EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 2, Article Number 5994
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Unique Fixed Point Results via Geraghty Type Contraction with Applications

Kamaleldin Abodayeh[1], Badshah-e-Rome[2], Sidra Nizam[3], Muhammad Sarwar[1,3,*]

[1] *Department of Mathematics and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia*
[2] *Department of Mathematics, Government Degree College Kabal Swat, Khyber Pakhtunkhwa, Pakistan*
[3] *Department of Mathematics, University of Malakand, Chakdara Dir(L), 18000, Khyber Pakhtunkhwa, Pakistan*

Abstract. In this research work, using the concept of Banach contraction principle and Geraghty type contractions, some unique fixed point results are established in the context of μ-extended fuzzy b-metric spaces. The developed results are applied to ensure the existence of solution to integral equation and non-linear fractional order differential equation.

2020 Mathematics Subject Classifications: 34A08, 34A12, 34B15

Key Words and Phrases: Fixed point results, Geraghty type contraction, fuzzy b-metric spaces, integral equation, fractional differential equation

1. Introduction

L. Zadeh [1] was first to propose the concept of fuzzy set in 1965. Fuzzy set theory includes the ordinary set theory as a special case. Membership degree of an element in fuzzy set is described by a real number from the unit interval as opposed to conventional set theory where it is 1 or 0.

There is evidence that the concept of distance measurement is fuzzy instead of classical. Improving this idea, Kramosil and Michalek [2] introduced the notion of fuzzy metric. Fuzzy metric is generalization of probabilistic metric. George and Veeramani [3] modified Kramosil's definition of fuzzy metric and proved that every metric induces a fuzzy metric. The idea of fixed point in fuzzy metric space ($\mathcal{FW}\tilde{\mathcal{S}}$) has been studied and explored by several authors. In 1988, Grabiec [4] proved fuzzy version of the well known Banach contraction principle. Gregori and Sapena [5] presented the concept of fuzzy contraction and proved a fixed point theorem in $\mathcal{FW}\tilde{\mathcal{S}}$ in the sense of George and

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i2.5994

Email addresses: kamal@psu.edu.sa (K. Abodayeh), baadeshah1@gmail.com (B. Rome), sidranizam2244@gmail.com (S. Nizam), sarwarswati@gmail.com (M. Sarwar)

Veeramani. In [6] existence and uniqueness of fixed points in modified intuitionistic fuzzy metric spaces is studied. V. Gupta at al.[7] established coupled fixed point theorems on V -fuzzy metric spaces, while in [8] authors studied new results on modified intuitionistic generalized fuzzy metric spaces. Nădăban [9] initiated fuzzy b-metric space ($\mathcal{F}_b\mathfrak{W}\tilde{\mathcal{S}}$) extending $\mathcal{F}\mathfrak{W}\tilde{\mathcal{S}}$. Mehmood *etal.*[10] generalized fuzzy b-metric by coming up with the concepts of extended fuzzy b-metric space($\mathcal{E}\mathcal{F}_b\mathfrak{W}\tilde{\mathcal{S}}$) and fuzzy rectangular b-metric [11]. Rome *et al.*[12] introduced μ-extended fuzzy b-metric space ($\mu\mathcal{E}\mathcal{F}_b\mathfrak{W}\tilde{\mathcal{S}}$) and generalized several fixed point results to this newly introduced fuzzy space. It was M. Geraghty [13] who first replaced the condition of Cauchy convergence for a contraction, in a complete metric space, by an equivalent functional condition. J.Martínez-Moreno*et al.*[14] obtained some new common fixed point theorems for Geraghty's type contraction mappings using the monotone property with two metrics. Ashraf *et al.*[15] established some fixed point results for Geraghty-type contraction in fuzzy b-metric space.

The aim of this paper is to explore Geraghty-type contractions in the setting of $\mu\mathcal{E}\mathcal{F}_b\mathfrak{W}\tilde{\mathcal{S}}$. Besides Banach contraction principle some well known fixed point results are generalized to $\mu\mathcal{E}\mathcal{F}_b\mathfrak{W}\tilde{\mathcal{S}}$. To demonstrate validity of the main work, a supporting example and a couple of applications to the existence of solution to non-linear fractional order differential equation and integral equation are provided.

2. Preliminaries

In the following some specific definitions and terms are recalled which are essential for main work of this paper. $\mathbb{R}$ and $\mathbb{N}$ will be used to represent sets of real numbers and positive integers respectively. $\mathfrak{S}$ will represent a non-empty set.

Definition 1. *[16] A binary operation *, mapping* $[0,1]^2$ *onto* $[0,1]$*, is called continuous* t*-norm, if for all* $\varkappa, \chi$ *and* $\tilde{z}$ *in* $[0,1]$*, it satisfies:*
(t1)$\varkappa * y = y * \varkappa$;
(t2)$(\varkappa * \chi) * \tilde{z} = \varkappa * (\chi * \tilde{z})$;
(t3) $*$ *is continuous;*
(t4) $\varkappa * 1 = \varkappa$ *for all* $\varkappa \in [0,1]$;
(t5) $\varkappa * \tilde{z} \leqslant \chi * \tilde{w}$ *whenever* $\varkappa \leqslant \chi$ *and* $\tilde{z} \leqslant \tilde{w}$.

Below are some instances of the continuous t-norm in usage: $\varkappa *_L \chi = \max\{\varkappa + \chi - 1, 0\}$, $\varkappa *_P \chi = \varkappa\chi$ and $\varkappa *_M \chi = \min\{\varkappa, \chi\}$, are respectively called Lukasievicz, product and minimum t-norm.
The relation among these t-norms is:
$*_M \geqslant *_P \geqslant *_L$. In the sequel, $*$ will stand for an arbitrary continuous t-norm.

Definition 2. *[2] A 3-tuple*$(\mathfrak{S}, \mathfrak{W}, *)$ *is called a* $\mathcal{F}\mathfrak{W}\tilde{\mathcal{S}}$*, if* $\mathfrak{W}$ *is a fuzzy set on* $\mathfrak{S}^2 x[0, +\infty)$ *satisfying the following conditions for all* $\varkappa, \tilde{y}, \tilde{z} \in \tilde{X}$ *and* $\zeta, s > 0$*:*
(Fm1) $\mathfrak{W}(\varkappa, \tilde{y}, 0) = 0$;
(Fm2) $\mathfrak{W}(\varkappa, \tilde{y}, \zeta) = 1$ *if and only if* $\varkappa = \tilde{y}$;
(Fm3) $\mathfrak{W}(\varkappa, \tilde{y}, \zeta) = \mathfrak{W}(\tilde{y}, \varkappa, \zeta)$;

(Fm4) $\mathfrak{W}(\varkappa, \tilde{y}, \zeta) * \mathfrak{W}(\tilde{y}, \tilde{z}, \tilde{s}) \leqslant \mathfrak{W}(\varkappa, \tilde{z}, \zeta + \tilde{s})$;
(Fm5) $\mathfrak{W}(\varkappa, \tilde{y}, \cdot) : [0, +\infty) \to [0, 1]$ *is left continuous.*

Definition 3. *[9] For a given* $b \in [0, +\infty)$, *the fuzzy set* $\mathfrak{W}$: $\mathfrak{S}^2 \times [0, +\infty) \to [0, 1]$ *is called* $\mathcal{F}_b\mathfrak{W}$ *if for all* $\varkappa, \tilde{y}, \tilde{z} \in \mathfrak{S}, \zeta, \tilde{s} > 0$ *it satisfies:*
(F_bm1) $\mathfrak{W}_b(\varkappa, \tilde{y}, 0) = 0$;
(F_bm2) $\mathfrak{W}_b(\varkappa, \tilde{y}, \zeta) = 1$ *if and only if* $\varkappa = \tilde{y}$;
(F_bm3) $\mathfrak{W}_b(\varkappa, \tilde{y}, \zeta) = \mathfrak{W}(\tilde{y}, \varkappa, \zeta)$;
(F_bm4) $\mathfrak{W}_b(\varkappa, \tilde{y}, \frac{\zeta}{b}) * \mathfrak{W}_b(\tilde{y}, \tilde{z}, \frac{\tilde{s}}{b}) \leqslant \mathfrak{W}_b(\varkappa, \tilde{z}, \zeta + \tilde{s})$;
(F_bm5) $\mathfrak{W}_b(\varkappa, \tilde{y}, \cdot) : (0, +\infty) \to [0, 1]$ *is continuous and* $\lim_{n \to +\infty} \mathfrak{W}_\mu(\varkappa, \tilde{y}, \zeta) = 1$.

Remark 1. $\mathcal{F}_b\mathfrak{W}\tilde{\mathcal{S}}$ *becomes* $\mathcal{F}\mathfrak{W}\tilde{\mathcal{S}}$ *for* $b = 1$.

Remark 2. *[17] Generally speaking, a* $\mathcal{F}_b\mathfrak{W}\tilde{\mathcal{S}}$ *is not continuous.*

Example 1 demonstrates that it is not necessary for a $\mathcal{F}_b\mathfrak{W}$ on $\mathfrak{S}$ to be a $\mathcal{F}\mathfrak{W}$ on $\mathfrak{S}$.

Definition 4. *[10] The ordered triple* $(\mathfrak{S}, \mathfrak{W}_e, *)$ *is called* $\mathcal{E}\mathcal{F}_b\mathfrak{W}\tilde{\mathcal{S}}$ *by function* $\alpha : \mathfrak{S} \times \mathfrak{S} \to [1, +\infty)$, *where* $\mathfrak{W}_e : \mathfrak{S} \times \mathfrak{S} \to [0, +\infty)$ *is fuzzy set which for all* $\varkappa, \tilde{y}, \tilde{z} \in \mathfrak{S}$ *satisfies the conditons:*
(F_ebm1) $\mathfrak{W}_e(\varkappa, \tilde{y}, 0) = 0$;
(F_ebm_2) $\mathfrak{W}_e(\varkappa, \tilde{y}, \zeta) = 1$, *for all* $\zeta > 0$ *if and only if* $\tilde{y} = \varkappa$;
(F_ebm_3) $\mathfrak{W}_e(\varkappa, \tilde{y}, \zeta) = \mathfrak{W}_e(\tilde{y}, \varkappa, \zeta)$;
(F_ebm_4) $\mathfrak{W}_e(\varkappa, \tilde{z}, \alpha(\varkappa, \tilde{z})(\zeta + \tilde{s}) \geqslant \mathfrak{W}_e(\varkappa, \tilde{y}, \zeta) * \mathfrak{W}_e(\tilde{y}, \tilde{z}, \tilde{s})$, *for all* $s, \zeta > 0$;
(F_ebm_5) $\mathfrak{W}_e(\varkappa, \tilde{y}, .) : (0, +\infty) \to [0, 1]$ *is continuous and* $\lim_{\zeta \to +\infty} \mathfrak{W}_e(\varkappa, \tilde{y}, \zeta) = 1$.

Definition 5. *[12] Let* $\alpha, \mu : \mathfrak{S} \times \mathfrak{S} \to [1, +\infty)$ *be bounded functions. A fuzzy set* $\mathfrak{W}_\mu : \mathfrak{S} \times \mathfrak{S} \times [0, +\infty)$ *is said to be* $\mu\mathcal{E}\mathcal{F}_b\mathfrak{W}$ *if for all* $\varkappa, \tilde{y}, \tilde{z} \in \mathfrak{S}$, *the following conditions are satisfied:*
(μe_1) $\mathfrak{W}_\mu(\varkappa, \tilde{y}, 0) = 0$;
(μe_2) $\mathfrak{W}_\mu(\varkappa, \tilde{y}, \zeta) = 1$, *for all* $\zeta > 0$ *if and only if* $\varkappa = \tilde{y}$;
(μe_3) $\mathfrak{W}_\mu(\varkappa, \tilde{y}, \zeta) = \mathfrak{W}_\mu(\tilde{y}, \varkappa, \zeta)$;
(μe_4) $\mathfrak{W}_\mu(\varkappa, \tilde{z}, \alpha(\varkappa, \tilde{z})\zeta + \mu(\varkappa, \tilde{z})\jmath) \geqslant \mathfrak{W}_\mu(\varkappa, \tilde{y}, \zeta) * \mathfrak{W}_\mu(\tilde{y}, \tilde{z}, \jmath)$, *for all* $\jmath, \zeta > 0$;
(μe_5) $\mathfrak{W}_\mu(\varkappa, \tilde{y}, .) : (0, +\infty) \to [0, 1]$ *is continuous and* $\lim_{\zeta \to +\infty} \mathfrak{W}_\mu(\varkappa, \tilde{y}, \zeta) = 1$. *Then* $(\mathfrak{S}, \mathfrak{W}_\mu, *, \alpha, \mu)$ *is called* $\mu\mathcal{E}\mathcal{F}_b\mathfrak{W}\tilde{\mathcal{S}}$,

Remark 3. *It is important to note that* $\mathcal{F}_b\mathfrak{W}\tilde{\mathcal{S}}$ *and* $\mathcal{E}\mathcal{F}_b\mathfrak{W}\tilde{\mathcal{S}}$ *are specific forms of* $\mu\mathcal{E}\mathcal{F}_b\mathfrak{W}\tilde{\mathcal{S}}$ *when* $\alpha(\varkappa, \tilde{y}) = \mu(\varkappa, \tilde{y}) = b$, *for some* $b \geqslant 1$ $\alpha(\varkappa, \tilde{y}) = \mu(\varkappa, \tilde{y})$, *respectively.*

Example 1. *[18] Let* $\mathfrak{W}(\varkappa, \tilde{y}, \zeta) = e^{-|\varkappa - \tilde{y}|^{\mathfrak{p}}/\zeta}$, *where* $\mathfrak{p} \in (1, +\infty)$. *Then* $\mathfrak{W}$ *is a* $\mathcal{F}_b\mathfrak{W}\tilde{\mathcal{S}}$ *with* $b = 2^{\mathfrak{p}-1}$.

$(\mathfrak{S}, \mathfrak{W}, *)$ in the above example is not a $\mathcal{F}\mathfrak{W}\tilde{\mathcal{S}}$ for $\mathfrak{p} = 2$.

Example 2. *[12] Let* $\mathfrak{S} = [0,1]$ *and* $\mathfrak{W}_\mu(\varkappa, \tilde{y}, \zeta) = e^{\frac{-|\varkappa-\tilde{y}|}{\zeta}}$, *for all* $\varkappa, \tilde{y} \in \mathfrak{S}$. *It can be readily confirmed that* $(\mathfrak{S}, \mathfrak{W}_\mu, *, \alpha, \mu)$ *is a G-complete* $\mu\mathcal{E}\mathcal{F}_b\mathfrak{W}\tilde{\mathcal{S}}$ *with mappings* $\alpha, \mu : \mathfrak{S} \times \mathfrak{S} \to [1, +\infty)$ *defined by* $\alpha(\varkappa, \tilde{y}) = 1 + \varkappa\tilde{y}$ *and* $\mu(\varkappa, \tilde{y}) = 1 + \varkappa + \tilde{y}$, *respectively and continuous t-norm* $*$ *as usual product.*

Definition 6. *Let* $\{\varkappa_\mathfrak{m}\}$ *be a sequence in* $(\mathfrak{S}, \mathfrak{W}_\mu, *, \alpha, \mu)$, *then:*
(1) $\{\varkappa_\mathfrak{m}\}$ *is called G-convergent to* $\varkappa_\mathfrak{o} \in \mathfrak{S}$, *if*

$$\lim_{\mathfrak{m}\to+\infty} \mathfrak{W}_\mu(\varkappa_\mathfrak{m}, \varkappa_\mathfrak{o}, \zeta) = 1 \text{ for all } \zeta > 0.$$

(2) $\{\varkappa_\mathfrak{m}\}$ *is G-Cauchy if*

$$\lim_{\mathfrak{m}\to+\infty} \mathfrak{W}_\mu(\varkappa_\mathfrak{m}, \varkappa_{\mathfrak{m}+\mathfrak{p}}, \zeta) = 1 \text{ for all } \zeta, p > 0.$$

(3) $\mathfrak{S}$ *is G-complete, if every Cauchy sequence converges to some point of* $\mathfrak{S}$.

3. Main results

Let F_b denote the collection of all mappings $\beta : [0, +\infty) \to [0, \frac{1}{b})$, that satisfy the following condition:

$$\beta(\zeta_\mathfrak{n}) \to \frac{1}{b} \quad \text{implies} \quad \zeta_\mathfrak{n} \longrightarrow 0 \text{ as } \mathfrak{n} \to +\infty, \quad \text{for a given real number} b > 1. \tag{3.1}$$

Our main results make use of the following Lemma.

Lemma 1. *[19] Let* $(\mathfrak{S}, \mathfrak{W}, *)$ *be a complete* $\mathcal{F}\mathfrak{W}\tilde{\mathcal{S}}$ *and* $\mathfrak{W}(\varkappa, \tilde{y}, k\zeta) \geqslant \mathfrak{W}(\varkappa, \tilde{y}, \zeta)$ *for all* $\varkappa, \tilde{y} \in \mathfrak{S}$. *With* $k \in (0,1)$ *and* $\zeta > 0$, *then* $\varkappa = \tilde{y}$.

Theorem 1. *Let* $(\mathfrak{S}, \mathfrak{W}, *, \alpha, \mu)$ *be a G-complete* $\mu\mathcal{E}\mathcal{F}_b\mathfrak{W}\tilde{\mathcal{S}}$ *with mappings* $\alpha, \mu : \mathfrak{S} \times \mathfrak{S} \to [1, +\infty)$. *Let* g *be a self mapping satisfying*

$$\mathfrak{W}_\mu(g\varkappa_1, g\varkappa_2, \beta(\mathfrak{W}_\mu(\varkappa_1, \varkappa_2, \zeta))\zeta) \geqslant \mathfrak{W}_\mu(\varkappa_1, \varkappa_2, \zeta), \quad \text{for all } \varkappa_1, \varkappa_2 \in \mathfrak{S} \text{ with } \beta \in F_b$$

and

$$\max\left\{\sup_{\mathfrak{p}\geqslant 1} \lim_{\mathfrak{m}\to+\infty} \alpha(\varkappa_\mathfrak{m}, \varkappa_{\mathfrak{n}+\mathfrak{p}}), \sup_{\mathfrak{p}\geqslant 1} \lim_{\mathfrak{m}\to+\infty} \mu(\varkappa_\mathfrak{m}, \varkappa_{\mathfrak{m}+\mathfrak{p}})\right\} < b, \quad \mathfrak{m}, \mathfrak{p} \in \mathbb{N}.$$

Then g *has a fixed point.*

Proof. Set $\varkappa_{\mathfrak{m}+1} = g\varkappa_\mathfrak{m}$, where $\mathfrak{m} = 0, 1, 2, \cdots$, and $\varkappa_0 \in \mathfrak{S}$ is fixed.

$$\begin{aligned} &\mathfrak{W}_\mu(\varkappa_\mathfrak{m}, \varkappa_{\mathfrak{m}+1}, \zeta) \\ &= \mathfrak{W}_\mu\left(g\varkappa_{\mathfrak{m}-1}, g\varkappa_\mathfrak{m}, \zeta\right) \\ &\geqslant \mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \zeta))}) \end{aligned} \tag{3.2}$$

$$
\begin{aligned}
&\geqslant \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-2}, \varkappa_{\mathfrak{m}-1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-2}, \varkappa_{\mathfrak{m}-1}, \zeta))}\right)\\
&\geqslant \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-3}, \varkappa_{\mathfrak{m}-2}, \frac{\zeta}{\prod_{\mathfrak{i}=1}^{3}\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-\mathfrak{i}}, \varkappa_{\mathfrak{m}+1-\mathfrak{i}}, \zeta))}\right)\\
&\vdots\\
&\geqslant \mathfrak{W}_\mu\left(\varkappa_0, \varkappa_1, \frac{\zeta}{\prod_{i=1}^{\mathfrak{m}}\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-\mathfrak{i}}, \varkappa_{\mathfrak{m}+1-\mathfrak{i}}, \zeta))}\right).\\
\text{implies}\quad &\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+1}, \zeta)\\
&\geqslant \mathfrak{W}_\mu\left(\varkappa_0, \varkappa_1, \frac{\zeta}{\prod_{i=1}^{\mathfrak{m}}\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-\mathfrak{i}}, \varkappa_{\mathfrak{m}+1-\mathfrak{i}}, \zeta))}\right). \qquad (3.3)
\end{aligned}
$$

For any $\mathfrak{p} \in \mathbb{N}$, writing $\zeta = \frac{\mathfrak{p}\zeta}{\mathfrak{p}} = \frac{\zeta}{\mathfrak{p}} + \frac{\zeta}{\mathfrak{p}} + \cdots + \frac{\zeta}{\mathfrak{p}}$, and applying (μ_4) repeatedly, we have

$$
\begin{aligned}
&\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+\mathfrak{p}}, \zeta)\\
&\geqslant \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+1}, \frac{\zeta}{p\alpha(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+\mathfrak{p}})}\right) * \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}+1}, \varkappa_{\mathfrak{m}+2}, \frac{\zeta}{p\mu(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+\mathfrak{p}})\alpha(\varkappa_{\mathfrak{m}+1}, \varkappa_{\mathfrak{m}+\mathfrak{p}})}\right) * \cdots *\\
&\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}+(\mathfrak{p}-1)}, \varkappa_{\mathfrak{m}+\mathfrak{p}}, \frac{\zeta}{\mathfrak{p}\mu(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+\mathfrak{p}})\mu(\varkappa_{\mathfrak{m}+1}, \varkappa_{\mathfrak{m}+\mathfrak{p}})\cdots\mu(\varkappa_{\mathfrak{m}+(\mathfrak{p}-3)}, \varkappa_{\mathfrak{m}+\mathfrak{p}})\alpha(\varkappa_{\mathfrak{m}+(\mathfrak{p}-2)}, \varkappa_{\mathfrak{m}+\mathfrak{p}})}\right)
\end{aligned}
$$

using inequality (3.2), we get

$$
\begin{aligned}
&\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+\mathfrak{p}}, \zeta)\\
&\geqslant \mathfrak{W}_\mu\left(\varkappa_0, \varkappa_1, \frac{\zeta}{p\alpha(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+\mathfrak{p}})\prod_{\mathfrak{i}=1}^{\mathfrak{m}}\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-\mathfrak{i}}, \varkappa_{\mathfrak{m}+1-\mathfrak{i}}, \zeta))}\right) *\\
&\mathfrak{W}_\mu\left(\varkappa_0, \varkappa_1, \frac{\zeta}{p\mu(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+\mathfrak{p}})\alpha(\varkappa_{\mathfrak{m}+1}, \varkappa_{\mathfrak{m}+\mathfrak{p}})\prod_{\mathfrak{i}=1}^{\mathfrak{m}+1}\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-\mathfrak{i}}, \varkappa_{\mathfrak{m}+1-\mathfrak{i}}, \zeta))}\right) *\\
&\mathfrak{W}_\mu\left(\varkappa_0, \varkappa_1, \frac{\zeta}{p\mu(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+\mathfrak{p}})\mu(\varkappa_{\mathfrak{m}+1}, \varkappa_{\mathfrak{m}+\mathfrak{p}})\alpha(\varkappa_{\mathfrak{n}+2}, \varkappa_{\mathfrak{m}+\mathfrak{p}})\prod_{\mathfrak{i}=1}^{\mathfrak{m}+2}\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-\mathfrak{i}}, \varkappa_{\mathfrak{m}+1-\mathfrak{i}}, \zeta))}\right) * \cdots *\\
&\mathfrak{W}_\mu\left(\varkappa_0, \varkappa_1, \frac{\zeta}{p\prod_{\mathfrak{i}=1}^{\mathfrak{p}-2}\mu(\varkappa_{\mathfrak{n}+\mathfrak{i}-1}, \varkappa_{\mathfrak{m}+\mathfrak{p}})\alpha(\varkappa_{\mathfrak{n}+(\mathfrak{p}-2)}, \varkappa_{\mathfrak{m}+\mathfrak{p}})\prod_{\mathfrak{i}=1}^{\mathfrak{m}+(\mathfrak{p}-1)}\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-\mathfrak{i}}, \varkappa_{\mathfrak{m}+1-\mathfrak{i}}, \zeta))}\right).
\end{aligned}
$$

The hypothesis $\max\left\{\sup_{\mathfrak{p}\geqslant 1}\lim_{\mathfrak{m}\to+\infty}\alpha(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+\mathfrak{p}}), \sup_{\mathfrak{p}\geqslant 1}\lim_{\mathfrak{m}\to+\infty}\mu(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+\mathfrak{p}})\right\} < b$, along with (3.1), gives

$$
\lim_{\mathfrak{m}\to+\infty}\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+\mathfrak{p}}, \zeta) \geqslant \mathfrak{W}_\mu\left(\varkappa_0, \varkappa_1, \frac{\zeta b^{\mathfrak{m}-1}}{\mathfrak{p}}\right) * \mathfrak{W}_\mu\left(\varkappa_0, \varkappa_1, \frac{\zeta b^{\mathfrak{m}-1}}{\mathfrak{p}}\right) * \cdots * \mathfrak{W}_\mu\left(\varkappa_0, \varkappa_1, \frac{\zeta b^{\mathfrak{m}-1}}{\mathfrak{p}}\right).
$$

Therefore $\lim_{\mathfrak{n}\to+\infty}\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+\mathfrak{p}}, \zeta) = 1 * 1 * ... * 1 = 1$. Therefore $\{\varkappa_{\mathfrak{m}}\}$ is Cauchy. The completeness of matric space $(\mathfrak{S}, \mathfrak{W}_u, *, \alpha, \mu)$ guarantees the existence of some $\tilde{z}$ in $(\mathfrak{S}, \mathfrak{W}_u, *, \alpha, \mu)$, such that $\lim_{\mathfrak{m}\to+\infty}\varkappa_{\mathfrak{m}} = \tilde{z}$. Thus we have

$$
\mathfrak{W}_\mu(g\tilde{z}, \tilde{z}, \zeta) \qquad \geqslant \mathfrak{W}_\mu\left(g\tilde{z}, g\varkappa_{\mathfrak{m}}, \frac{\zeta}{2\alpha(g\tilde{z}, \tilde{z})}\right) * \mathfrak{W}_\mu\left(g\varkappa_{\mathfrak{m}}, \tilde{z}, \frac{\zeta}{2\mu(g\tilde{z}, \tilde{z})}\right)
$$

$$\geqslant \lim_{\mathfrak{m}\to+\infty} \mathfrak{W}_\mu\left(\tilde{z}, \varkappa_{\mathfrak{m}}, \frac{\zeta}{2\alpha(g\tilde{z},\tilde{z})\beta(\mathfrak{W}_\mu(\tilde{z},\varkappa_{\mathfrak{m}}))}\right) * \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}+1}, \varkappa_{\mathfrak{m}}, \frac{\zeta}{2\mu(g\tilde{z},\tilde{z})}\right)$$
$$\longrightarrow 1 * 1 = 1.$$

Hence $g\tilde{z} = \tilde{z}$ is fixed point. For uniqueness assume that $g\tilde{s} = \tilde{s} \neq \tilde{z} = g\tilde{z}$, for some $\tilde{s}\epsilon\mathfrak{S}$, then

$$\begin{aligned}
\mathfrak{W}_\mu(\tilde{s},\tilde{z},\zeta) &= \mathfrak{W}_\mu(g\tilde{s}, g\tilde{z}, \zeta)\\
&\geqslant \mathfrak{W}_\mu(\tilde{s},\tilde{z}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\tilde{s},\tilde{z},\zeta))})\\
&= \mathfrak{W}_\mu(g\tilde{s}, g\tilde{z}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\tilde{s},\tilde{z},\zeta))})\\
&\geqslant \mathfrak{W}_\mu(\tilde{s},\tilde{z}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\tilde{s},\tilde{z},\zeta))^2})\\
&\vdots\\
&\geqslant \mathfrak{W}_\mu(\tilde{s},\tilde{z}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\tilde{s},\tilde{z},\zeta))^{\mathfrak{m}}})\\
&= \mathfrak{W}_\mu(\tilde{s},\tilde{z}, b^{\mathfrak{m}}\zeta).
\end{aligned}$$

That is $\mathfrak{W}_\mu(\tilde{s},\tilde{z},\zeta) \to 1 \quad$ as $\quad \mathfrak{m} \to +\infty$.
Thus $\tilde{s} = \tilde{z}$ and hence fixed point is unique.

Letting $\beta(\mathfrak{W}_\mu(\varkappa_1,\varkappa_2,\zeta)) = k$ for some $k, \epsilon[0, \frac{1}{b})$. The following Theorem's corollary, which generalizes the widely known Banach contraction theorem in the context of $\mu\mathcal{E}\mathcal{F}_b\mathfrak{W}\tilde{\mathcal{S}}$, is obtained.

Corollary 1. *Let $(\mathfrak{S}, \mathfrak{W}_u, *, \alpha, \mu)$ be a G-complete $\mu\mathcal{E}\mathcal{F}_b\mathfrak{W}\tilde{\mathcal{S}}$ with mappings $\alpha, \mu : \mathfrak{S}\times\mathfrak{S} \to [1, +\infty)$. Let g be a self mapping satisfying*

$$\mathfrak{W}_\mu(g\varkappa_1, g\varkappa_2, k\zeta) \geqslant \mathfrak{W}_\mu(\varkappa_1,\varkappa_2,\zeta), \quad \textit{for all } \varkappa_1,\varkappa_2 \in \mathfrak{S} \textit{where } \beta \in F_b.$$

Then g has a fixed point.

An example will be provided for the aforementioned Theorem in order to validate it

Example 3. *Let $\mathfrak{S} = [0, 1]$, Let $\mathfrak{W}_\mu : \mathfrak{S}\times\mathfrak{S}\times[1, +\infty) \to [0, 1]$, defined by $\mathfrak{W}_\mu(\varkappa_1,\varkappa_2,\zeta) = e^{\frac{-|\varkappa_1-\varkappa_2|}{\zeta}}$. $(\mathfrak{S}, \mathfrak{W}_u, *, \alpha, \mu)$ is $\mu\mathcal{E}\mathcal{F}_b\mathfrak{W}\tilde{\mathcal{S}}$ with b=2. Consider the mapping $g : \mathfrak{S} \to \mathfrak{S}$ defined by $g(\varkappa) = 1-\frac{1}{6}\varkappa$, and $\beta : [0, 1] \to [0, \frac{1}{2})$ as $\beta(\zeta) = \frac{1}{4}$. obviously $\beta \in F_2$. For all $\varkappa_1,\varkappa_2 \in \mathfrak{S}$, $\zeta > 0$, we have*

$$\begin{aligned}
\mathfrak{W}_\mu(g\varkappa_1, g\varkappa_2, \beta(\mathfrak{W}_\mu(\varkappa_1,\varkappa_2,\zeta))\zeta) &= e^{\frac{-|g\varkappa_1-g\varkappa_2|}{\beta(\mathfrak{W}_\mu(\varkappa_1,\varkappa_2,\zeta)\zeta)}}\\
&= e^{\frac{-|1-\frac{1}{6}\varkappa_1-(1-\frac{1}{6}\varkappa_2)|}{\frac{1}{4}\zeta}}\\
&= e^{\frac{-|-\frac{1}{6}\varkappa_1+\frac{1}{6}\varkappa_2|}{\frac{1}{4}\zeta}}
\end{aligned}$$

$$= e^{\frac{-\frac{1}{6}|\varkappa_1-\varkappa_2|}{\frac{1}{4}\zeta}}$$
$$= e^{\frac{-\frac{2}{3}|\varkappa_1-\varkappa_2|}{\zeta}}$$
$$\geqslant e^{\frac{-|\varkappa_1-\varkappa_2|}{\zeta}}.$$

This implies that

$$\mathfrak{W}_\mu(g\varkappa_1, g\varkappa_2, \beta(\mathfrak{W}_\mu(\varkappa_1,\varkappa_2,\zeta))\zeta) \geqslant \mathfrak{W}_\mu(\varkappa_1,\varkappa_2,\zeta).$$

And $g(\frac{6}{7}) = \frac{6}{7}$*. Consequently,* $\frac{6}{7}$ *is an unique fixed point of* g*. Since Theorem 1 criteria are all fulfilled.*

Theorem 2. *Consider a G-complete* $\mu\mathcal{E}\mathcal{F}_b\mathfrak{W}\tilde{\mathcal{S}}$*,* $(\mathfrak{S}, \mathfrak{W}_u, *, \alpha, \mu)$ *with mappings* $\alpha, \mu : \mathfrak{S} \times \mathfrak{S} \to [1, +\infty)$*.*
Let g *be a self mapping on* $\mathfrak{S}$ *such that*

$$\begin{aligned}
&\mathfrak{W}_\mu(g\varkappa_1, g\varkappa_2, \beta(\mathfrak{W}_\mu(\varkappa_1,\varkappa_2,\zeta))\zeta)\\
&\geqslant \min\Big\{\mathfrak{W}_\mu(\varkappa_1,\varkappa_2,\zeta), \mathfrak{W}_\mu(\varkappa_1, g\varkappa_1,\zeta), \mathfrak{W}_\mu(\varkappa_2, g\varkappa_2,\zeta),\\
&\mathfrak{W}_\mu(\varkappa_1, g\varkappa_2, (\alpha(\varkappa_1,\varkappa_2)+\mu(\varkappa_1,\varkappa_2))\zeta) * \mathfrak{W}_\mu(\varkappa_2, g\varkappa_1, (\alpha(\varkappa_1,\varkappa_2)+\mu(\varkappa_1,x_2))\zeta)\Big\},
\end{aligned}$$

for all $\varkappa_1, \varkappa_2 \in \mathfrak{S}$*. Then there is unique fixed point of* g *in* $\mathfrak{S}$*. Here* $* = *_\mathfrak{m}$*.*

Proof. Beginning in the same manner as in Theorem 1, we have

$$\begin{aligned}
&\mathfrak{W}_\mu(\varkappa_\mathfrak{m}, \varkappa_{\mathfrak{m}+1}, \zeta)\\
&= \mathfrak{W}_\mu(g\varkappa_{\mathfrak{m}-1}, g\varkappa_\mathfrak{m}, \zeta)\\
&\geqslant \min\left\{\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right), \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, g\varkappa_{\mathfrak{m}-1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right),\right.\\
&\mathfrak{W}_\mu\left(\varkappa_\mathfrak{m}, g\varkappa_\mathfrak{m}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right), \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, g\varkappa_\mathfrak{m}, \frac{(\alpha(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m})+\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m}))\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right) *\\
&\left.\mathfrak{W}_\mu\left(\varkappa_\mathfrak{m}, g\varkappa_{\mathfrak{m}-1}, \frac{(\alpha(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m})+\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m}))\zeta)}{\beta(\mathfrak{W}_\mu(a_{\mathfrak{m}-1},a_\mathfrak{m},\zeta))}\right)\right\}\\
&\geqslant \min\left\{\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right), \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right),\right.\\
&\mathfrak{W}_\mu\left(\varkappa_\mathfrak{m}, \varkappa_{\mathfrak{m}+1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right), \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}+1}, \frac{(\alpha(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m})+\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m}))\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right) *\\
&\left.\mathfrak{W}_\mu\left(\varkappa_\mathfrak{m}, \varkappa_\mathfrak{m}, \frac{(\alpha(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m})+\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m}))\zeta)}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right)\right\}\\
&\geqslant \min\left\{\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right), \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right),\right.
\end{aligned}$$

$$\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right), \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}+1}, \frac{(\alpha(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}) + \mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}))}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right) * 1\Bigg\}$$

$$\geqslant \min\left\{\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right), \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right),\right.$$
$$\left.\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}+1}, \frac{(\alpha(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}) + \mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}))\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right)\right\}$$

$$\geqslant \min\left\{\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right), \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right),\right.$$
$$\left.\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right) * \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right)\right\}. \quad (3.4)$$

If

$$\min\left\{\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right), \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right)\right\}$$
$$= \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right),$$

then inequality (3.4) implies that

$$\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+1}, \zeta\right) \geqslant \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right),$$

the remaining part of the proof is the consequence of the Lemma 1.
If

$$\min\left\{\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right), \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right)\right\}$$
$$= \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right)$$

then from (3.4) we have

$$\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+1}, \zeta\right) \geqslant \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right),$$

continuing in this way, we get

$$\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+1}, \zeta\right) \geqslant \left(\varkappa_0, \varkappa_1, \frac{\zeta}{\prod_{\mathfrak{i}=1}^{\mathfrak{m}} \beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-\mathfrak{i}}, \varkappa_{\mathfrak{m}+1-\mathfrak{i}}, \zeta))}\right).$$

The same method employed after inequality (3.2) in Theorem 1 can be used to finish the proof of this result.

Letting $\min\Big\{\mathfrak{W}_\mu(\varkappa_1, \varkappa_2, \zeta), \mathfrak{W}_\mu(\varkappa_1, g\varkappa_1, \zeta), \mathfrak{W}_\mu(\varkappa_2, g\varkappa_2, \zeta),$

$\left(\mathfrak{W}_\mu(\varkappa_1, g\varkappa_2, (\alpha(\varkappa_1,\varkappa_2)+\mu(\varkappa_1,\varkappa_2))\zeta) * \mathfrak{W}_\mu(\varkappa_2, g\varkappa_1, (\alpha(\varkappa_1,\varkappa_2)+\mu(\varkappa_1,\varkappa_2))\zeta)\right)\Big\}$
$= \mathfrak{W}_\mu(\varkappa_1,\varkappa_2,\zeta).$
We have deduced the following corollary from Theorem 2, which reduce the above theorem to Theorem 1.

Corollary 2. *Let $(\mathfrak{S}, \mathfrak{W}_u, *, \alpha, \mu)$ be a G-complete $\mu\mathcal{E}\mathcal{F}_b\mathfrak{W}\tilde{\mathcal{S}}$ with mappings $\alpha, \mu : \mathfrak{S}\times\mathfrak{S} \to [1, +\infty)$. Let g be a self mapping satisfying*

$$\mathfrak{W}_\mu(g\varkappa_1, g\varkappa_2, \beta(\mathfrak{W}_\mu(\varkappa_1,\varkappa_2,\zeta))\zeta) \geqslant \mathfrak{W}_\mu(\varkappa_1,\varkappa_2,\zeta), \quad \textit{for all } \varkappa_1,\varkappa_2 \in \mathfrak{S} \textit{ where} \beta \in F_b.$$

Then g has a fixed point.

Theorem 3. *Let $(\mathfrak{S}, \mathfrak{W}_u, *, \alpha, \mu)$ be a G-complete$\mu\mathcal{E}\mathcal{F}_b\mathfrak{W}\tilde{\mathcal{S}}$ with mappings $\alpha, \mu : \mathfrak{S}\times\mathfrak{S} \to [1, +\infty)$ and $g : \mathfrak{S} \to \mathfrak{S}$ be a mapping satisfying*

$$\begin{aligned}&\mathfrak{W}_\mu(g\varkappa_1, g\varkappa_2, \beta(\mathfrak{W}_\mu(\varkappa_1,\varkappa_2,\zeta))\zeta)\\ &\geqslant \min\left\{\mathfrak{W}_\mu(g\varkappa_1, g\varkappa_2,\zeta), \mathfrak{W}_\mu(\varkappa_1, g\varkappa_1,\zeta), \mathfrak{W}_\mu(\varkappa_2, g\varkappa_2,\zeta), \mathfrak{W}_\mu(\varkappa_1,\varkappa_2,\zeta)\right\},\end{aligned}$$

for all $\varkappa_1,\varkappa_2 \in \mathfrak{S}$ and $\beta \in F_b$. Then there is a unique $\varkappa_0$ in $\mathfrak{S}$ such that $g(\varkappa_0) = \varkappa_0$.

Proof. Beginning in the same manner as in Theorem 1, we have

$$\begin{aligned}&\mathfrak{W}_\mu(\varkappa_\mathfrak{m}, \varkappa_{\mathfrak{m}+1},\zeta)\\ &= \mathfrak{W}_\mu(g\varkappa_{\mathfrak{m}-1}, g\varkappa_\mathfrak{m},\zeta)\\ &\geqslant \min\left\{\mathfrak{W}_\mu\left(g\varkappa_{\mathfrak{m}-1}, g\varkappa_\mathfrak{m}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right), \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, g\varkappa_{\mathfrak{m}-1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right),\right.\\ &\left.\mathfrak{W}_\mu\left(\varkappa_\mathfrak{m}, g\varkappa_\mathfrak{m}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right), \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right)\right\}\\ &\geqslant \min\left\{\mathfrak{W}_\mu\left(\varkappa_\mathfrak{m}, \varkappa_{\mathfrak{m}+1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right), \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right),\right.\\ &\left.\mathfrak{W}_\mu\left(\varkappa_\mathfrak{m}, \varkappa_{\mathfrak{m}+1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right), \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right)\right\}\\ &\geqslant \min\left\{\mathfrak{W}_\mu\left(\varkappa_\mathfrak{m}, \varkappa_{\mathfrak{m}+1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right), \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right)\right\},\end{aligned}$$

employing the identical method as applied in Theorem 2 subsequent to the inequality (3.4), the existence and uniqueness of fixed point can be shown.

Theorem 4. *Let $(\mathfrak{S}, \mathfrak{W}_u, *, \alpha, \mu)$ be a G-complete $\mu\mathcal{E}\mathcal{F}_b\mathfrak{W}\tilde{\mathcal{S}}$ with mappings $\alpha, \mu : \mathfrak{S}\times\mathfrak{S} \to [1, +\infty)$. Let $g : \mathfrak{S} \to \mathfrak{S}$ be a mapping satisfying*

$$\mathfrak{W}_\mu(g\varkappa_1, g\varkappa_2, \beta(\mathfrak{W}_\mu(\varkappa_1,\varkappa_2,\zeta))\zeta) \geqslant \frac{\eta(\varkappa_1,\varkappa_2,\zeta)}{\max\left\{\mathfrak{W}_\mu(\varkappa_1, g\varkappa_1,\zeta), \mathfrak{W}_\mu(\varkappa_2, g\varkappa_2,\zeta)\right\}},$$

where $\eta(\varkappa_1, \varkappa_2, t) = \min\left\{\mathfrak{W}_\mu(g\varkappa_1, g\varkappa_2, \zeta)\mathfrak{W}_\mu(\varkappa_1, \varkappa_2, \zeta), \mathfrak{W}_\mu(\varkappa_1, g\varkappa_1, \zeta)\mathfrak{W}_\mu(\varkappa_2, g\varkappa_2, \zeta)\right\},$

for all $\varkappa_1, x_2 \in \mathfrak{S}$*, and* $\beta \in F_b$*. Then there is a unique* $\varkappa_0$ *in* $\mathfrak{S}$ *such that* $g(\varkappa_0) = \varkappa_0$*..*

Proof. Arguing as in Theorem 1, we have

$$\begin{aligned}
&\mathfrak{W}_\mu(\varkappa_\mathfrak{m}, \varkappa_{\mathfrak{m}+1}, \zeta) \\
&= \mathfrak{W}_\mu(g\varkappa_{\mathfrak{m}-1}, g\varkappa_\mathfrak{m}, \zeta) \\
&\geqslant \frac{\eta\left(\varkappa_n, \varkappa_{\mathfrak{m}+1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \zeta))}\right)}{\max\left\{\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, g\varkappa_{\mathfrak{m}-1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \zeta))}\right), \mathfrak{W}_\mu\left(\varkappa_\mathfrak{m}, g\varkappa_\mathfrak{m}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \zeta))}\right)\right\}} \quad (3.5)
\end{aligned}$$

Now,

$$\begin{aligned}
&\eta\left(\varkappa_n, \varkappa_{\mathfrak{m}+1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \zeta))}\right) \\
&= \min\left\{\mathfrak{W}_\mu\left(g\varkappa_{\mathfrak{m}-1}, g\varkappa_\mathfrak{m}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \zeta))}\right)\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \zeta))}\right),\right. \\
&\left.\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, g\varkappa_{\mathfrak{m}-1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \zeta))}\right)\mathfrak{W}_\mu\left(\varkappa_\mathfrak{m}, g\varkappa_\mathfrak{m}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \zeta))}\right)\right\} \\
&= \min\left\{\mathfrak{W}_\mu\left(\varkappa_\mathfrak{m}, \varkappa_{\mathfrak{m}+1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \zeta))}\right)\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \zeta))}\right),\right. \\
&\left.\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \zeta))}\right)\mathfrak{W}_\mu\left(\varkappa_\mathfrak{m}, \varkappa_{\mathfrak{m}+1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \zeta))}\right)\right\}. \\
&\eta\left(\varkappa_\mathfrak{m}, \varkappa_{\mathfrak{m}+1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \zeta))}\right) \\
&= \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \zeta))}\right)\mathfrak{W}_\mu\left(\varkappa_\mathfrak{m}, \varkappa_{\mathfrak{m}+1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \zeta))}\right). \quad (3.6)
\end{aligned}$$

By substituting (3.6) in (3.5), we get

$$\begin{aligned}
&\mathfrak{W}_\mu(\varkappa_\mathfrak{m}, \varkappa_{\mathfrak{m}+1}, \zeta) \\
&\geqslant \frac{\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \zeta))}\right)\mathfrak{W}_\mu\left(\varkappa_\mathfrak{m}, \varkappa_{\mathfrak{m}+1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \zeta))}\right)}{\max\left\{\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \zeta))}\right), \mathfrak{W}_\mu\left(\varkappa_\mathfrak{m}, \varkappa_{\mathfrak{m}+1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{n,\zeta}))}\right)\right\}}. \quad (3.7)
\end{aligned}$$

If

$$\begin{aligned}
&\max\left\{\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(x_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \zeta))}\right), \mathfrak{W}_\mu\left(\varkappa_\mathfrak{m}, \varkappa_{\mathfrak{m}+1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{n,\zeta}))}\right)\right\} \\
&= \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_\mathfrak{m}, \zeta))}\right),
\end{aligned}$$

as inequality (3.7) implies that

$$\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+1}, \zeta) \geqslant \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right),$$

then there is nothing to prove by the assertion stated in lemma 1.
If

$$\max\left\{\mathfrak{W}_\mu\big(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\big), \mathfrak{W}_\mu\big(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{n,\zeta}))}\big)\right\}$$
$$= \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right),$$

then from inequality (3.7) , we have

$$\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+1}, \zeta) \geqslant \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right)$$
$$\geqslant \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-2}, \varkappa_{\mathfrak{m}-1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta)).\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-2}, \varkappa_{\mathfrak{m}-1}, \zeta))}\right)$$
$$\vdots$$
$$\geqslant \mathfrak{W}_\mu\left(\varkappa_0, \varkappa_1, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta)).\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-2}, \varkappa_{\mathfrak{m}-1}, \zeta)) \ldots \beta(\mathfrak{W}_\mu(\varkappa_0, \varkappa_1, \zeta))}\right).$$

The theorem can be proven using the same method as demonstrated in Theorem 1. After considering inequality (3.2), we are able to demonstrate the fixed point's existence and uniqueness.

Theorem 5. *Let* $(\mathfrak{S}, \mathfrak{W}_u, *, \alpha, \mu)$ *be a G-complete* $\mu\mathcal{E}\mathcal{F}_b\mathfrak{W}\tilde{\mathcal{S}}$ *with mappings* $\alpha, \mu : \mathfrak{S}\times\mathfrak{S} \to [1, +\infty)$. *Let* g *be a self mapping satisfying*

$$\mathfrak{W}_\mu(g\varkappa_1, g\varkappa_2, \beta(\mathfrak{W}_\mu(\varkappa_1, \varkappa_2, \zeta))\zeta) \geqslant \lambda(\varkappa_1, \varkappa_2, \zeta) * \gamma(\varkappa_1, \varkappa_2, \zeta),$$

where

$$\lambda\big(\varkappa_1, \varkappa_2, t\big) = \min\left\{\mathfrak{W}_\mu(g\varkappa_1, g\varkappa_2, \zeta), \mathfrak{W}_\mu(\varkappa_1, g\varkappa_1, \zeta), \mathfrak{W}_\mu(\varkappa_2, g\varkappa_2, \zeta), \mathfrak{W}_\mu(\varkappa_1, \varkappa_2, \zeta)\right\}$$
$$\gamma\big(\varkappa_1, \varkappa_2, \zeta\big) = \max\left\{\mathfrak{W}_\mu(\varkappa_1, g\varkappa_2, \zeta), \mathfrak{W}_\mu(g\varkappa_1, \varkappa_2, \zeta)\right\},$$

for all $\varkappa_1, \varkappa_2 \in \mathfrak{S}$, *and* $\beta \in F_b$. *Then there will be a unique fixed point of* g *in* $\mathfrak{S}$.

Proof. Using similar arguments as used in the proof of Theorem 1, we have

$$\mathfrak{W}_\mu(x_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+1}, \zeta)$$

$$= \mathfrak{W}_\mu(g\varkappa_{\mathfrak{m}-1}, g\varkappa_{\mathfrak{m}}, \zeta)$$
$$\geqslant \lambda\left(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{n,\zeta}))}\right) * \gamma\left(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{n,\zeta}))}\right). \quad (3.8)$$

Now,

$$\lambda\left(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right)$$
$$= \min\left\{\mathfrak{W}_\mu\left(g\varkappa_{\mathfrak{m}-1}, g\varkappa_{\mathfrak{m}}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right), \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, g\varkappa_{\mathfrak{m}-1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right),\right.$$
$$\left.\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}}, g\varkappa_{\mathfrak{m}}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right), \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right)\right\}.$$
$$= \min\left\{\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right), \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right),\right.$$
$$\left.\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right), \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right)\right\}.$$
$$\lambda\left(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{n,\zeta}))}\right)$$
$$= \min\left\{\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right), \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right)\right\}. \quad (3.9)$$

Also

$$\gamma\left(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right)$$
$$= \max\left\{\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, g\varkappa_{\mathfrak{m}}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right), \mathfrak{W}_\mu\left(g\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right)\right\}$$
$$= \max\left\{\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}+1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right), \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right)\right\}$$
$$= \max\left\{\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}+1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right), 1\right\}$$
$$\gamma(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}) = 1. \quad (3.10)$$

Using equation(3.9) and (3.10) in inequality (3.8), we have

$$\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+1}, \zeta)$$
$$\geqslant \min\left\{\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right), \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right)\right\} * 1$$
$$\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+1}, \zeta)$$
$$\geqslant \min\left\{\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+1}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right), \mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right)\right\} \quad (3.11)$$

If

$$\min\left\{\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}},\varkappa_{\mathfrak{m}+1},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\zeta))}\right),\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\zeta))}\right)\right\}$$
$$=\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}},\varkappa_{\mathfrak{m}+1},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\zeta))}\right).$$

Inequality (3.11) implies that

$$\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}},\varkappa_{\mathfrak{m}+1},\zeta)\geqslant\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}},\varkappa_{\mathfrak{m}+1},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\zeta))}\right).$$

The remaining portion of the proof follows from Lemma 1.
If

$$\min\left\{\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}},\varkappa_{\mathfrak{m}+1},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\zeta))}\right),\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\zeta))}\right)\right\}$$
$$=\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\zeta))}\right),$$

then from inequality (3.8), we have

$$\begin{aligned}
&\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}},\varkappa_{\mathfrak{m}+1},\zeta)\geqslant\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\zeta))}\right)\\
&\geqslant\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-2},\varkappa_{\mathfrak{m}-1},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\zeta))\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-2},\varkappa_{\mathfrak{m}-1},\zeta))}\right)\\
&\vdots\\
&\geqslant\mathfrak{W}_\mu\left(\varkappa_0,\varkappa_1,\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\zeta))\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-2},\varkappa_{\mathfrak{m}-1},\zeta))\ldots\beta(\mathfrak{W}_\mu(\varkappa_0,\varkappa_1,\zeta))}\right).
\end{aligned}$$

Using the identical process as applied in Theorem 2 following the inequality (3.4), we can show the existence and uniqueness of fixed point.

Theorem 6. *Let* $(\mathfrak{S},\mathfrak{W}_u,*,\alpha,\mu)$ *be a G-complete* $\mu\mathcal{E}\mathcal{F}_b\mathfrak{W}\tilde{\mathcal{S}}$ *and* $b\geqslant 1$. *Let* g *be a self mapping satisfying*

$$\mathfrak{W}_\mu(g\varkappa_1,g\varkappa_2,\beta(\mathfrak{W}_\mu(\varkappa_1,\varkappa_2,\zeta))\zeta)\geqslant\frac{\lambda(\varkappa_1,\varkappa_2,\zeta)*\gamma(\varkappa_1,\varkappa_2,\zeta)}{\eta(\varkappa_1,\varkappa_2,\zeta)},$$

where

$$\lambda(\varkappa_1,\varkappa_2,\zeta)=\min\left\{\mathfrak{W}_\mu(g\varkappa_1,g\varkappa_2,\zeta)\mathfrak{W}_\mu(\varkappa_1,\varkappa_2,\zeta),\mathfrak{W}_\mu(\varkappa_1,g\varkappa_1,\zeta)\mathfrak{W}_\mu(\varkappa_2,g\varkappa_2,\zeta)\right\}$$
$$\gamma(\varkappa_1,\varkappa_2,\zeta)=\max\left\{\mathfrak{W}_\mu(\varkappa_1,g\varkappa_1,\zeta)\mathfrak{W}_\mu(\varkappa_1,g\varkappa_2,\zeta),(\mathfrak{W}_\mu(\varkappa_2,g\varkappa_1,\zeta))^2\right\}$$

$$\eta\Big(\varkappa_1,\varkappa_2,\zeta\Big) = \max\Bigg\{\mathfrak{W}_\mu(\varkappa_1, g\varkappa_1,\zeta), \mathfrak{W}_\mu(\varkappa_2, g\varkappa_2,\zeta)\Bigg\}.$$

For all $\varkappa_1,\varkappa_2 \in \mathfrak{S}$, *with* $\beta \in F_b$. *Then* g *has a unique fixed point.*

Proof. By utilizing the identical method as outlined in the previous Theorems, we arrive at the same result.

$$\begin{aligned}
&\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}},\varkappa_{\mathfrak{m}+1},\zeta)\\
&= \mathfrak{W}_\mu(g\varkappa_{\mathfrak{m}-1}, g\varkappa_{\mathfrak{m}},\zeta)\\
&\geqslant \frac{\lambda\Bigg(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\zeta))}\Bigg) * \gamma\Bigg(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\zeta))}\Bigg)}{\eta\Bigg(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\zeta))}\Bigg)}.
\end{aligned} \tag{3.12}$$

Now

$$\begin{aligned}
&\lambda\Bigg(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\zeta))}\Bigg)\\
&= \min\Bigg\{\mathfrak{W}_\mu\Bigg(gx_{\mathfrak{m}-1}, g\varkappa_{\mathfrak{m}},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\zeta))}\Bigg)\mathfrak{W}_\mu\Bigg(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\frac{\zeta}{\beta(\mathfrak{W}_\mu(x_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\zeta))}\Bigg),\\
&\mathfrak{W}_\mu\Bigg(\varkappa_{\mathfrak{m}-1}, g\varkappa_{\mathfrak{m}-1},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\zeta))}\Bigg)\mathfrak{W}_\mu\Bigg(\varkappa_{\mathfrak{m}}, g\varkappa_{\mathfrak{m}},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\zeta))}\Bigg)\Bigg\}\\
&= \min\Bigg\{\mathfrak{W}_\mu\Bigg(\varkappa_{\mathfrak{m}},\varkappa_{\mathfrak{m}+1},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\zeta))}\Bigg)\mathfrak{W}_\mu\Bigg(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\zeta))}\Bigg),\\
&\mathfrak{W}_\mu\Bigg(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_{n},\zeta))}\Bigg)\mathfrak{W}_\mu\Bigg(\varkappa_{\mathfrak{m}},\varkappa_{\mathfrak{m}+1},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\zeta))}\Bigg)\Bigg\}\\
&\lambda\Bigg(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\zeta))}\Bigg)\\
&= \mathfrak{W}_\mu\Bigg(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\zeta))}\Bigg)\mathfrak{W}_\mu\Bigg(\varkappa_{\mathfrak{m}},\varkappa_{\mathfrak{m}+1},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\zeta))}\Bigg). \qquad (3.13)\\
&\gamma\Bigg(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\zeta))}\Bigg)\\
&= \max\Bigg\{\mathfrak{W}_\mu\Bigg(\varkappa_{\mathfrak{m}-1}, g\varkappa_{\mathfrak{m}-1},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\zeta))}\Bigg)\mathfrak{W}_\mu\Bigg(\varkappa_{\mathfrak{m}-1}, g\varkappa_{\mathfrak{m}},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\zeta))}\Bigg),\\
&\Bigg(\mathfrak{W}_\mu\Bigg(\varkappa_{\mathfrak{m}}, g\varkappa_{\mathfrak{m}-1},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\zeta))}\Bigg)\Bigg)^2\Bigg\}\\
&= \max\Bigg\{\mathfrak{W}_\mu\Bigg(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\zeta))}\Bigg)\mathfrak{W}_\mu\Bigg(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}+1},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\zeta))}\Bigg),\\
&\Bigg(\mathfrak{W}_\mu\Bigg(\varkappa_{\mathfrak{m}},\varkappa_{\mathfrak{m}},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}},\zeta))}\Bigg)\Bigg)^2\Bigg\}
\end{aligned}$$

$$= \max\left\{\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right)\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1},\varkappa_{\mathfrak{m}+1},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right),1\right\}$$

$$\gamma\left(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right)=1. \tag{3.14}$$

$$\begin{aligned}&\eta\left(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\frac{\zeta}{\beta(\mathfrak{W}_\mu(x_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right)\\&=\max\left\{\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1},g\varkappa_{\mathfrak{m}-1},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right),\mathfrak{W}_\mu\left(\varkappa_\mathfrak{m},g\varkappa_\mathfrak{m},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right)\right\}\\&=\max\left\{\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right),\mathfrak{W}_\mu\left(\varkappa_\mathfrak{m},\varkappa_{\mathfrak{m}+1},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right)\right\}\end{aligned} \tag{3.15}$$

Using equation (3.13),(3.14)and (3.15) in equation(3.12), we get

$$\begin{aligned}&\mathfrak{W}_\mu(g\varkappa_{\mathfrak{m}-1},g\varkappa_\mathfrak{m},\zeta)\\&\geqslant\frac{\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right)\mathfrak{W}_\mu\left(\varkappa_\mathfrak{m},\varkappa_{\mathfrak{m}+1},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right)}{\max\left\{\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right),\mathfrak{W}_\mu\left(\varkappa_\mathfrak{m},\varkappa_{\mathfrak{m}+1},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right)\right\}}\end{aligned} \tag{3.16}$$

If

$$\begin{aligned}&\max\left\{\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right),\mathfrak{W}_\mu\left(\varkappa_\mathfrak{m},\varkappa_{\mathfrak{m}+1},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right)\right\}\\&=\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right),\end{aligned}$$

in inequality (3.16), implies that

$$\mathfrak{W}_\mu\left(\varkappa_\mathfrak{m},\varkappa_{\mathfrak{m}+1},\zeta\right)\geqslant\mathfrak{W}_\mu\left(\varkappa_\mathfrak{m},\varkappa_{\mathfrak{m}+1},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right),$$

then there is nothing to prove by Lemma 1.
If

$$\begin{aligned}&\max\left\{\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right),\mathfrak{W}_\mu\left(\varkappa_\mathfrak{m},\varkappa_{\mathfrak{m}+1},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right)\right\}\\&=\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right),\end{aligned}$$

then inequality (3.16), implies that

$$\mathfrak{W}_\mu\left(\varkappa_\mathfrak{m},\varkappa_{\mathfrak{m}+1},\zeta\right)\geqslant\mathfrak{W}_\mu\left(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\frac{\zeta}{\beta(\mathfrak{W}_\mu(\varkappa_{\mathfrak{m}-1},\varkappa_\mathfrak{m},\zeta))}\right).$$

Continuing in the same way, we get

$$\begin{aligned}
&\mathfrak{W}_{\mu}(\varkappa_{\mathfrak{m}}, \varkappa_{\mathfrak{m}+1}, \zeta) \geqslant \mathfrak{W}_{\mu}\left(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \frac{\zeta}{\beta(\mathfrak{W}_{\mu}(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))}\right)\\
&\geqslant \mathfrak{W}_{\mu}\left(\varkappa_{\mathfrak{m}-2}, \varkappa_{\mathfrak{m}-1}, \frac{\zeta}{\beta(\mathfrak{W}_{\mu}(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))\beta(\mathfrak{W}_{\mu}(\varkappa_{\mathfrak{m}-2}, \varkappa_{\mathfrak{m}-1}, \zeta))}\right)\\
&\vdots\\
&\geqslant \mathfrak{W}_{\mu}\left(\varkappa_{0}, \varkappa_{1}, \frac{\zeta}{\beta(\mathfrak{W}_{\mu}(\varkappa_{\mathfrak{m}-1}, \varkappa_{\mathfrak{m}}, \zeta))\beta(\mathfrak{W}_{\mu}(\varkappa_{\mathfrak{m}-2}, \varkappa_{\mathfrak{m}-1}, \zeta))\cdots\beta(\mathfrak{W}_{\mu}(\varkappa_{0}, \varkappa_{1}, \zeta))}\right)
\end{aligned}$$

using the identical method applied subsequent to Theorem 1 right after the inequality (3.2), we have the capability to demonstrate both the presence and exclusivity of a fixed point.

4. Application to Nonlinear Fractional Differential Equations

In this section, existence of solution to a nonlinear fractional order differential equation ($\mathcal{NFDE}$) is investigated via the results established in this manuscript. The set $\tilde{\mathcal{S}} \equiv C([0,1],\mathbb{R})$, furnished with norm $\|\varkappa\| := \sup_{s\in[0,1]} |\varkappa(s)|$ is Banach space.

Consider

$$D^{\xi}_{0+}(\tilde{\mathrm{u}}(\zeta)) = \mathtt{g}(\zeta, \tilde{\mathrm{u}}(\zeta)), \quad \zeta \in]0,1[, \tag{4.1}$$

with conditions

$$\varkappa(0) + \acute{\varkappa}(0) = 0, \quad \varkappa(1) + \acute{\varkappa}(1) = 0,$$

here $\varkappa \in \tilde{\mathcal{S}}, \xi \in]1,2]$ and $\mathtt{g} : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous mapping.

Now $\varkappa \in \tilde{\mathcal{S}}$ will be a solution to (4.1) if it solves the following equation

$$\begin{aligned}
\varkappa(\tilde{\mathtt{s}}) =& \frac{1}{\Gamma(\xi)} \int_0^1 (1-\tilde{r})^{\xi-1}(1-\tilde{s})\mathtt{g}(\tilde{r}, \varkappa(\tilde{\tau}))\mathtt{d}\tilde{r} + \\
& \frac{1}{\Gamma(\xi-1)} \int_0^1 (1-\tilde{r})^{\xi-2}(1-\tilde{s})\mathtt{g}(\tilde{r}, \varkappa(\tilde{r}))\mathtt{d}\tilde{r} + \frac{1}{\Gamma(\xi)} \int_0^{\tilde{s}} (\tilde{s}-\tilde{r})^{\xi-1}\mathtt{g}(\tilde{r}, \varkappa(\tilde{r}))\mathtt{d}\tilde{r}.
\end{aligned} \tag{4.2}$$

For further detail and comprehensive discussion of the problem setting, [20–23] should be consulted.

Define an integral operator $\tilde{\mathfrak{J}} : \tilde{\mathcal{S}} \to \tilde{\mathcal{S}}$ by

$$\begin{aligned}
\tilde{\mathfrak{J}}\varkappa(\tilde{\mathtt{s}}) =& \frac{1}{\Gamma(\xi)} \int_0^1 (1-\tilde{r})^{\xi-1}(1-\tilde{s})\mathtt{g}(\tilde{r}, \varkappa(\tilde{r}))\mathtt{d}\tilde{r} + \frac{1}{\Gamma(\xi-1)} \int_0^1 (1-\tilde{r})^{\xi-2}(1-\tilde{s})\mathtt{g}(\tilde{r}, \varkappa(\tilde{r}))\mathtt{d}\tilde{r} \\
& + \frac{1}{\Gamma(\xi)} \int_0^{\tilde{s}} (\tilde{s}-\tilde{r})^{\xi-1}\mathtt{g}(\tilde{r}, \varkappa(\tilde{r}))\mathtt{d}\tilde{r}.
\end{aligned} \tag{4.3}$$

where $\tilde{\mathcal{S}}$ is an $\mu\mathcal{E}\mathcal{F}_b\mathfrak{W}\tilde{\mathcal{S}}$, $\mathfrak{W}_\mu$ is a $\mu\mathcal{E}\mathcal{F}_b\mathfrak{W}$ with functions $\mu = 1 + \varkappa(\tilde{s}) + \tilde{y}(\tilde{s})$ and $\alpha = 1 + \varkappa(\tilde{s})\tilde{y}(\tilde{s})$, for all $\zeta > 0$ and $\tilde{u}, \tilde{y} \in \tilde{\mathcal{S}}$.

$$\mathfrak{W}_\mu(\varkappa, \tilde{y}, \zeta) = e^{-\sup_{\tilde{s}\in[0,1]} \frac{|\varkappa(\tilde{s})-\tilde{y}(\tilde{s})|}{\zeta}},$$

Theorem 7. *Define* $\{g_{\mathfrak{m}}\} \subset \tilde{\mathcal{S}}$, *by* $g_{\mathfrak{m}+1} = \tilde{\mathfrak{J}}(g_{\mathfrak{m}})$, *where* $\mathfrak{m} \in \mathbb{N} \cup \{0\}$ *and* $g \in \tilde{\mathcal{S}}$. *Let there be* $\beta(\mathfrak{W}_\mu(\varkappa, \tilde{y}, \zeta)) \in (0, \frac{1}{k^2})$, *with* $k = \limsup_{\mathfrak{m},\mathfrak{n}\to+\infty} \Omega(g_{\mathfrak{m}}, g_{\mathfrak{n}})$, *for a bounded function* $\Omega : \tilde{\mathcal{S}} \times \tilde{\mathcal{S}} \to [1, +\infty)$ *and the following conditions are satisfied:*

- $|\mathsf{g}(\tilde{r}, \varkappa(\tilde{r})) - \mathsf{g}(\tilde{r}, \tilde{\mathsf{y}}(\tilde{r}))| \leqslant |\varkappa(\tilde{r}) - \tilde{\mathsf{y}}(\tilde{r})|$, *for all* $\varkappa, \tilde{y} \in \tilde{\mathcal{S}}$
- $\sup_{\zeta\in[0,1]} \left\{ \frac{1-\tilde{s}}{\Gamma(\xi+1)} + \frac{1-\tilde{s}}{\Gamma(\xi)} + \frac{s^\xi}{\Gamma(\xi+1)} \right\} \leqslant \beta(\mathfrak{W}_\mu(\varkappa, \tilde{y}, \zeta)) < 1.$

Then $(\mathcal{NFDE})$ *(4.1) has unique solution*

Proof.

$$
\begin{aligned}
\left|\tilde{\mathfrak{J}}\varkappa(\tilde{\mathrm{s}})-\tilde{\mathfrak{J}}\tilde{\mathrm{y}}(\tilde{\mathrm{s}})\right| =& \left|\frac{1-\tilde{s}}{\Gamma(\xi)}\int_0^1(1-\tilde{r})^{\xi-1}\left[\mathrm{g}(\tilde{r},\varkappa(\tilde{r}))-\mathrm{g}(\tilde{r},\tilde{\mathrm{y}}(\tilde{r}))\right]\mathrm{d}\tilde{r}\right.\\
&+\frac{1-\tilde{s}}{\Gamma(\xi-1)}\int_0^1(1-\tilde{r})^{\xi-2}\left[\mathrm{g}(\tilde{r},\varkappa(\tilde{r}))-\mathrm{g}(\tilde{r},\tilde{\mathrm{y}}(\tilde{r}))\right]\mathrm{d}\tilde{r}\\
&\left.+\frac{1}{\Gamma(\xi)}\int_0^{\tilde{s}}(\tilde{s}-\tilde{r})^{\xi-1}\left[\mathrm{g}(\tilde{r},\varkappa(\tilde{r}))-\mathrm{g}(\tilde{r},\tilde{\mathrm{y}}(\tilde{r}))\right]\mathrm{d}\tilde{r}\right|\\
\leqslant& \frac{1-\tilde{s}}{\Gamma(\xi)}\int_0^1(1-\tilde{r})^{\xi-1}\left|\mathrm{g}(\tilde{r},\varkappa(\tilde{r}))-\mathrm{g}(\tilde{r},\tilde{\mathrm{y}}(\tilde{r}))\right|\mathrm{d}\tilde{r}\\
&+\frac{1-\tilde{s}}{\Gamma(\xi-1)}\int_0^1(1-\tilde{r})^{\xi-2}\left|\mathrm{g}(\tilde{r},\varkappa(\tilde{r}))-\mathrm{g}(\tilde{r},\tilde{\mathrm{y}}(\tilde{r}))\right|\mathrm{d}\tilde{r}\\
&+\frac{1}{\Gamma(\xi)}\int_0^{\tilde{s}}(\tilde{s}-\tilde{r})^{\xi-1}\left|\mathrm{g}(\tilde{r},\varkappa(\tilde{r}))-\mathrm{g}(\tilde{r},\tilde{\mathrm{y}}(\tilde{r}))\right|\mathrm{d}\tilde{r}\\
\leqslant& \frac{1-\tilde{s}}{\Gamma(\xi)}\int_0^1(1-\tilde{r})^{\xi-1}\left|\varkappa(\tilde{r})-\tilde{\mathrm{y}}(\tilde{r})\right|\mathrm{d}\tilde{r}\\
&+\frac{1-\tilde{s}}{\Gamma(\xi-1)}\int_0^1(1-\tilde{r})^{\xi-2}\left|\varkappa(\tilde{r})-\tilde{\mathrm{y}}(\tilde{r})\right|\mathrm{d}\tilde{r}\\
&+\frac{1}{\Gamma(\xi)}\int_0^{\tilde{s}}(\tilde{s}-\tilde{r})^{\xi-1}\left|\varkappa(\tilde{r})-\tilde{\mathrm{y}}(\tilde{r})\right|\mathrm{d}\tilde{r}\\
\leqslant& \sup_{\tilde{\mathrm{s}}\in[0,1]}\left|\varkappa(\tilde{s})-\tilde{\mathrm{y}}(\tilde{s})\right|\left(\frac{1-\tilde{s}}{\Gamma(\xi)}\int_0^1(1-\tilde{r})^{\xi-1}\mathrm{d}\tilde{r}+\frac{1-\tilde{s}}{\Gamma(\xi-1)}\int_0^1(1-\tilde{r})^{\xi-2}\mathrm{d}\tilde{r}\right.\\
&\left.+\frac{1}{\Gamma(\xi)}\int_0^{\tilde{s}}(\tilde{s}-\tilde{r})^{\xi-1}\mathrm{d}\tilde{r}\right)\\
&=\sup_{\tilde{\mathrm{s}}\in[0,1]}\left|\varkappa(\tilde{s})-\tilde{\mathrm{y}}(\tilde{s})\right|\left(\frac{1-\tilde{s}}{\Gamma(\xi+1)}+\frac{1-\tilde{s}}{\Gamma(\xi)}+\frac{s^{\xi}}{\Gamma(\xi+1)}\right)\\
&=\beta(\mathfrak{W}_\mu(\varkappa,\tilde{y},\zeta))\sup_{\tilde{\mathrm{s}}\in[0,1]}\left|\varkappa(\tilde{s})-\tilde{\mathrm{y}}(\tilde{s})\right|,
\end{aligned}
$$

where $\beta(\mathfrak{W}_\mu(\varkappa,\tilde{y},\zeta))=\dfrac{1-\tilde{s}}{\Gamma(\xi+1)}+\dfrac{1-\tilde{s}}{\Gamma(\xi)}+\dfrac{s^{\xi}}{\Gamma(\xi+1)}$.

From the above inequality, it becomes apparent that

$$\sup_{\tilde{\mathfrak{s}}\in[0,1]}\left|\tilde{\mathfrak{J}}\varkappa(\tilde{\mathfrak{s}})-\tilde{\mathfrak{J}}\tilde{\mathfrak{y}}(\tilde{\mathfrak{s}})\right| \leqslant \beta(\mathfrak{W}_\mu(\varkappa,\tilde{y},\zeta))\sup_{\tilde{\mathfrak{s}}\in[0,1]}\left|\varkappa(\tilde{s})-\tilde{\mathfrak{y}}(\tilde{s})\right|$$

$$\Rightarrow \frac{-\sup_{\tilde{\mathfrak{s}}\in[0,1]}\left|\tilde{\mathfrak{J}}\varkappa(\tilde{\mathfrak{s}})-\tilde{\mathfrak{J}}\tilde{\mathfrak{y}}(\tilde{\mathfrak{s}})\right|}{\beta(\mathfrak{W}_\mu(\varkappa,\tilde{y},\zeta))\zeta} \geqslant \frac{-\beta(\mathfrak{W}_\mu(\varkappa,\tilde{y},\zeta))\sup_{\tilde{\mathfrak{s}}\in[0,1]}\left|\varkappa(\tilde{s})-\tilde{\mathfrak{y}}(\tilde{s})\right|}{\beta(\mathfrak{W}_\mu(\varkappa,\tilde{y},\zeta))\zeta}$$

$$\Rightarrow e^{\frac{-\sup_{\tilde{\mathfrak{s}}\in[0,1]}\left|\tilde{\mathfrak{J}}\varkappa(\tilde{\mathfrak{s}})-\tilde{\mathfrak{J}}\tilde{\mathfrak{y}}(\tilde{\mathfrak{s}})\right|}{\beta(\mathfrak{W}_\mu(\varkappa,\tilde{y},\zeta))\zeta}} \geqslant e^{\frac{-\sup_{\tilde{\mathfrak{s}}\in[0,1]}\left|\varkappa(\tilde{s})-\tilde{\mathfrak{y}}(\tilde{s})\right|}{\zeta}}$$

$$\Rightarrow \mathfrak{W}_\mu\left(\tilde{\mathfrak{J}}\varkappa,\tilde{\mathfrak{J}}\tilde{\mathfrak{y}},\beta(\mathfrak{W}_\mu(\varkappa,\tilde{y},\zeta))\zeta\right) \geqslant \mathfrak{W}_\mu\left(\varkappa,\tilde{\mathfrak{y}},\zeta\right).$$

Theorem 1 implies that $\tilde{\mathfrak{J}}$ has a fixed point $v* \in \tilde{\mathcal{S}}$, and as a result, the $\mathcal{NFDE}$ (4.1) has a unique solution in $\tilde{\mathcal{S}}$.

5. Application to Integral Equations

In this section, we use our established results to ensure that integral equations of the type

$$u(\zeta) = g(\zeta) + \int_0^\zeta H(\zeta,\tilde{s},\varkappa(\tilde{s}))ds,\ \zeta\in[0,b], \tag{5.1}$$

have solution in $\mathcal{C}\equiv C([0,b],\mathbb{R})$, being a Banach space, with norm $\|\varkappa\| := \sup_{\tilde{s}\in[0,b]}|\varkappa(s)|$, where b is some given positive real number.

Consider a complete $\mu\mathcal{EF}_b\mathfrak{W}\tilde{\mathcal{S}}$, $(\mathcal{C},\mathfrak{W}_\mu,*_p,\alpha,\mu)$ where

$$\mathfrak{W}_\mu(\varkappa,\tilde{y},\zeta) = e^{-\frac{\sup_{\tilde{s}\in[0,b]}|\varkappa(\tilde{s})-\tilde{y}(\tilde{s})|^2}{\zeta}},$$

for all $\varkappa,\tilde{y}\in\mathcal{C}$ and $\beta(\mathfrak{W}_\mu(\varkappa,\tilde{y},\zeta))>0$, with t-norm $*_p$ and bounded functions $\mu,\alpha:\mathcal{C}\times\mathcal{C}\to[1,+\infty)$. The existence of a solution to the integral equation of the form (5.1) is ensured by the following theorem.

Theorem 8. *Let $I:\mathcal{C}\to\mathcal{C}$ be an integral operator given by*

$$[I(u)](\zeta) = g(\zeta) + \int_0^\zeta H(\zeta,\tilde{s},\varkappa(\tilde{s}))d\tilde{s},\ u\in\mathcal{C},\ \zeta\in[0,b].$$

Let $\{g_{\mathfrak{m}}\}\subset\mathcal{C}$, be defined by $g_{\mathfrak{m}+1}=I(g_{\mathfrak{m}})$, $\mathfrak{m}\in\mathbb{N}\cup\{0\}$, for $g\in\mathcal{C}$. Suppose there exists $\beta(\mathfrak{W}_\mu(\varkappa,\tilde{y},\zeta))\in(0,\frac{1}{k^2})$, with $\kappa=\limsup_{\mathfrak{m},\mathfrak{n}\to+\infty}\Omega(g_{\mathfrak{m}},g_{\mathfrak{n}})$, where $\Omega:\mathcal{C}\times\mathcal{C}\to[1,+\infty)$ is a bounded mapping and let $H\in C([0,b]\times[0,b]\times\mathbb{R},\mathbb{R})$ holds the following condition:

$$|H(\zeta,\tilde{s},\varkappa(\tilde{s}))-H(\zeta,\tilde{s},\tilde{y}(\tilde{s}))| \leqslant \sqrt{\beta(\mathfrak{W}_\mu(\varkappa,\tilde{y},\zeta))}|\varkappa(\tilde{r})-\tilde{y}(\tilde{r})|.$$

Then, the integral equation (5.1) has a solution $u^*\in\mathcal{C}$.

Proof. For all $\varkappa, \tilde{y} \in \mathcal{C}$, and $\beta(\mathfrak{W}(\varkappa, \tilde{y}, \zeta)) > 0$, we have

$$\begin{aligned}
\mathfrak{W}_\mu(I(\varkappa), I(\tilde{y}), \beta(\mathfrak{W}_\mu(\varkappa, \tilde{y}, \zeta))\zeta) &= e^{-\frac{\sup_{\tilde{r}\in[0,b]} |[I(\varkappa)](\tilde{s}) - [I(\tilde{y})](\tilde{s})|^2}{\beta(\mathfrak{W}_\mu(\varkappa,\tilde{y},\zeta))\zeta}} \\
&= e^{-\frac{\sup_{\tilde{r}\in[0,b]} |\int_0^{\tilde{s}} H(\tilde{s},\tilde{r},\varkappa(\tilde{r}))d\tilde{r} - \int_0^{\tilde{s}} H(\tilde{s},\tilde{r},\tilde{y}(\tilde{r}))d\tilde{r}|^2}{\beta(\mathfrak{W}_\mu(\varkappa,\tilde{y},\zeta))\zeta}} \\
&\geqslant e^{-\frac{\sup_{\tilde{r}\in[0,b]} \int_0^{\tilde{s}} |H(\tilde{s},\tilde{r},\varkappa(\tilde{r})) - H(\tilde{s},\tilde{r},\tilde{y}(\tilde{r}))|^2 d\tilde{r}}{\beta(\mathfrak{W}_\mu(\varkappa,\tilde{y},\zeta))\zeta}} \\
&\geqslant e^{-\frac{\sup_{\tilde{r}\in[0,b]} \int_0^{\tilde{s}} \left(\sqrt{\beta(\mathfrak{W}_\mu(\varkappa,\tilde{y},\zeta))}|\varkappa(\tilde{r}) - \tilde{y}(\tilde{r})|\right)^2 d\tilde{r}}{\beta(\mathfrak{W}_\mu(\varkappa,\tilde{y},\zeta))\zeta}} \\
&\geqslant e^{-\frac{\sup_{\tilde{r}\in[0,b]} \int_0^{\tilde{s}} \beta(\mathfrak{W}_\mu(\varkappa,\tilde{y},\zeta))|\varkappa(\tilde{r}) - \tilde{y}(\tilde{r})|^2 d\tilde{r}}{\beta(\mathfrak{W}_\mu(\varkappa,\tilde{y},\zeta))\zeta}} \\
&\geqslant e^{-\frac{\sup_{\tilde{r}\in[0,b]} \beta(\mathfrak{W}_\mu(\varkappa,\tilde{y},\zeta))|\varkappa(\tilde{r}) - \tilde{y}(\tilde{r})|^2}{\beta(\mathfrak{W}_\mu(\varkappa,\tilde{y},\zeta))\zeta}} \\
&= e^{-\frac{\sup_{\tilde{r}\in[0,b]} |\varkappa(\tilde{r}) - \tilde{y}(\tilde{r})|^2}{\zeta}} \\
&= \mathfrak{W}_\mu(\varkappa, \tilde{y}, \zeta). \\
\Rightarrow \mathfrak{W}_\mu(I(\varkappa), I(\tilde{y}), \beta(\mathfrak{W}_\mu(\varkappa, \tilde{y}, \zeta))\zeta) &\geqslant \mathfrak{W}_\mu(\varkappa, \tilde{y}, \zeta).
\end{aligned}$$

As a result, the unique fixed point $u^* \in \mathcal{C}$ of I, is unique solution to the integral equation (5.1).

Acknowledgements

The authors would to thank Prince Sultan University for paying the APC and for the support through the TAS research lab.

Author Contributions

Conceptualization, Writing original Draft(B.Rome, S.Nizam, M.Sarwar), Writing-Review and Editing original Draft, (B.Rome, S.Nizam, M.Sarwar, K.Abodayeh), Supervision(M. Sarwar, K.Abodayeh)

Conflicts of Interest

The authors disclose that they have no competing interests.

References

[1] L. A. Zadeh. Fuzzy sets. *Information and Control*, 8:16, 1965.

[2] I. Kramosil and J. Michalek. Fuzzy metric and statistical metric spaces. *Kybernetika*, 11:9, 1975.

[3] A. George and P. Veeramani. On some result in fuzzy metric space. *Fuzzy Sets and Systems*, 64:5, 1994.

[4] M. Grabiec. Fixed points in fuzzy metric spaces. *Fuzzy Sets and Systems*, 27:5, 1988.

[5] V. Gregori and A. Sapena. On fixed point theorems in fuzzy metric spaces. *Fuzzy Sets and Systems*, 125:8, 2002.

[6] S. Chauhana, W. Shatanawi, S. Kumar, and S. Radenovic. Existence and uniqueness of fixed points in modified intuitionistic fuzzy metric spaces. *Journal of Nonlinear Science and Applications*, 7:14, 2014.

[7] V. Gupta, W. Shatanawi, and A. Kanwar. Coupled fixed point theorems employing clr-property on v -fuzzy metric spaces. *Mathematics*, 8, 2020.

[8] W. Shatanawi, V. Gupta, and A. Kanwar. New results on modified intuitionistic generalized fuzzy metric spaces by employing e.a property and common e.a property for coupled maps. *Journal of Intelligent & Fuzzy Systems*, 38:8, 2020.

[9] S. Nădăban. Fuzzy b-metric spaces. *International Journal of Computers, Communications & Control*, 11:9, 2016.

[10] F. Mehmood, R. A. Ali, C. Ionescu, and T. A. Kamran. Extended fuzzy b-metric spaces. *Journal of Mathematical Analysis*, 8:8, 2017.

[11] F. Mehmood, R. Ali, and N. Hussain. Contractions in fuzzy rectangular b-metric spaces with application. *Journal of Intelligent and Fuzzy Systems*, 8:8, 2017.

[12] B. Rome, M. Sarwar, and T. Abdeljawad. μ-extended fuzzy b-metric spaces and related fixed point results. *AIMS Mathematics*, 5:9, 2020.

[13] M. Geraghty. On contractive mappings. *Proceedings of the American Mathematical Society*, 40:5, 1973.

[14] J. Martínez, W. Sintunavarat, and Y. J. Cho. Common fixed point theorems for geraghty's type contraction mappings using the monotone property with two metrics. *Fixed Point Theory and Applications*, 2015:15, 2015.

[15] M. S. Ashraf, R. Ali, and N. Hussain. Geraghty type contractions in fuzzy b-metric spaces with application to integral equations. *Filomat*, 34:16, 2020.

[16] E. Peter Klement, R. Mesiar, and E. Pap. Triangular norms. *Fuzzy Sets and Systems*, 8, 2013.

[17] B. Rome, M. Sarwar, K. Shah, B. Abdalla, and T. Abdeljawad. Some generalized fixed point results of banach and ćirić type in extended fuzzy b-metric spaces with applications. *AIMS Mathematics*, 7:22, 2022.

[18] T. Došenovic, A. Javaheri, S. Sedghi, and N. Shobe. Coupled fixed point theorem in b-fuzzy metric spaces. *Novi Sad Journal of Mathematics*, 47:12, 2017.

[19] S. N. Mishra, S. N. Sharma, and S. L. Singh. Common fixed point of maps on fuzzy metric spaces. *International Journal of Mathematics and Mathematical Sciences*, 17:6, 1994.

[20] A. Elhamed and M. Gehad. Fixed point results for (β, α)-implicit contractions in two generalized b-metric spaces. *Journal of Nonlinear Science and Applications*, 14:9, 2010.

[21] N. K. Alharbi. Hyers–ulam and hyers-ulam-rassias stability for a class of fractional evolution differential equations with neutral time delay. *Symmetry*, 17, 2025.

[22] T. T. Phong and L. D. Long. Well-posed results for nonlocal fractional parabolic equation involving caputo-fabrizio operator. *Journal of Mathematical and Computational Science*, 26:11, 2022.

[23] S. Zhang. Positive solutions for boundary-value problems of nonlinear fractional differential equation. *Electronic Journal of Differential Equations*, 36:12, 2006.