)$. So, P_1 is an $\mathcal{SSH}$-subgroup, by hypothesis. Then $P_1/\Phi(P)$ is an $\mathcal{SSH}$-subgroup by Lemma 2(iii). If $Q^*/\Phi(P)$ is a maximal subgroup of the non-cyclic Sylow q-subgroup of $Q\Phi(P)/\Phi(P)$ of $F^*(G)/\Phi(P)$, where Q is the non-cyclic Sylow q-subgroup of $F^*(G)$ and $q \neq p$, then we can denoted $Q^* = Q_1\Phi(P)$, where Q_1 is a maximal subgroup of the non-cyclic Sylow q-subgroup of Q of $F^*(G)$. Now, Q_1 is an $\mathcal{SSH}$-subgroup (by hypothesis and by Lemma 2 (iii)), implies that $Q^*/\Phi(P)$ is an $\mathcal{SSH}$-subgroup in $G/\Phi(P)$. By minimality of G, $G/\Phi(P)$ is supersolvable. But $P \trianglelefteq G$, then $\Phi(P) \leqslant \Phi(G)$, and we get $G/\Phi(G)$ is supersolvable. By Huppert's Theorem [13, p. 713, Satz 8.6], G is supersolvable, a contradiction.

(4) There is no subgroup of prime order normal in G.
If not, let P_0 be a normal subgroup of G of prime order p. Then $P_0 \leqslant P$ as $P \trianglelefteq G$. Since $P_0 \leqslant Z(P) \leqslant Z(F(G))$, it follows that $F(G) \leqslant C_G(P_0) \leqslant G$. By (2) and Lemma 8((i) and (ii)), $F^*(G) \leqslant F^*(C_G(P_0))$. But $F^*(C_G(P_0)) \leqslant F^*(G)$. Therefore, by the fact that $C_G(P_0) \trianglelefteq G$ and Lemma 8(i), $F^*(C_G(P_0)) = F^*(G) = F(G)$. If further $C_G(P_0) < G$, then $C_G(P_0)$ is supersolvable by (1). Since $G/C_G(P_0)$ is isomorphic to a subgroup of $Aut(P_0)$, which is cyclic, we get that $G/C_G(P_0)$ is cyclic and hence solvable. But $G/C_G(P_0)$ is solvable, then G is solvable. Applying Theorem 1 implies that G is supersolvable, a con-

tradiction. If $C_G(P_0) = G$, then $P_0 \leqslant Z(G)$. By Lemma 8(iii), $F^*(G/P_0) = F^*(G)/P_0$. By using similar argument in (3), we get that all maximal subgroups of the non-cyclic Sylow subgroups of $F^*(G/P_0)$ are $\mathcal{SSH}$-subgroups in G/P_0. The minimal choice of G implies that G/P_0 is supersolvable. Therefore, by Lemma 9, G is supersolvable, a contradiction.

(5) The Final contradiction.
From (3), $\Phi(P) = 1$. Then by Lemma 3, $F(G)$ is a direct product of minimal normal subgroups of G which are contained in $F(G)$. Let P be a non-cyclic Sylow p-subgroup of $F(G)$. Then $P = R_1 \times R_2 \times R_3 \times \cdots \times R_t$, where $R_i (i = 1, \ldots, t)$ is a minimal normal subgroup of G. Let P be a non-cyclic Sylow p-subgroup of $F(G)$ and P is characteristic in $F(G) \trianglelefteq G$, then $P \trianglelefteq G$. Then there exists a maximal subgroup P_1 of P, and by hypothesis, P_1 is an $\mathcal{SSH}$-subgroup in G. Then there exists an S-permutable subgroup K of G such that $(P_1)^{SG} = P_1K$ and $(P_1)^g \cap N_K(P_1) \leqslant P_1$, for all $g \in G$. Assuming that $K = P$, then we have $(P_1)^g \cap N_G(P_1) = (P_1)^g \cap K \cap N_G(P_1) = (P_1)^g \cap N_K(P_1) \leqslant P_1$. Then we get P_1 is an $\mathcal{H}$-subgroup in G and $P_1 \trianglelefteq P$. Applying Lemma 4, we get $P_1 \trianglelefteq G$. Let Q be a non-cyclic Sylow q-subgroup of $F(G)$ such that $(p, |Q|) = 1$. Now, $P_1Q \leqslant G$ and since P_1 is a normal Hall subgroup of P_1Q, it follows that P_1 is a characteristic subgroup of P_1Q. In particular P_1 is a normal subgroup of P_1Q. Hence $Q \leqslant N_G(P_1)$ for all Sylow q-subgroup Q of $F(G)$, where $(p, |Q|) = 1$. Since P_1 is a normal subgroup of P and P_1 is a normal subgroup of P_1Q, we get P_1 is a normal subgroup of PQ. Thus we have that every maximal subgroup of P is a normal subgroup of PQ by Lemma 7(ii). Since P is an elementary abelian p-group and P is a non-cyclic Sylow p-subgroup, so every subgroup of order p is a normal subgroup in PQ, where $(p, |Q|) = 1$. On the other hand, we know that $R_i \cap Z(P) \neq 1$, where $(i = 1, ..., t)$. Let L_i be subgroup of $R_i \cap Z(P)$ of order p, , where $(i = 1, ..., t)$. Then L_i is normal in P and we have L_i is subnormal in G. Now, if $L_i = P_1$, then L_i is normal in G. Also, if L_i is a proper subgroup of P_1, then L_i is an $\mathcal{H}$-subgroup in G. Applying Lemma 4, we get $L_i \trianglelefteq G$. Since R_i is a minimal normal subgroup of G, it follows that $|L_i| = |R_i| = p$ is a cyclic group of order p, for any i, which contradict (4) completing the proof of the theorem.

As an application of Theorem 4, we have:

Theorem 5. *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. A group G lies in $\mathfrak{F}$ if and only if it has a solvable normal subgroup H such that $G/H \in \mathfrak{F}$ and all maximal subgroups of the non-cyclic Sylow subgroups of $F^\star(H)$ are $\mathcal{SSH}$-subgroups in G.*

Proof. We need only to prove the part "if". We use induction on $|G|$. By hypothesis and Lemma 2(i), we have that all maximal subgroups of the non-cyclic Sylow subgroups of $F^\star(H)$ are $\mathcal{SSH}$-subgroups of H. Then $F^\star(H) = F(H)$ as H is supersolvable by Theorem 4. Therefore, H is solvable normal subgroup of G with $G/H \in \mathfrak{F}$ and all maximal subgroups of the non-cyclic Sylow subgroups of $F(H)$ are $\mathcal{SSH}$-subgroups in G. Applying Theorem 3 yield $G \in \mathfrak{F}$. This completes the proof of the theorem.

Remark 2. *(i) Theorem 5 is not true if we omit the solvability of H. Set $G = N \times M$,* *where $N = SL(2,5)$, the special linear group of degree 2 and $M \in \mathfrak{U}$. Then $F^\star(N) = N$ and $G/N \cong M \in \mathfrak{U}$, but G does not belong to $\mathfrak{U}$.*

(ii) Theorem 5 is not true for non-saturated formation. For example, let $\mathfrak{F}$ be the formation composed of all groups G such that $G^{\mathfrak{U}}$, the supersolvable residual, is elementary Abelian. It is clear that $\mathfrak{U} \subseteq \mathfrak{F}$ and $\mathfrak{F}$ is not a saturated formation. Let $G = SL(2,3)$ and $N = Z(G)$. Then $G/N \cong A_4$, so $G/N \in \mathfrak{F}$. But G does not belong to $\mathfrak{U}$.

4. Conclusion

Due to the importance of finite groups theory and its application in abstract algebra, our study in this article focused on the structure of a finite group G assuming that some subgroups of prime power order are $\mathcal{SSH}$-subgroups. In the current article, we have reached the following results: If G is solvable and the maximal subgroups of the non-cyclic Sylow subgroups of $F(G)$ are $\mathcal{SSH}$-subgroups, then G is supersolvable. Also, let G be a group with a normal subgroup H such that G/H is supersolvable. If all maximal subgroups of the non-cyclic Sylow subgroups of $F^\star(H)$ are $\mathcal{SSH}$-subgroups of G, then G is supersolvable. Finally, several recent and classical results were generalized through the theory of formations.

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