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New Extension of Inequalities through Extended Version of Fractional Operators for s-Convexity with Applications

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Abstract. Fractional integral inequalities play a significant role in both pure and applied mathematics, contributing to the advancement and extension of various mathematical techniques. An accurate formulation of such inequalities is essential to establish the existence and uniqueness of fractional methods. Additionally, convexity theory serves as a fundamental component in the study of fractional integral inequalities due to its defining characteristics and properties. Moreover, there is a strong interconnection between convexity and symmetric theories, allowing results from one to be effectively applied to the other. This correlation has become particularly evident in recent decades, further enhancing their importance in mathematical research. This article investigate two innovative approaches of differentiable functions to modify Hermite-Hadamard inequalities and their refinements by implementation of generalized fractional operators through the s-convex functions. The study aims to extend and refine existing inequalities with a fractional operator that has extended the Bessel-Maitland functions as a kernel, providing a more generalized framework. By incorporating these special functions, the results encompass and improve numerous classical inequalities found in the literature, offering deeper insights and broader applicability in mathematical analysis.

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Key Words and Phrases: Convex function, Extended Bessel-Maitland function; Fractional operators

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1. Introduction

Mathematical inequalities are now widely recognized as one of the most valuable and applicable branches of mathematics. Their versatility and effectiveness have been demonstrated in various scientific and engineering disciplines, with significant applications in information theory, economics, finance, and engineering. The theory of inequalities plays a fundamental role in nearly all areas of pure and applied mathematics. The study of convex functions has been instrumental in advancing inequality theory, tracing back to the pioneering work of Jensen in 1905–1906. Many modern analytical inequalities stem directly from the properties of convex functions. In addition, convex functions and various forms of convexity serve as powerful tools to derive numerous important and practical inequalities. Due to its expanding range of applications, the study of inequalities remains one of the most actively researched fields in mathematical analysis.

The broad spectrum of applications of fractional calculus [1] in domains such as fluid dynamics, mathematical biology, and mathematical physics has made it an essential field of ongoing research [2–5]. In order to validate various solutions in theoretical and practical contexts, numerous researchers have generated fractional integral inequalities using fractional operators [6, 7]. Recent research has explored fractional integral inequalities in extensive detail, examining their various manifestations and possible uses. These studies have tremendously broadened the discipline and provided new tools and insights.

A notable advancement in this field is the formulation of integral expressions that include special functions. Fractional integral operators that utilize particular kernel functions are essential in many fields of study [8, 9]. The Bessel-Maitland function, first proposed by Daniel Bernoulli, is related to the linear differential equation. The fractional calculus, with its extensive applications, extends its scope. Mathematical analysis and fractional theory have greatly benefited from its extensions. More research and development in the topic is being motivated by quick developments in fractional calculus, which have brought attention to the need for creative transformations and generalized fractional operators.

Convex analysis plays a crucial role in variational analysis, as it encompasses a generalized differentiation theory applicable to mathematical models that do not require differentiability assumptions. Convex optimization is well known to be just one of many areas in which convex analysis has demonstrated its significance. The convexity of a problem allows for the development of efficient numerical algorithms to solve convex optimization problems, even in the presence of non-differentiable data, while also enabling an in-depth study of the qualitative properties of optimal solutions. The impact of convex analysis and optimization continues to expand across various mathematical disciplines and practical applications, including estimation, control systems, communications, networks, signal processing, data analysis, electrical circuit design, finance, statistics, economics, and mathematical modeling.Inequalities have applications as tools in various fields of mathematics, such as differential and integral equations, and these ideas serve as a basis for their analysis. One of the most well-known inequalities is the Hermite-Hadamard inequality, which was first introduced by Charles Hermite and Jacques Hadamard and describes how convex functions behave and is used extensively in mathematical modeling, optimization, and numerical analysis. In recent years the researchers have extended the classical convexity into h-convexity, h-Godunova-Levin convexity, s-convexity and (η_1, η_2) convex functions etc. For s-convex functions, the integral identities linked to Hermite-Hadamard inequality were first presented by Barsam et al. [10].

Sattarzadeh and Barsam [11] discovered the Hermite-Hadamard type problems with fractional integrals for functions that are uniformly convex. It is meant to explore the HermiteHadamard type inequalities involving fractional integrals due to the numerous applications of fractional calculus [12–16] and Hermite-Hadamard type inequalities [17–21]. The inequalities of fractional integral by convex functions with respect to increasing functions was obtained by Mohammed [22]. Abdeljawad et al. [23] proposed new Simpson-type inequalities for (s, m)-convex functions . Some inequalities for s-convex functions with fractional integrals were presented by I¸scan [24]. Trapezoid type inequalities for s-convex functions involving generalized fractional operators established by Usta et al. [25] . Butt et al. [26] introduced integral identity; by using that identity, new inequalities were obtained via a general form of fractional integral operators. Agarwal et al. [27] proposed Hermite-Hadamard type inequalities for generalized k-fractional integrals.

In our present work we use the s-convex function for the class of first odder derivatives and second order derivatives and modified the Hermite-Hadamard (H-H) integral inequalities by utilizing the Bessel-Maitland function as its kernel.

2. Preliminaries

In this section, we discuss the basic definitions which will help us to understand our main work.

Definition 1. *[28, 29] Let $\phi : \mathbb{R} \to \mathbb{R}$ be real valued function, is said to be convex if the following inequalities holds:*

$$\phi(\wp\alpha + (1-\wp)\omega) \le \wp\phi(\alpha) + (1-\wp)\phi(\omega), \tag{1}$$

where $\wp \in [0,1]$ and $\forall\ \omega, \alpha \in \mathbb{R}$.

Definition 2. *[30] Let $\phi : \mathbb{R} \to \mathbb{R}$ be real valued function, is said to be s-convex function, if the following relation holds:*

$$\phi(\wp\alpha + (1-\wp)\omega) \le \wp^s\phi(\alpha) + (1-\wp)^s\phi(\omega), \tag{2}$$

where $\alpha, \omega \in \mathbb{R}$ and $\wp \in (0,1)$ and $s \in (0,1]$.

Definition 3. *[29, 31–33] The Hermite-Hadamard type inequality for convex function $\phi : \mathbb{R} \to \mathbb{R}$, is defined as follows;*

$$\phi\left(\frac{\alpha+\omega}{2}\right) \le \frac{1}{\omega-\alpha}\int_\alpha^\omega \phi(\wp)d\wp \le \frac{\phi(\alpha)+\phi(\omega)}{2}. \tag{3}$$

where $\alpha, \omega \in \mathbb{R}, \alpha < \omega$.

Definition 4. *[34] The gamma function in integral type is defined for* $\Re(t) > 0$, *as follows*

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx. \tag{4}$$

Definition 5. *[34] The Pochammer's symbol is defined as follows:*

$$(\eth)_\eta = \left\{ \begin{array}{ll} 1, & \text{for } \eta = 0, \eth \neq 0 \\ \eth(\eth+1)\cdots(\eth+\eta-1), & \text{for } \eta \geq 1, \end{array} \right\} \tag{5}$$

For $\eta \in \mathbb{N}$ *and* $\eth \in \mathbb{C}$ *Here, is some relations of gamma functions.*

$$\begin{aligned} (\wp)_n &= \frac{\Gamma(\wp+n)}{\Gamma(\wp)}, \\ (\wp)_{kn} &= \frac{\Gamma(\wp+kn)}{\Gamma(\wp)}. \end{aligned}$$

where Γ *being the gamma notation.*

Definition 6. *[35] The beta function is defined for* $\Re(m) > 0$ *and* $\Re(n) > 0$ *as follows:*

$$\begin{aligned} \mathbf{B}(g,h) &= \int_0^1 \eta^{g-1}(1-\eta)^{h-1} d\eta, \\ &= \frac{\Gamma(g)\Gamma(h)}{\Gamma(g+h)}. \end{aligned} \tag{6}$$

Definition 7. *[36] The extended form of beta function is define for* $\Re(g) > 0$, $\Re(h) > 0$, $\Re(p) > 0$ *as follows*

$$\mathbf{B}_p(g,h) = \int_0^1 z^{g-1}(1-z)^{h-1} exp\Big(\frac{-p}{z(1-z)}\Big) dz. \tag{7}$$

Taking the value of $p = 1$, *the extended beta function becomes the classical beta function.*

Definition 8. *[37] The Bessel-Maitland function is defined for* $\phi, \psi, \upsilon, \chi, \varpi \in \mathbb{C}$ *and* $\Re(\phi) > 0, \Re(\xi) > 0, \Re(\upsilon) > 0, \Re(\chi) > 0, \Re(\varpi) > 0, \rho, \eta, m \geq 0$ *and* $m, \eta > \Re(\phi) + \eta$ *;*

$$\mathbb{J}^{\psi,\upsilon,\chi,\varpi}_{\phi,\rho,m,\eta}(y) = \sum_{p=0}^{\infty} \frac{(\upsilon)_{\rho p}(\varpi)_{\eta p}(-y)^p}{\Gamma(\phi p + \psi + 1)(\chi)_{mp}}. \tag{8}$$

Definition 9. *[38] The extended version of Bessel-Maitland is defined for* $\mu, \xi, \zeta, \varsigma, c, \kappa_1 \in \mathbb{C}, \Re(\mu) > 0, \Re(\xi) > 0, \Re(\zeta) > 0, \Re(\varsigma) > 0, \Re(\kappa_1) > 0, \rho, m, \eta \geq 0$ *and* $m, \rho > \Re(\mu) + \eta$, *as follows*

$$\mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa, p) = \sum_{n=0}^{\infty} \frac{\beta_p(\zeta + \rho n, c - \zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta, c-\zeta)\Gamma(\mu n + \xi + 1)(\varsigma)_{mn}} (-\kappa)^n. \tag{9}$$

Definition 10. *[39] The left and right sided of extended version of Bessel-Maitland function is defined for the same assumption of definition* (9)*, as follows:*

$$\left(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,p^+}f\right)(x,r)=\int_p^x (x-t)^{\xi}\mathbb{J}^{\mu,\rho,m,\eta}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa(x-t)^{\mu};r)f(t)dt, \tag{10}$$

$$\left(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,q^-}f\right)(x,r)=\int_x^q (t-x)^{\xi}\mathbb{J}^{\mu,\rho,m,\eta}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa(t-x)^{\mu};r)f(t)dt. \tag{11}$$

Remark 1. *If we replace $r=0,\kappa=0,$ and $\xi=\xi-1$ in the definition* (10)*, we obtain the left-and right-sided Riemann fractional operators.*

3. Applications of differentiable function with the refinements of Hermite-Hadamard $(H-H)$ inequalities

Here, we discuss the refinements of Hermite-Hadamard type fractional inequalities with the differentiable functions by implementations of extended version of fractional operators for s-convexity.

Lemma 1. *Let $\phi: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I^o and $\omega,\tau\in I^o$ with $\omega<\tau$. If $\phi^{'}\in L[\omega,\tau]$, and $\mu,\xi,\zeta,\varsigma,c,\kappa_1\in\mathbb{C},\Re(\mu)>0,\Re(\xi)>0,\Re(\zeta)>0,\Re(\varsigma)>0,\Re(\kappa_1)>0,\rho,m,\eta\geq 0$ and $m,\rho>\Re(\mu)+\eta$, then following integral equality holds;*

$$\phi(\alpha)\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa,p)\big[\frac{1}{\alpha-\omega}+\frac{1}{\alpha-\tau}\big]-(\xi^{'}+\mu n)$$
$$\left[\frac{1}{(\alpha-\omega)^{\xi^{'}+1}}\big(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\alpha^+}\phi\big)(\omega,\kappa_1)+\frac{1}{(\alpha-\tau)^{\xi^{'}+1}}\big(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\tau^-}\phi\big)(\omega,\kappa)\right]$$
$$=\int_0^1\wp^{\xi^{'}}\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa\wp^{\mu};p)\phi^{'}(\wp\alpha+(1-\wp)\omega d\wp+\int_0^1\wp^{\xi^{'}}\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa\wp^{\mu};p)\phi^{'}(\wp\alpha+(1-\wp)\tau d\wp.$$

Proof. Consider the integral

$$I_1=\int_0^1\wp^{\xi^{'}}\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa\wp^{\mu};p)\phi^{'}(\wp\alpha+(1-\wp)\omega d\wp$$
$$=\sum_{n=0}^{\infty}\frac{\beta_p(\zeta+\rho n,c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta,c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n\int_0^1\wp^{\xi^{'}+\mu n}\phi^{'}(\wp\alpha+(1-\wp)\omega d\wp$$

$$=\sum_{n=0}^{\infty}\frac{\beta_p(\zeta+\rho n,c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta,c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n\Bigg[\int_0^1\frac{\wp^{\xi^{'}+\mu n}}{\alpha-\omega}\phi(\wp\alpha+(1-\wp)\omega\Big|_0^1-$$
$$\int_0^1\frac{\xi^{'}+\mu n}{\alpha-\omega}\wp^{\xi^{'}+\mu n-1}\phi(\wp\alpha+(1-\wp)\omega d\wp\Bigg]$$

$$=\sum_{n=0}^{\infty}\frac{\beta_p(\zeta+\rho n,c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta,c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n\left[\frac{\phi(\alpha)}{\alpha-\omega}-\frac{\xi^{'}+\mu n}{\alpha-\omega}\int_0^1\wp^{\xi^{'}+\mu n-1}\phi(\wp\alpha+(1-\wp)\omega d\wp\right]. \tag{12}$$

Using substitution $\wp\alpha + (1-\wp)\omega = x$ in equation (12), we get

$$I_1 = \frac{\phi(\alpha)}{\alpha-\omega}\mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa,p) - \frac{\xi^{'}+\mu n}{(\alpha-\omega)^{\xi^{'}+1}}\big(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\alpha^+}\phi\big)(\omega,\kappa). \tag{13}$$

Now, consider the integral I_2, we have

$$I_2 = \int_0^1 \wp^{\xi^{'}}\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa\wp^{\mu};p)\phi^{'}(\wp\alpha+(1-\wp)\tau d\wp.$$

Proceeding same as I_1 and we get the result,

$$I_2 = \frac{\phi(\alpha)}{\alpha-\tau}\mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa,p) - \frac{\xi^{'}+\mu n}{(\alpha-\tau)^{\xi^{'}+1}}\big(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\tau^-}\phi\big)(\omega,\kappa). \tag{14}$$

By combining equation (13) and (14) we get the required result.

Corollary 1. *If we replace $p=0,\kappa=0$, and $\xi=\xi-1$ in the Lemma (1), we have a result [19].*

Theorem 1. *Let $\phi:[\omega,\tau]\to\mathbb{R}$ be a positive function with $0\le\omega<\alpha<\tau$ and $\phi\in L[\omega,\tau]$. If ϕ is s-convex function on $[\omega,\tau]$, and $\mu,\xi,\zeta,\varsigma,c,\kappa_1\in\mathbb{C},\Re(\mu)>0,\Re(\xi)>0,\Re(\zeta)>0,\Re(\varsigma)>0,\Re(\kappa_1)>0,\rho,m,\eta\ge 0$ and $m,\rho>\Re(\mu)+\eta$, then we get the following inequality*

$$\begin{aligned}&\frac{\big(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\alpha^+}\phi\big)(\omega,\kappa)}{(\alpha-\omega)^{\xi'+\mu n+1}} + \frac{\big(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\tau^-}\phi\big)(\omega,\kappa)}{(\alpha-\tau)^{\xi'+\mu n+1}}\\ &\le \phi(\alpha)\mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa,p)\big[\beta(\xi^{'}+\mu n+s+1,1)+\beta(\xi^{'}+s,1)\big]\\ &+\big[\phi(\omega)+\phi(\tau)\big]\mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa,p)\beta(\xi^{'}+1,s+1).\end{aligned}$$

Proof. From definition of s-convexity of ϕ,

$$\phi(\wp\alpha+(1-\wp)\omega)+\phi(\wp\alpha+(1-\wp)\tau)\le\wp^s\phi(\alpha)+(1-\wp)^s\phi(\omega)+\wp^s\phi(\alpha)+(1-\wp)^s\phi(\tau)$$

Multiplying both sides with $\wp^{\xi^{'}}\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa\wp^{\mu};p)$ and integrate w.r.t $\wp$ on $[0,1]$

$$\begin{aligned}&\int_0^1\wp^{\xi^{'}}\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa\wp^{\mu};p)\phi(\wp\alpha+(1-\wp)\omega)+\int_0^1\wp^{\xi^{'}}\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa\wp^{\mu};p)\phi(\wp\alpha+(1-\wp)\tau)\\ &\le\int_0^1\wp^{\xi^{'}+s}\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa\wp^{\mu};p)\phi(\alpha)+\int_0^1\wp^{\xi^{'}}\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa\wp^{\mu};p)(1-\wp)^s\phi(\omega)+\\ &\int_0^1\wp^{\xi^{'}+s}\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa\wp^{\mu};p)\phi(\alpha)+\int_0^1\wp^{\xi^{'}}\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa\wp^{\mu};p)(1-\wp)^s\phi(\tau)\end{aligned} \tag{15}$$

Now, solving the following integral

$$\int_0^1\wp^{\xi^{'}+s}\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa\wp^{\mu};p)\phi(\alpha)=\sum_{n=0}^{\infty}\frac{\beta_p(\zeta+\rho n,c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta,c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n\Big[\phi(\alpha)\int_0^1\wp^{\xi^{'}+\mu n+s}d\wp\Big]$$

$$= \frac{\phi(\alpha)}{\xi' + \mu n + s + 1}\mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa, p)$$
$$= \phi(\alpha)\mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa, p)\beta(\xi' + \mu n + s + 1, 1). \tag{16}$$

Similarly, we resolve another fractional integral

$$\int_0^1 \wp^{\xi'}\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa\wp^{\mu}; p)(1-\wp)^s\phi(\omega) = \phi(\omega)\mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa, p)\beta(\xi' + \mu n + 1, s + 1) \tag{17}$$

$$\int_0^1 \wp^{\xi'+s}\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa\wp^{\mu}; p)\phi(\alpha) = \phi(\alpha)\mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa, p)\beta(\xi' + \mu n + s, 1) \tag{18}$$

$$\int_0^1 \wp^{\xi'}\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa\wp^{\mu}; p)(1-\wp)^s\phi(\tau) = \phi(\tau)\mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa, p)\beta(\xi' + \mu n + 1, s + 1) \tag{19}$$

Putting the values (16), (17), (18) and (19) in equation (15), we have

$$\int_0^1 \wp^{\xi'}\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa\wp^{\mu}; p)\phi(\wp\alpha + (1-\wp)\omega) + \int_0^1 \wp^{\xi'}\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa\wp^{\mu}; p)\phi(\wp\alpha + (1-\wp)\tau)$$
$$\leq \phi(\alpha)\mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa, p)\big[\beta(\xi' + \mu n + s + 1, 1) + \beta(\xi' + \mu n + s, 1)\big] +$$
$$\big[\phi(\omega) + \phi(\tau)\big]\mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa, p)\beta(\xi' + \mu n + 1, s + 1). \tag{20}$$

By substitution $\wp\alpha + (1-\wp)\omega = x$, and $\wp\alpha + (1-\wp)\tau = \ell$ in left side of equation (20), and then simplify, we have the required result.

Corollary 2. *If we replace $p = 0, \kappa = 0$, and $\xi = \xi - 1$ in the Theorem (1), we have a result [19].*

Theorem 2. *Let $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^o and $\omega, \tau \in I^o$ with $\omega < \alpha < \tau$ such that $\phi' \in L[\omega, \tau]$. If $|\phi'|$ is s-convex function on $[\omega, \tau]$, and $\mu, \xi, \zeta, \varsigma, c, \kappa_1 \in \mathbb{C}, \Re(\mu) > 0, \Re(\xi) > 0, \Re(\zeta) > 0, \Re(\varsigma) > 0, \Re(\kappa_1) > 0, \rho, m, \eta \geq 0$ and $m, \rho > \Re(\mu) + \eta$, then we have the following integral inequality in result*

$$\left|\phi(\alpha)\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa, p)\big[\tfrac{1}{\alpha-\omega} + \tfrac{1}{\alpha-\tau}\big] - (\xi' + \mu n)\left[\tfrac{1}{(\alpha-\omega)^{\xi'+1}}\big(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\alpha^+}\phi\big)(\omega, \kappa) + \tfrac{1}{(\alpha-\tau)^{\xi'+1}}\big(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\tau^-}\phi\big)(\omega, \kappa)\right]\right|$$
$$\leq 2\big|\phi'(\alpha)\big|\mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa, p)\beta(\xi' + \mu n + s + 1, 1) + \big[\big|\phi'(\omega)\big| + \big|\phi'(\tau)\big|\big]\mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa, p)\beta(\xi' + \mu n + 1, s + 1)$$

Proof. Using Lemma 1

$$\left|\phi(\alpha)\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa, p)\big[\tfrac{1}{\alpha-\omega} + \tfrac{1}{\alpha-\tau}\big] - (\xi' + \mu n)\left[\tfrac{1}{(\alpha-\omega)^{\xi'+1}}\big(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\alpha^+}\phi\big)(\omega, \kappa_1) + \tfrac{1}{(\alpha-\tau)^{\xi'+1}}\big(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\tau^-}\phi\big)(\omega, \kappa_1)\right]\right|$$
$$= \int_0^1 \wp^{\xi'}\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa\wp^{\mu}; p)\big|\phi'(\wp\alpha + (1-\wp)\omega\big|d\wp + \int_0^1 \wp^{\xi'}\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa\wp^{\mu}; p)\big|\phi'(\wp\alpha + (1-\wp)\tau\big|d\wp$$

Utilizing the definition of s-convex function on $|\phi^{'}|$, we have

$$\leq \sum_{n=0}^{\infty}\left|\frac{\beta_p(\zeta+\rho n, c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta, c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n\right|\left[\left|\phi^{'}(\alpha)\right|\int_0^1 \wp^{\xi^{'}+\mu n+s}d\wp+\left|\phi^{'}(\omega)\right|\int_0^1 \wp^{\xi^{'}+\mu n}(1-\wp)^s d\wp\right]+$$
$$\sum_{n=0}^{\infty}\left|\frac{\beta_p(\zeta+\rho n, c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta, c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n\right|\left[\left|\phi^{'}(\alpha)\right|\int_0^1 \wp^{\xi^{'}+\mu n+s}d\wp+\left|\phi^{'}(\tau)\right|\int_0^1 \wp^{\xi^{'}+\mu n}(1-\wp)^s d\wp\right]$$

$$\leq \left|\phi^{'}(\alpha)\right|\mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa,p)\beta(\xi^{'}+\mu n+s+1,1)+\left|\phi^{'}(\omega)\right|\mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa,p)\beta(\xi^{'}+\mu n+1,s+1)+$$
$$\left|\phi^{'}(\alpha)\right|\mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa,p)\beta(\xi^{'}+\mu n+s,1)+\left|\phi^{'}(\tau)\right|\mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa,p)\beta(\xi^{'}+\mu n+1,s+1)$$

$$\leq 2\left|\phi^{'}(\alpha)\right|\mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa,p)\beta(\xi^{'}+\mu n+s+1,1)+\left[\left|\phi^{'}(\omega)\right|+\left|\phi^{'}(\tau)\right|\right]\mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa,p)\beta(\xi^{'}+\mu n+1,s+1)$$

The proof is completed.

Corollary 3. *If we replace $p=0, \kappa=0$, and $\xi=\xi-1$ in the Theorem (2), we have a result [19].*

Theorem 3. *Let $\phi: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I^o and $\omega,\tau\in I^o$ with $\omega<\alpha<\tau$ such that $\phi^{'}\in L[\omega,\tau]$. If $|\phi^{'}|^q (q>0)$ is s-convex function on $[\omega,\tau]$, and $\mu,\xi,\zeta,\varsigma,c,\kappa_1\in\mathbb{C}, \Re(\mu)>0, \Re(\xi)>0, \Re(\zeta)>0, \Re(\varsigma)>0, \Re(\kappa_1)>0, \rho,m,\eta\geq 0$ and $m,\rho>\Re(\mu)+\eta$, then we have the following integral inequality in result;*

$$\left|\phi(\alpha)\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa,p)\left[\frac{1}{\alpha-\omega}+\frac{1}{\alpha-\tau}\right]-(\xi^{'}+\mu n)\left[\frac{1}{(\alpha-\omega)^{\xi^{'}+1}}\left(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\alpha^+}\phi\right)(\omega,\kappa)+\right.\right.$$
$$\left.\left.\frac{1}{(\alpha-\tau)^{\xi^{'}+1}}\left(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\tau^-}\phi\right)(\omega,\kappa)\right]\right|\leq$$
$$\mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa,p)\left(\frac{1}{\xi^{'}+\mu n+1}\right)^{1-\frac{1}{q}}\left[\left(\frac{\left|\phi^{'}(\alpha)\right|^q}{\xi^{'}+\mu n+s+1}+\left|\phi^{'}(\omega)\right|^q\beta(\xi^{'}+\mu n+s+1,1)\right)^{\frac{1}{q}}+\right.$$
$$\left.\left(\frac{\left|\phi^{'}(\alpha)\right|^q}{\xi^{'}+\mu n+s+1}+\left|\phi^{'}(\tau)\right|^q\beta(\xi^{'}+\mu n+s+1,1)\right)^{\frac{1}{q}}\right].$$

Proof. By using Lemma 1

$$\aleph:=\left|\phi(\alpha)\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa,p)\left[\tfrac{1}{\alpha-\omega}+\tfrac{1}{\alpha-\tau}\right]-(\xi^{'}+\mu n)\left[\tfrac{1}{(\alpha-\omega)^{\xi^{'}+1}}\left(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\alpha^+}\phi\right)(\omega,\kappa)+\right.\right.$$
$$\left.\left.\tfrac{1}{(\alpha-\tau)^{\xi^{'}+1}}\left(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\tau^-}\phi\right)(\omega,\kappa)\right]\right|$$
$$\leq \int_0^1 \wp^{\xi^{'}}\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa\wp^{\mu};p)\left|\phi^{'}(\wp\alpha+(1-\wp)\omega\right|d\wp+\int_0^1 \wp^{\xi^{'}}\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa\wp^{\mu};p)\left|\phi^{'}(\wp\alpha+(1-\wp)\tau\right|d\wp$$

Applying Power mean inequality;

$$\aleph \leq \sum_{n=0}^{\infty}\left|\frac{\beta_p(\zeta+\rho n, c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta, c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n\right|$$
$$\left[\left(\int_0^1 \wp^{\xi'+\mu n} d\wp\right)^{1-\frac{1}{q}}\left(\int_0^1 \wp^{\xi'+\mu n}\left|\phi'(\wp\alpha+(1-\wp)\omega\right|^q d\wp\right)^{\frac{1}{q}}\right]+$$
$$\sum_{n=0}^{\infty}\left|\frac{\beta_p(\zeta+\rho n, c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta, c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n\right|\left[\left(\int_0^1 \wp^{\xi'+\mu n} d\wp\right)^{1-\frac{1}{q}}\left(\int_0^1 \wp^{\xi'+\mu n}\left|\phi'(\wp\alpha+(1-\wp)\tau\right|^q d\wp\right)^{\frac{1}{q}}\right]$$

As $\left|\phi'\right|^q$ is s-convex function;

$$\aleph \leq \sum_{n=0}^{\infty}\left|\frac{\beta_p(\zeta+\rho n, c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta, c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n\right|\left[\left(\frac{1}{\xi'+\mu n+1}\right)^{1-\frac{1}{q}}\left(\left|\phi'(\alpha)\right|^q\int_0^1 \wp^{\xi'+\mu n+s} d\wp+\right.\right.$$
$$\left.\left.\left|\phi'(\omega)\right|^q\int_0^1 \wp^{\xi'+\mu n}(1-\wp)^s d\wp\right)^{\frac{1}{q}}\right]$$
$$+\sum_{n=0}^{\infty}\left|\frac{\beta_p(\zeta+\rho n, c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta, c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n\right|\left[\left(\frac{1}{\xi'+\mu n+1}\right)^{1-\frac{1}{q}}\left(\left|\phi'(\alpha)\right|^q\int_0^1 \wp^{\xi'+\mu n+s} d\wp\right.\right.$$
$$\left.\left.+\left|\phi'(\tau)\right|^q\int_0^1 \wp^{\xi'+\mu n}(1-\wp)^s d\wp\right)^{\frac{1}{q}}\right].$$

After simplification, we have

$$\aleph \leq \mathbb{J}_{\xi,\zeta,\varsigma,\kappa_1}^{\mu,\rho,m,\eta,c}(\kappa,p)\left[\left(\frac{1}{\xi'+\mu n+1}\right)^{1-\frac{1}{q}}\left(\frac{\left|\phi'(\alpha)\right|^q}{\xi'+\mu n+s+1}+\left|\phi'(\omega)\right|^q\beta(\xi'+\mu n+s+1,1)\right)^{\frac{1}{q}}\right]$$
$$+\mathbb{J}_{\xi,\zeta,\varsigma,\kappa_1}^{\mu,\rho,m,\eta,c}(\kappa,p)\left[\left(\frac{1}{\xi'+\mu n+1}\right)^{1-\frac{1}{q}}\left(\frac{\left|\phi'(\alpha)\right|^q}{\xi'+\mu n+s+1}+\left|\phi'(\tau)\right|^q\beta(\xi'+\mu n+s+1,1)\right)^{\frac{1}{q}}\right].$$

Now, the final result is obtained as follows

$$\aleph \leq \mathbb{J}_{\xi,\zeta,\varsigma,\kappa_1}^{\mu,\rho,m,\eta,c}(\kappa,p)\left(\frac{1}{\xi'+\mu n+1}\right)^{1-\frac{1}{q}}\left[\left(\frac{\left|\phi'(\alpha)\right|^q}{\xi'+\mu n+s+1}+\left|\phi'(\omega)\right|^q\beta(\xi'+\mu n+s+1,1)\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{\left|\phi'(\alpha)\right|^q}{\xi'+\mu n+s+1}+\left|\phi'(\tau)\right|^q\beta(\xi'+\mu n+s+1,1)\right)^{\frac{1}{q}}\right].$$

Corollary 4. *If we replace $p=0, \kappa=0,$ and $\xi=\xi-1$ in Theorem (3), we have a result [19].*

Theorem 4. *Let $\phi: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I^o and $\omega, \tau \in I^o$ with $\omega<\alpha<\tau$ such that $\phi' \in L[\omega, \tau]$. If $|\phi'|^q$ is s-convex function on $[\omega, \tau]$, and $\mu, \xi, \zeta, \varsigma, c, \kappa_1 \in \mathbb{C}, \Re(\mu)>0, \Re(\xi)>0, \Re(\zeta)>0, \Re(\varsigma)>0, \Re(\kappa_1)>0, \rho, m, \eta \geq 0$ and $m, \rho>\Re(\mu)+\eta$, then following fractional integral inequality holds;*

$$\Big|\phi(\alpha)\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa,p)\big[\frac{1}{\alpha-\omega}+\frac{1}{\alpha-\tau}\big]-(\xi^{'}+\mu n)\Big[\frac{1}{(\alpha-\omega)^{\xi^{'}+1}}\big(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\alpha^+}\phi\big)(\omega,\kappa)$$
$$+\frac{1}{(\alpha-\tau)^{\xi^{'}+1}}\big(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\tau^-}\phi\big)(\omega,\kappa)\Big]\Big|$$
$$\leq \mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa,p)\bigg(\frac{1}{p\xi^{'}+p\mu n+1}\bigg)^{\frac{1}{p}}\bigg[\bigg(\big|\phi^{'}(\alpha)\big|^q\frac{1}{s+1}+\big|\phi^{'}(\omega)\big|^q\frac{1}{s+1}\bigg)^{\frac{1}{q}}$$
$$+\bigg(\big|\phi^{'}(\alpha)\big|^q\frac{1}{s+1}+\big|\phi^{'}(\tau)\big|^q\frac{1}{s+1}\bigg)^{\frac{1}{q}}\bigg].$$

Proof. According to Lemma 1

$$\aleph=\Big|\phi(\alpha)\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa,p)\big[\frac{1}{\alpha-\omega}+\frac{1}{\alpha-\tau}\big]-(\xi^{'}+\mu n)\Big[\frac{1}{(\alpha-\omega)^{\xi^{'}+1}}\big(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\alpha^+}\phi\big)(\omega,\kappa)+$$
$$\frac{1}{(\alpha-\tau)^{\xi^{'}+1}}\big(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\tau^-}\phi\big)(\omega,\kappa)\Big]\Big|$$
$$\leq\int_0^1\wp^{\xi^{'}}\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa\wp^{\mu};p)\big|\phi^{'}(\wp\alpha+(1-\wp)\omega\big|d\wp+\int_0^1\wp^{\xi^{'}}\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa\wp^{\mu};p)\big|\phi^{'}(\wp\alpha+(1-\wp)\tau\big|d\wp.$$

Applying Holder's inequality;

$$\aleph\leq\sum_{n=0}^{\infty}\bigg|\frac{\beta_p(\zeta+\rho n,c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta,c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n\bigg|\bigg[\bigg(\int_0^1\wp^{p\xi^{'}+p\mu n}d\wp\bigg)^{\frac{1}{p}}\bigg(\int_0^1\big|\phi^{'}(\wp\alpha+(1-\wp)\omega\big|^qd\wp\bigg)^{\frac{1}{q}}\bigg]$$
$$+\sum_{n=0}^{\infty}\bigg|\frac{\beta_p(\zeta+\rho n,c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta,c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n\bigg|\bigg[\bigg(\int_0^1\wp^{p\xi^{'}+p\mu n}d\wp\bigg)^{\frac{1}{q}}\bigg(\int_0^1\big|\phi^{'}(\wp\alpha+(1-\wp)\tau\big|^qd\wp\bigg)^{\frac{1}{q}}\bigg].$$

As $\big|\phi^{'}\big|^q$ is s-convex function;

$$\aleph\leq\sum_{n=0}^{\infty}\bigg|\frac{\beta_p(\zeta+\rho n,c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta,c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n\bigg|\bigg[\bigg(\frac{1}{p\xi^{'}+p\mu n+1}\bigg)^{\frac{1}{p}}\bigg(\big|\phi^{'}(\alpha)\big|^q\int_0^1\wp^sd\wp$$
$$+\big|\phi^{'}(\omega)\big|^q\int_0^1(1-\wp)^sd\wp\bigg)^{\frac{1}{q}}\bigg]$$
$$+\sum_{n=0}^{\infty}\bigg|\frac{\beta_p(\zeta+\rho n,c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta,c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n\bigg|\bigg[\bigg(\frac{1}{p\xi^{'}+p\mu n+1}\bigg)^{\frac{1}{p}}\bigg(\big|\phi^{'}(\alpha)\big|^q\int_0^1\wp^sd\wp$$
$$+\big|\phi^{'}(\tau)\big|^q\int_0^1(1-\wp)^sd\wp\bigg)^{\frac{1}{q}}\bigg].$$

$$\aleph\leq\mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa,p)\bigg(\frac{1}{p\xi^{'}+p\mu n+1}\bigg)^{\frac{1}{p}}\bigg[\big|\phi^{'}(\alpha)\big|^q\frac{1}{s+1}+\big|\phi^{'}(\omega)\big|^q\frac{1}{s+1}\bigg]^{\frac{1}{q}}+$$

$$\mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa,p)\left(\frac{1}{p\xi'+p\mu n+1}\right)^{\frac{1}{p}}\left[\left|\phi'(\alpha)\right|^q\frac{1}{s+1}+\left|\phi'(\tau)\right|^q\frac{1}{s+1}\right]^{\frac{1}{q}}$$

The proof is completed.

Corollary 5. *If we replace $p=0,\kappa=0$, and $\xi=\xi-1$ in Theorem (4), we have an inequality [19].*

4. Behavior of Hermite-Hadamard type fractional integral inequalities for the class of twice differentiable function

In this section, we develop the lemma for twice differentiable s-convex function with extended Bessel-Maitland function as a kernel which is helpful to prove our main results.

Lemma 2. *Let $\phi: I\subseteq\mathbb{R}\to\mathbb{R}$ be twice differentiable mapping on I^o and let, if $\omega,\tau\in I^o$ with $\omega<\alpha<\tau$ such that $\phi''\in L[\omega,\tau]$, and $\mu,\xi,\zeta,\varsigma,c,\kappa_1\in\mathbb{C},\Re(\mu)>0,\Re(\xi)>0,\Re(\zeta)>0,\Re(\varsigma)>0,\Re(\kappa_1)>0,\rho,m,\eta\geq 0$ and $m,\rho>\Re(\mu)+\eta$, then we get the following result;*

$$(\xi')\mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa,p)\big[\frac{\phi(\alpha)+\phi(\omega)}{2}\big]+\left[\frac{(\mu n+1)(\mu n)}{2(\alpha-\omega)^{\mu n}}-\frac{(\xi'+\mu n+1)(\xi'+\mu n)}{2(\alpha-\omega)^{\xi'+\mu n}}\right]\times$$
$$\left[\big(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\omega^+}\phi\big)(\alpha,\kappa)+\big(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\alpha^-}\phi\big)(\omega,\kappa)\right]$$
$$=\frac{(\alpha-\omega)^2}{2}\int_0^1\wp(1-\wp^{\xi'})\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa\wp^{\mu};p)\big[\phi''\big(\wp\omega+(1-\wp)\alpha+\phi''\big((1-\wp)\omega+\wp\alpha\big)\big]d\wp.$$

Proof. Consider the integral

$$I=\int_0^1\wp(1-\wp^{\xi'})\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa\wp^{\mu};p)\big[\phi''\big(\wp\omega+(1-\wp)\alpha+\phi''\big((1-\wp)\omega+\wp\alpha\big)\big]d\wp$$

$$\begin{aligned}I=&\int_0^1\wp(1-\wp^{\xi'})\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa\wp^{\mu};p)\phi''\big(\wp\omega+(1-\wp)\alpha\big)d\wp\\&+\int_0^1\wp(1-\wp^{\xi'})\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa\wp^{\mu};p)\phi''\big((1-\wp)\omega+\wp\alpha\big)d\wp\end{aligned}\tag{21}$$

$I=I_1+I_2$.
Consider the integral I_1

$$I_1=\int_0^1\wp(1-\wp^{\xi'})\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa\wp^{\mu};p)\phi''\big(\wp\omega+(1-\wp)\alpha\big)d\wp$$

$$=\sum_{n=0}^{\infty}\frac{\beta_p(\zeta+\rho n,c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta,c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n\int_0^1(\wp-\wp^{\xi'+1})\wp^{\mu n}\phi''\big(\wp\omega+(1-\wp)\alpha\big)d\wp$$

$$= \sum_{n=0}^{\infty} \frac{\beta_p(\zeta+\rho n, c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta, c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n \int_0^1 (\wp^{\mu n+1} - \wp^{\xi'+\mu n+1})\phi''\big(\wp\omega+(1-\wp)\alpha\big)d\wp$$
$$= \sum_{n=0}^{\infty} \frac{\beta_p(\zeta+\rho n, c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta, c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n \Bigg[(\wp^{\mu n+1} - \wp^{\xi'+\mu n+1})\frac{\phi'\big(\wp\omega+(1-\wp)\alpha\big)}{\omega-\alpha}\Bigg|_0^1$$
$$- \int_0^1 \Big((\mu n+1)\wp^{\mu n} - (\xi'+\mu n+1)\wp^{\xi'+\mu n}\Big)\frac{\phi'\big(\wp\omega+(1-\wp)\alpha\big)}{\omega-\alpha}d\wp\Bigg].$$

After simplification, we have

$$I_1 = \sum_{n=0}^{\infty} \frac{\beta_p(\zeta+\rho n, c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta, c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n \left[\frac{(\mu n+1)}{\alpha-\omega}\int_0^1 \wp^{\mu n}\phi'\big(\wp\omega+(1-\wp)\alpha\big)d\wp\right] -$$
$$\sum_{n=0}^{\infty} \frac{\beta_p(\zeta+\rho n, c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta, c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n \left[\frac{(\xi'+\mu n+1)}{\alpha-\omega}\int_0^1 \wp^{\xi'+\mu n}\phi'\big(\wp\omega+(1-\wp)\alpha\big)d\wp\right] \quad (22)$$

$I_1 = I_3 - I_4$
Taking I_3 from above equation

$$I_3 = \sum_{n=0}^{\infty} \frac{\beta_p(\zeta+\rho n, c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta, c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n \left[\frac{(\mu n+1)}{\alpha-\omega}\int_0^1 \wp^{\mu n}\phi'\big(\wp\omega+(1-\wp)\alpha\big)d\wp\right]$$

$$= \sum_{n=0}^{\infty} \frac{\beta_p(\zeta+\rho n, c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta, c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n \Bigg[\frac{(\mu n+1)}{\alpha-\omega}\Bigg\{\wp^{\mu n}\frac{\phi\big(\wp\omega+(1-\wp)\alpha\big)}{\omega-\alpha}\Bigg|_0^1$$
$$- \int_0^1 (\mu n)\wp^{\mu n-1}\frac{\phi\big(\wp\omega+(1-\wp)\alpha\big)}{\omega-\alpha}d\wp\Bigg\}\Bigg].$$

$$I_3 = \sum_{n=0}^{\infty} \frac{\beta_p(\zeta+\rho n, c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta, c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n \left[\frac{(\mu n+1)}{\alpha-\omega}\left\{-\frac{\phi(\omega)}{\alpha-\omega} + \int_0^1 (\mu n)\wp^{\mu n-1}\frac{\phi\big(\wp\omega+(1-\wp)\alpha\big)}{\alpha-\omega}d\wp\right\}\right]$$

$$= \mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa, p)\left[\frac{(\mu n+1)}{(\alpha-\omega)^2}\Big(-\phi(\omega)\Big)\right] + \sum_{n=0}^{\infty} \frac{\beta_p(\zeta+\rho n, c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta, c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n \Bigg[\frac{(\mu n+1)(\mu n)}{(\alpha-\omega)^2} \times$$
$$\int_0^1 \wp^{\mu n-1}\phi\big(\wp\omega+(1-\wp)\alpha\big)d\wp\Bigg]$$

Use substitution $\wp\omega + (1-\wp)\alpha = \ell$ in above equation and after solving, we get the result;

$$I_3 = \mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa, p)\left[\frac{(\mu n+1)}{(\alpha-\omega)^2}\Big(-\phi(\omega)\Big)\right] + \frac{(\mu n+1)(\mu n)}{(\alpha-\omega)^{2+\mu n}}\big(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\omega^+}\phi\big)(\alpha, \kappa_1) \quad (23)$$

Now, solving I_4 from equation (22)

$$I_4=\sum_{n=0}^{\infty}\frac{\beta_p(\zeta+\rho n,c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta,c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n\left[\frac{(\xi^{'}+\mu n+1)}{\alpha-\omega}\int_0^1\wp^{\xi^{'}+\mu n}\phi^{'}\big(\wp\omega+(1-\wp)\alpha\big)d\wp\right]$$

$$=\sum_{n=0}^{\infty}\frac{\beta_p(\zeta+\rho n,c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta,c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n\left[\frac{(\xi^{'}+\mu n+1)}{\alpha-\omega}\left\{\wp^{\xi^{'}+\mu n}\frac{\phi\big(\wp\omega+(1-\wp)\alpha\big)}{\omega-\alpha}\Bigg|_0^1\right.\right.$$
$$\left.\left.-\int_0^1(\xi^{'}+\mu n)\wp^{\xi^{'}+\mu n-1}\frac{\phi\big(\wp\omega+(1-\wp)\alpha\big)}{\omega-\alpha}d\wp\right\}\right]$$

$$=\sum_{n=0}^{\infty}\frac{\beta_p(\zeta+\rho n,c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta,c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n\left[\frac{(\xi^{'}+\mu n+1)}{\alpha-\omega}\left\{-\frac{\phi(\omega)}{\alpha-\omega}\right.\right.$$
$$\left.\left.+\int_0^1(\xi^{'}+\mu n)\wp^{\xi^{'}+\mu n-1}\frac{\phi\big(\wp\omega+(1-\wp)\alpha\big)}{\alpha-\omega}d\wp\right\}\right]$$

$$=\mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa,p)\left[\frac{(\xi^{'}+\mu n+1)}{(\alpha-\omega)^2}\Big(-\phi(\omega)\Big)\right]+\sum_{n=0}^{\infty}\frac{\beta_p(\zeta+\rho n,c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta,c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n\times$$
$$\left[\frac{(\xi^{'}+\mu n+1)(\xi^{'}+\mu n)}{(\alpha-\omega)^2}\int_0^1\wp^{\xi^{'}+\mu n-1}\phi\big(\wp\omega+(1-\wp)\alpha\big)d\wp\right]$$

Use substitution $\wp\omega+(1-\wp)\alpha=\ell$ in above equation and after solving, we get the result;

$$I_4=\mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa,p)\left[\frac{(\xi^{'}+\mu n+1)}{(\alpha-\omega)^2}\Big(-\phi(\omega)\Big)\right]+\frac{(\xi^{'}+\mu n+1)(\xi^{'}+\mu n)}{(\alpha-\omega)^{2+\xi^{'}+\mu n}}\big(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\omega^+}\phi\big)(\alpha,\kappa_1)\tag{24}$$

Using equation (23), (24) in (22), we have

$$I_1=\big(\frac{\xi^{'}}{(\alpha-\omega)^2}\big)\mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa,p)[\phi(\omega)]+\big(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\omega^+}\phi\big)(\alpha,\kappa_1)\left[\frac{(\mu n+1)(\mu n)}{(\alpha-\omega)^{2+\mu n}}-\frac{(\xi^{'}+\mu n+1)(\xi^{'}+\mu n)}{(\alpha-\omega)^{2+\xi^{'}+\mu n}}\right].\tag{25}$$

Similarly, we solve I_2 and get the result;

$$I_2=\big(\frac{\xi^{'}}{(\alpha-\omega)^2}\big)\mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa,p)[\phi(\alpha)]+\big(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\alpha^-}\phi\big)(\omega,\kappa)\left[\frac{(\mu n+1)(\mu n)}{(\alpha-\omega)^{2+\mu n}}-\frac{(\xi^{'}+\mu n+1)(\xi^{'}+\mu n)}{(\alpha-\omega)^{2+\xi^{'}+\mu n}}\right]\tag{26}$$

Combining equations (25) and (26), then simplify, we have

$$I=\big(\xi^{'}\big)\mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa,p)\big[\frac{\phi(\alpha)+\phi(\omega)}{2}\big]+\left[\frac{(\mu n+1)(\mu n)}{2(\alpha-\omega)^{\mu n}}-\frac{(\xi^{'}+\mu n+1)(\xi^{'}+\mu n)}{2(\alpha-\omega)^{\xi^{'}+\mu n}}\right]\times$$
$$\left[\big(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\omega^+}\phi\big)(\alpha,\kappa)+\big(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\alpha^-}\phi\big)(\omega,\kappa)\right].$$

Hence the required result is proved.

Corollary 6. *If we replace $p = 0, \kappa = 0$, and $\xi = \xi - 1$ in the Lamma (2), we have a result [19].*

Theorem 5. *Let $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be twice differentiable mapping on I^o and $\omega, \tau \in I^o$ with $\omega < \alpha < \tau$ such that $\phi^{''} \in L[\omega, \tau]$. If $|\phi^{''}|$ is s-convex function on I, and $\mu, \xi, \zeta, \varsigma, c, \kappa_1 \in \mathbb{C}, \Re(\mu) > 0, \Re(\xi) > 0, \Re(\zeta) > 0, \Re(\varsigma) > 0, \Re(\kappa_1) > 0, \rho, m, \eta \geq 0$ and $m, \rho > \Re(\mu) + \eta$, then following fractional integral inequality holds;*

$$\Big|(\xi^{'})\mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa, p)\big[\frac{\phi(\alpha)+\phi(\omega)}{2}\big] + \Big[\frac{(\mu n+1)(\mu n)}{2(\alpha-\omega)^{\mu n}} - \frac{(\xi^{'}+\mu n+1)(\xi^{'}+\mu n)}{2(\alpha-\omega)^{\xi^{'}+\mu n}}\Big] \times$$
$$\Big[\big(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\omega^+}\phi\big)(\alpha,\kappa) + \big(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\alpha^-}\phi\big)(\omega,\kappa)\Big]\Big|$$
$$\leq \sum_{n=0}^{\infty}\Big|\frac{\beta_p(\zeta+\rho n, c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta, c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n\Big|\frac{(\alpha-\omega)^2}{2}\Bigg[\bigg(|\phi^{''}(\omega)+|\phi^{''}(\alpha)|\bigg)\big[\beta(\mu n+2+s, \xi^{'}+1)$$
$$+\beta(\mu n+2, \xi^{'}+s+1)\big]\Bigg].$$

Proof. By using Lemma 2

$$\beth := \Big|(\xi^{'})\mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa, p)\big[\frac{\phi(\alpha)+\phi(\omega)}{2}\big] + \Big[\frac{(\mu n+1)(\mu n)}{2(\alpha-\omega)^{\mu n}} - \frac{(\xi^{'}+\mu n+1)(\xi^{'}+\mu n)}{2(\alpha-\omega)^{\xi^{'}+\mu n}}\Big] \times$$
$$\Big[\big(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\omega^+}\phi\big)(\alpha,\kappa) + \big(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\alpha^-}\phi\big)(\omega,\kappa)\Big]\Big|$$
$$\leq \frac{(\alpha-\omega)^2}{2}\int_0^1 \wp(1-\wp^{\xi^{'}})\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa\wp^{\mu}; p)\big|\big[\phi^{''}\big(\wp\omega+(1-\wp)\alpha+\phi^{''}\big((1-\wp)\omega+\wp\alpha\big)\big]\big|d\wp$$

$$\aleph \leq \sum_{n=0}^{\infty}\Big|\frac{\beta_p(\zeta+\rho n, c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta, c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n\Big|\frac{(\alpha-\omega)^2}{2} \times$$
$$\Big[\int_0^1 \wp^{\mu n+1}(1-\wp^{\xi^{'}})\big|\big[\phi^{''}\big(\wp\omega+(1-\wp)\alpha+\phi^{''}\big((1-\wp)\omega+\wp\alpha\big)\big]\big|\Big]d\wp$$

$$\beth \leq \sum_{n=0}^{\infty}\Big|\frac{\beta_p(\zeta+\rho n, c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta, c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n\Big|\frac{(\alpha-\omega)^2}{2} \times$$
$$\Big[\int_0^1 \wp^{\mu n+1}(1-\wp^{\xi^{'}})\big|\big[\phi^{''}\big(\wp\omega+(1-\wp)\alpha\big)\big]\big|d\wp + \int_0^1 \wp^{\mu n+1}(1-\wp^{\xi^{'}})\big|\big[\phi^{''}\big((1-\wp)\omega+\wp\alpha\big)\big]\big|\Big]d\wp$$

As $|\phi^{''}|$ is s-convex function;

$$\beth \leq \sum_{n=0}^{\infty}\Big|\frac{\beta_p(\zeta+\rho n, c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta, c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n\Big|\frac{(\alpha-\omega)^2}{2} \times$$

$$\Big[\int_0^1 \wp^{\mu n+1}(1-\wp^{\xi'})\big|\big[\wp^s|\phi''(\omega)|+(1-\wp)^s|\phi''(\alpha)|\big]d\wp+$$
$$\int_0^1 \wp^{\mu n+1}(1-\wp^{\xi'})\big|\big[(1-\wp)^s|\phi''(\omega)|+\wp^s|\phi''(\alpha)|\big]\Big]d\wp$$

Since $\wp^{\xi'}\geq\wp,\xi'\in(0,1]$ and $\wp\in[0,1]$, we have $-\wp^{\xi'}\leq\wp\Rightarrow 1-\wp^{\xi'}\leq 1-\wp\leq(1-\wp)^{\xi'}$

$$\beth\leq\sum_{n=0}^{\infty}\left|\frac{\beta_p(\zeta+\rho n,c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta,c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n\right|\frac{(\alpha-\omega)^2}{2}\Big[|\phi''(\omega)|\int_0^1\wp^{\mu n+1+s}(1-\wp)^{\xi'}d\wp+$$
$$|\phi''(\alpha)|\int_0^1\wp^{\mu n+1}(1-\wp)^{\xi'+s}d\wp+|\phi''(\omega)|\int_0^1\wp^{\mu n+1}(1-\wp)^{\xi'+s}d\wp+|\phi''(\alpha)|\int_0^1\wp^{\mu n+1+s}(1-\wp)^{\xi'}d\wp]\Big]$$

$$\beth\leq\sum_{n=0}^{\infty}\left|\frac{\beta_p(\zeta+\rho n,c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta,c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n\right|\frac{(\alpha-\omega)^2}{2}\Big[|\phi''(\omega)|\beta(\mu n+2+s,\xi'+1)+$$
$$|\phi''(\alpha)|\beta(\mu n+2,\xi'+s+1)+|\phi''(\omega)\beta(\mu n+2,\xi'+s+1)+|\phi''(\alpha)|\beta(\mu n+2+s,\xi'+1)\Big]$$

$$\beth\leq\sum_{n=0}^{\infty}\left|\frac{\beta_p(\zeta+\rho n,c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta,c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n\right|\frac{(\alpha-\omega)^2}{2}\Big[\Big(|\phi''(\omega)+|\phi''(\alpha)|\Big)[\beta(\mu n+2+s,\xi'+1)$$
$$+\beta(\mu n+2,\xi'+s+1)]\Big]$$

Corollary 7. *If we replace $p=0,\kappa=0$, and $\xi=\xi-1$ in Theorem (5), we have a result [19].*

Theorem 6. *Let $\phi: I\subseteq\mathbb{R}\to\mathbb{R}$ be twice differentiable mapping on I^o and $\omega,\tau\in I^o$ with $\omega<\alpha<\tau$ such that $\phi''\in L[\omega,\tau]$. If $|\phi''|^q(q>1)$ is s-convex function on I, and $\mu,\xi,\zeta,\varsigma,c,\kappa_1\in\mathbb{C},\Re(\mu)>0,\Re(\xi)>0,\Re(\zeta)>0,\Re(\varsigma)>0,\Re(\kappa_1)>0,\rho,m,\eta\geq 0$ and $m,\rho>\Re(\mu)+\eta$, then we have the following fractional inequality*

$$\Big|(\xi')\mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa,p)\big[\frac{\phi(\alpha)+\phi(\omega)}{2}\big]+\Big[\frac{(\mu n+1)(\mu n)}{2(\alpha-\omega)^{\mu n}}-\frac{(\xi'+\mu n+1)(\xi'+\mu n)}{2(\alpha-\omega)^{\xi'+\mu n}}\Big]\times$$
$$\Big[\big(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\omega^+}\phi\big)(\alpha,\kappa)+\big(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\alpha^-}\phi\big)(\omega,\kappa)\Big]\Big|$$
$$\leq\sum_{n=0}^{\infty}\left|\frac{\beta_p(\zeta+\rho n,c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta,c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n\right|(\alpha-\omega)^2\times$$
$$\Big[\Big(\beta(\mu np+p+1,\xi'p+1)\Big)^{\frac{1}{p}}\Big(\frac{|\phi''(\omega)|^q+|\phi''(\alpha)|^q}{s+1}\Big)^{\frac{1}{q}}\Big].\tag{27}$$

Proof. By using Lemma 2

$$\beth = \Bigg|(\xi^{'})\mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa,p)\big[\frac{\phi(\alpha)+\phi(\omega)}{2}\big] + \Bigg[\frac{(\mu n+1)(\mu n)}{2(\alpha-\omega)^{\mu n}} - \frac{(\xi^{'}+\mu n+1)(\xi^{'}+\mu n)}{2(\alpha-\omega)^{\xi^{'}+\mu n}}\Bigg]\times$$
$$\Bigg[\big(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\omega^+}\phi\big)(\alpha,\kappa) + \big(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\alpha^-}\phi\big)(\omega,\kappa)\Bigg]\Bigg|$$
$$\leq \frac{(\alpha-\omega)^2}{2}\int_0^1 \wp(1-\wp^{\xi^{'}})\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa\wp^{\mu};p)\big|\big[\phi^{''}\big(\wp\omega+(1-\wp)\alpha+\phi^{''}\big((1-\wp)\omega+\wp\alpha\big)\big]\big|d\wp$$

$$\beth \leq \sum_{n=0}^{\infty}\Bigg|\frac{\beta_p(\zeta+\rho n,c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta,c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n\Bigg|\frac{(\alpha-\omega)^2}{2}\times$$
$$\Bigg[\int_0^1 \wp^{\mu n+1}(1-\wp^{\xi^{'}})\big|\big[\phi^{''}\big(\wp\omega+(1-\wp)\alpha+\phi^{''}\big((1-\wp)\omega+\wp\alpha\big)\big]\big|\Bigg]d\wp$$

$$\beth \leq \sum_{n=0}^{\infty}\Bigg|\frac{\beta_p(\zeta+\rho n,c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta,c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n\Bigg|\frac{(\alpha-\omega)^2}{2}\times$$
$$\Bigg[\int_0^1 \wp^{\mu n+1}(1-\wp^{\xi^{'}})\big|\big[\phi^{''}\big(\wp\omega+(1-\wp)\alpha\big)\big]\big|d\wp + \int_0^1 \wp^{\mu n+1}(1-\wp^{\xi^{'}})\big|\big[\phi^{''}\big((1-\wp)\omega+\wp\alpha\big)\big]\big|\Bigg]d\wp.$$

Using Holder inequality,

$$\beth \leq \sum_{n=0}^{\infty}\Bigg|\frac{\beta_p(\zeta+\rho n,c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta,c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n\Bigg|\frac{(\alpha-\omega)^2}{2}\times$$
$$\Bigg[\Bigg(\int_0^1 \wp^{\mu np+p}(1-\wp^{\xi^{'}})^p\Bigg)^{\frac{1}{p}}\Bigg(\int_0^1 \big|\big[\phi^{''}\big(\wp\omega+(1-\wp)\alpha\big)\big]\big|^q\Bigg)^{\frac{1}{q}}d\wp +$$
$$\Bigg(\int_0^1 \wp^{\mu np+p}(1-\wp^{\xi^{'}})^p\Bigg)^{\frac{1}{p}}\Bigg(\int_0^1 \big|\big[\phi^{''}\big((1-\wp)\omega+\wp\alpha\big)\big]\big|^q\Bigg)^{\frac{1}{q}}d\wp\Bigg]d\wp \tag{28}$$

Since, $\wp^{\xi^{'}} \geq \wp, \xi^{'} \in (0,1]$ and $\wp \in [0,1]$, we have $-\wp^{\xi^{'}} \leq \wp \Rightarrow 1-\wp^{\xi^{'}} \leq 1-\wp \leq (1-\wp)^{\xi^{'}}$

$$\beth \leq \sum_{n=0}^{\infty}\Bigg|\frac{\beta_p(\zeta+\rho n,c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta,c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n\Bigg|\frac{(\alpha-\omega)^2}{2}\times$$
$$\Bigg[\Bigg(\int_0^1 \wp^{\mu np+p}(1-\wp)^{\xi^{'}p}\Bigg)^{\frac{1}{p}}\Bigg\{\Bigg(\int_0^1 \big|\big[\phi^{''}\big(\wp\omega+(1-\wp)\alpha\big)\big]\big|^q\Bigg)^{\frac{1}{q}}d\wp +$$
$$\Bigg(\int_0^1 \big|\big[\phi^{''}\big((1-\wp)\omega+\wp\alpha\big)\big]\big|^q\Bigg)^{\frac{1}{q}}d\wp\Bigg\}\Bigg] \tag{29}$$

As $|\phi^{''}|^q$ is s-convex function;

$$\int_0^1 \big|\big[\phi^{''}\big(\wp\omega+(1-\wp)\alpha\big)\big]\big|^q d\wp \leq |\phi^{''}(\omega)|^q\int_0^1 \wp^s d\wp + |\phi^{''}(\alpha)|^q\int_0^1(1-\wp)^s d\wp$$

$$= \frac{|\phi^{''}(\omega)|^q + |\phi^{''}(\alpha)|^q}{s+1} \tag{30}$$

$$\int_0^1 \left|\left[\phi^{''}\left((1-\wp)\omega + \wp\alpha\right)\right]\right|^q d\wp \leq \phi^{''}(\omega)|^q \int_0^1 (1-\wp)^s d\wp + |\phi^{''}(\alpha)|^q \int_0^1 \wp^s d\wp$$
$$= \frac{|\phi^{''}(\omega)|^q + |\phi^{''}(\alpha)|^q}{s+1} \tag{31}$$

$$\beta(\mu np + p + 1, \xi^{'} p + 1) = \int_0^1 \wp^{\mu np + p}(1-\wp)^{\xi^{'} p} d\wp \tag{32}$$

Substitute equations (30), (31),(32) in equation (29), then we get

$$\beth \leq \sum_{n=0}^{\infty} \left| \frac{\beta_p(\zeta + \rho n, c - \zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta, c-\zeta)\Gamma(\mu n + \xi + 1)(\varsigma)_{mn}} (-\kappa)^n \right| \frac{(\alpha - \omega)^2}{2} \times$$
$$\left[\left(\beta(\mu np + p + 1, \xi^{'} p + 1) \right)^{\frac{1}{p}} \left\{ \left(\frac{|\phi^{''}(\omega)|^q + |\phi^{''}(\alpha)|^q}{s+1} \right)^{\frac{1}{q}} + \left(\frac{|\phi^{''}(\omega)|^q + |\phi^{''}(\alpha)|^q}{s+1} \right)^{\frac{1}{q}} \right\} \right]$$

The proof is completed.

Corollary 8. *If we replace $p = 0, \kappa = 0,$ and $\xi = \xi - 1$ in Theorem (6), we have a result [19].*

Theorem 7. *Let $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be twice differentiable mapping on I^o and $\omega, \tau \in I^o$ with $\omega < \alpha < \tau$ such that $\phi^{''} \in L[\omega, \tau]$. If $|\phi^{''}|^q (q \geq 1)$ is s-convex function on I, and $\mu, \xi, \zeta, \varsigma, c, \kappa_1 \in \mathbb{C}, \Re(\mu) > 0, \Re(\xi) > 0, \Re(\zeta) > 0, \Re(\varsigma) > 0, \Re(\kappa_1) > 0, \rho, m, \eta \geq 0$ and $m, \rho > \Re(\mu) + \eta$, then following integral inequality holds as a result;*

$$\Bigg| (\xi^{'}) \mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa, p) [\frac{\phi(\alpha) + \phi(\omega)}{2}] + \left[\frac{(\mu n + 1)(\mu n)}{2(\alpha - \omega)^{\mu n}} - \frac{(\xi^{'} + \mu n + 1)(\xi^{'} + \mu n)}{2(\alpha - \omega)^{\xi^{'} + \mu n}} \right] \times$$
$$\left[\left(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\omega^+} \phi \right)(\alpha, \kappa) + \left(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\alpha^-} \phi \right)(\omega, \kappa) \right] \Bigg|$$
$$\leq \sum_{n=0}^{\infty} \left| \frac{\beta_p(\zeta + \rho n, c - \zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta, c-\zeta)\Gamma(\mu n + \xi + 1)(\varsigma)_{mn}} (-\kappa)^n \right| \frac{(\alpha - \omega)^2}{2} \left[\left(\frac{\xi^{'}}{(\mu n + 2)(\xi^{'} + \mu n + 2)} \right)^{1 - \frac{1}{q}} \times \right.$$
$$\left\{ \left(|\phi^{''}(\omega)|^q \beta(\mu n + s + 2, \xi^{'} + 1) + |\phi^{''}|^q(\alpha) \beta(\mu n + 2, \xi^{'} + s + 1) \right)^{\frac{1}{q}} + \right.$$
$$\left. \left. \left(|\phi^{''}(\omega)|^q \beta(\mu n + 2, \xi^{'} + s + 1) + |\phi^{''}|^q \beta(\mu n + s + 2, \xi^{'} + 1) \right)^{\frac{1}{q}} \right\} \right]$$

Proof. By using Lemma 2, we have

$$\beth = \Bigg|(\xi')\mathbb{J}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa,p)\big[\frac{\phi(\alpha)+\phi(\omega)}{2}\big] + \Bigg[\frac{(\mu n+1)(\mu n)}{2(\alpha-\omega)^{\mu n}} - \frac{(\xi'+\mu n+1)(\xi'+\mu n)}{2(\alpha-\omega)^{\xi'+\mu n}}\Bigg] \times$$
$$\Bigg[\big(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\omega^+}\phi\big)(\alpha,\kappa) + \big(\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1,\alpha^-}\phi\big)(\omega,\kappa)\Bigg]\Bigg|$$
$$\leq \frac{(\alpha-\omega)^2}{2}\int_0^1 \wp(1-\wp^{\xi'})\mathbb{E}^{\mu,\rho,m,\eta,c}_{\xi,\zeta,\varsigma,\kappa_1}(\kappa\wp^{\mu};p)\big|\big[\phi''\big(\wp\omega+(1-\wp)\alpha+\phi''\big((1-\wp)\omega+\wp\alpha)\big]\big|d\wp.$$

$$\beth \leq \sum_{n=0}^{\infty}\Bigg|\frac{\beta_p(\zeta+\rho n, c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta,c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n\Bigg|\frac{(\alpha-\omega)^2}{2}\times$$
$$\Bigg[\int_0^1 \wp^{\mu n+1}(1-\wp^{\xi'})\big|\big[\phi''\big(\wp\omega+(1-\wp)\alpha)\big]\big|d\wp + \int_0^1 \wp^{\mu n+1}(1-\wp^{\xi'})\big|\big[\phi''\big((1-\wp)\omega+\wp\alpha)\big]\big|\Bigg]d\wp.$$

Using Power mean inequality,

$$\beth \leq \sum_{n=0}^{\infty}\Bigg|\frac{\beta_p(\zeta+\rho n, c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta,c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n\Bigg|\frac{(\alpha-\omega)^2}{2}\times$$
$$\Bigg[\Bigg(\int_0^1 \wp^{\mu n+1}(1-\wp^{\xi'})\Bigg)^{1-\frac{1}{q}}\Bigg(\int_0^1 \wp^{\mu n+1}(1-\wp^{\xi'})\big|\big[\phi''\big(\wp\omega+(1-\wp)\alpha)\big]\big|^q\Bigg)^{\frac{1}{q}}d\wp +$$
$$\Bigg(\int_0^1 \wp^{\mu n+1}(1-\wp^{\xi'})\Bigg)^{1-\frac{1}{q}}\Bigg(\int_0^1 \wp^{\mu n+1}(1-\wp^{\xi'})\big|\big[\phi''\big((1-\wp)\omega+\wp\alpha)\big]\big|^q\Bigg)^{\frac{1}{q}}d\wp\Bigg] \quad (33)$$

Using Modulus property, we have

$$\beth \leq \sum_{n=0}^{\infty}\Bigg|\frac{\beta_p(\zeta+\rho n, c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta,c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n\Bigg|\frac{(\alpha-\omega)^2}{2}\times$$
$$\Bigg[\Bigg(\int_0^1 \wp^{\mu n+1}(1-\wp^{\xi'})\Bigg)^{1-\frac{1}{q}}\Bigg\{\Bigg(\int_0^1 \wp^{\mu n+1}(1-\wp^{\xi'})\big|\big[\phi''\big(\wp\omega+(1-\wp)\alpha)\big]\big|^q\Bigg)^{\frac{1}{q}}d\wp +$$
$$\Bigg(\int_0^1 \wp^{\mu n+1}(1-\wp^{\xi'})\big|\big[\phi''\big((1-\wp)\omega+\wp\alpha)\big]\big|^q\Bigg)^{\frac{1}{q}}d\wp\Bigg\}\Bigg] \quad (34)$$

Now, consider the integral

$$\int_0^1 \wp^{\mu n+1}(1-\wp^{\xi'})d\wp = \int_0^1(\wp^{\mu n+1}-\wp^{\xi'+\mu n+1})d\wp = \frac{\xi'}{(\mu n+2)(\xi'+\mu n+2)} \quad (35)$$

Putting the value (35) in equation (34), we have

$$\beth \leq \sum_{n=0}^{\infty}\Bigg|\frac{\beta_p(\zeta+\rho n, c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta,c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n\Bigg|\frac{(\alpha-\omega)^2}{2}\Bigg[\Bigg(\frac{\xi'}{(\mu n+2)(\xi'+\mu n+2)}\Bigg)^{1-\frac{1}{q}}\times$$

$$\Bigg\{\bigg(\int_0^1 \wp^{\mu n+1}(1-\wp^{\xi'})[|\phi''(\omega)|^q\wp^s+|\phi''(\alpha)|^q(1-\wp)^s]d\wp\bigg)^{\frac{1}{q}}+$$
$$\bigg(\int_0^1 \wp^{\mu n+1}(1-\wp^{\xi'})[|\phi''(\omega)|^q(1-\wp)^s+|\phi''(\alpha)|^q\wp^s]d\wp\bigg)^{\frac{1}{q}}\Bigg\}\Bigg] \tag{36}$$

Since $\wp^{\xi'}\geq\wp,\xi'\in(0,1]$ and $\wp\in[0,1]$, we have $-\wp^{\xi'}\leq\wp\Rightarrow 1-\wp^{\xi'}\leq 1-\wp\leq(1-\wp)^{\xi'}$
Simplify;

$$\int_0^1 \wp^{\mu n+1}(1-\wp)^{\xi'}[|\phi''(\omega)|^q\wp^s+|\phi''(\alpha)|^q(1-\wp)^s]d\wp=$$
$$|\phi''(\omega)|^q\int_0^1 \wp^{\mu n+s+1}(1-\wp)^{\xi'}d\wp+|\phi''(\alpha)|^q\int_0^1 \wp^{\mu n+1}(1-\wp)^{\xi'+s}$$
$$=|\phi''(\omega)|^q\beta(\mu n+s+2,\xi'+1)+|\phi''|^q(\alpha)\beta(\mu n+2,\xi'+s+1). \tag{37}$$

Consider the integral

$$\int_0^1 \wp^{\mu n+1}(1-\wp)^{\xi'}[|\phi''(\omega)|^q(1-\wp)^s+|\phi''(\alpha)|^q\wp^s]d\wp=$$
$$|\phi''(\omega)|^q\int_0^1 \wp^{\mu n+1}(1-\wp)^{\xi'+s}d\wp+|\phi''(\alpha)|^q\int_0^1 \wp^{\mu n+s+1}(1-\wp)^{\xi'}d\wp$$
$$=|\phi''(\omega)|^q\beta(\mu n+2,\xi'+s+1)+|\phi''|^q\beta(\mu n+s+2,\xi'+1) \tag{38}$$

Use equations (37) and (38) in (36) then we get;

$$\beth\leq\sum_{n=0}^{\infty}\left|\frac{\beta_p(\zeta+\rho n,c-\zeta)(c_{\rho n})(\kappa_1)_{\eta n}}{\beta(\zeta,c-\zeta)\Gamma(\mu n+\xi+1)(\varsigma)_{mn}}(-\kappa)^n\right|\frac{(\alpha-\omega)^2}{2}\Bigg[\bigg(\frac{\xi'}{(\mu n+2)(\xi'+\mu n+2)}\bigg)^{1-\frac{1}{q}}\times$$
$$\Bigg\{\bigg(|\phi''(\omega)|^q\beta(\mu n+s+2,\xi'+1)+|\phi''|^q(\alpha)\beta(\mu n+2,\xi'+s+1)\bigg)^{\frac{1}{q}}+$$
$$\bigg(|\phi''(\omega)|^q\beta(\mu n+2,\xi'+s+1)+|\phi''|^q\beta(\mu n+s+2,\xi'+1)\bigg)^{\frac{1}{q}}\Bigg\}\Bigg]$$

The proof is completed.

Corollary 9. *If we replace $p=0,\kappa=0$, and $\xi=\xi-1$ in Theorem (7), we have a result [19].*

5. Conclusion

This study introduced two innovative approaches to proving Hermite-Hadamard type inequalities using the extended Bessel-Maitland function as a kernel within the framework of s-convex functions. The first approach used differentiable functions and their first derivatives to derive a key identity that serves as the basis for establishing these inequalities. Taking a more comprehensive perspective, the second approach employs the integral of a second derivative as an alternative, yet equally effective, way to establish the fundamental identity. Beyond proving these inequalities, we explore various applications, particularly in relation to different types of means. Furthermore, this approach can be extended to other classes of convex functions, which enhances its broader mathematical significance. Ultimately, this study deepens the understanding of convexity-based inequalities and paves the way for further research in mathematical analysis.

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Availability of data and material

Data sharing is not applicable to this paper as no datasets were generated or analyzed during 218 the current study.

Competing interests

The authors declare that there is no conflict of interest regarding the publication of this article.

Author's contributions

All authors contributed equally to this manuscript. All authors read and approved the final manuscript.

References

[1] Rudolf Gorenflo and Francesco Mainardi. *Fractional calculus: integral and differential equations of fractional order*. Springer, 1997.

[2] Sunil Kumar, Kottakkaran Sooppy Nisar, Ranbir Kumar, Carlo Cattani, and Bessem Samet. A new rabotnov fractional-exponential function-based fractional derivative for diffusion equation under external force. *Mathematical Methods in the Applied Sciences*, 43(7):4460–4471, 2020.

[3] Behzad Ghanbari, Sunil Kumar, and Ranbir Kumar. A study of behaviour for immune and tumor cells in immunogenetic tumour model with non-singular fractional derivative. *Chaos, Solitons & Fractals*, 133:109619, 2020.

[4] Khalid K Ali, Mohamed A Abd El Salam, Emad MH Mohamed, Bessem Samet, Sunil Kumar, and MS Osman. Numerical solution for generalized nonlinear fractional integro-differential equations with linear functional arguments using chebyshev series. *Advances in Difference Equations*, 2020:1–23, 2020.

[5] Sunil Kumar, Surath Ghosh, Mansour SM Lotayif, and Bessem Samet. A model for describing the velocity of a particle in brownian motion by robotnov function based fractional operator. *Alexandria Engineering Journal*, 59(3):1435–1449, 2020.

[6] Gauhar Rahman, Kottakkaran Sooppy Nisar, Thabet Abdeljawad, and Muhammad Samraiz. Some new tempered fractional pólya-szegö and chebyshev-type inequalities with respect to another function. *Journal of Mathematics*, 2020(1):9858671, 2020.

[7] Muhammad Samraiz, Fakhra Nawaz, Sajid Iqbal, Thabet Abdeljawad, Gauhar Rahman, and Kottakkaran Sooppy Nisar. Certain mean-type fractional integral inequalities via different convexities with applications. *Journal of Inequalities and Applications*, 2020(1):208, 2020.

[8] Miguel Vivas-Cortez, Muhammad Aamir Ali, Artion Kashuri, and Hüseyin Budak. Generalizations of fractional hermite-hadamard-mercer like inequalities for convex functions. *AIMS Math*, 6(9):9397–9421, 2021.

[9] Tilak Raj Prabhakar. A singular integral equation with a generalized mittag leffler function in the kernel. *Yokohama Mathematical Journal= . D , *, 19(1):7–15, 1971.

[10] Hasan Barsam, Sayyed Mehrab Ramezani, and Yamin Sayyari. On the new hermite–hadamard type inequalities for s-convex functions. *Afrika Matematika*, 32(7):1355–1367, 2021.

[11] H Barsam and AR Sattarzadeh. Hermite-hadamard inequalities for uniformly convex functions and its applications in means. *Miskolc mathematical notes*, 21(2):621–630, 2020.

[12] Chun Zhu, Michal Feckan, and Jinrong Wang. Fractional integral inequalities for differentiable convex mappings and applications to special means and a midpoint formula. *Journal of Applied Mathematics, Statistics and Informatic*, 8(2), 2012.

[13] Pshtiwan Othman Mohammed and Mehmet Zeki Sarikaya. Hermite–hadamard type inequalities for f-convex function involving fractional integrals. *Journal of inequalities and applications*, 2018:1–33, 2018.

[14] Tieyan Lian, Wei Tang, and Rui Zhou. Fractional hermite–hadamard inequalities for (s, m)-convex or s-concave functions. *Journal of Inequalities and Applications*, 2018(1):240, 2018.

[15] Yousaf Khurshid, M Adil Khan, and Yu-Ming Chu. Conformable fractional integral inequalities for gg-and ga-convex function. *AIMS Math*, 5(5):5012–5030, 2020.

[16] Shahid Qaisar, Jamshed Nasir, Saad Ihsan Butt, and Sabir Hussain. On some fractional integral inequalities of hermite-hadamard's type through convexity. *Symmetry*, 11(2):137, 2019.

[17] M Emin Ozdemir, Saad I Butt, Bahtiyar Bayraktar, and Jamshed Nasir. Several integral inequalities for (α, s, m)-convex functions. *AIMS Mathematics*, 5(4):3906–3921, 2020.

[18] Hasan Barsam and Ali Sattarzadeh. Some results on hermite-hadamard inequalities.

Journal of Mahani Mathematical Research, 9(2):79–86, 2020.

[19] Khuram Ali Khan, Saeeda Fatima, Ammara Nosheen, and Rostin Matendo Mabela. New developments of hermite–hadamard type inequalities via s-convexity and fractional integrals. *Journal of Mathematics*, 2024(1):1997549, 2024.

[20] Bo-Yan Xi, Dan-Dan Gao, and Feng Qi. Integral inequalities of hermite–hadamard type for (\alpha, s) $-convexand($\alpha, s, m) $-convexfunctions. ItalianJournalofPureandAppliedMathematics, (44) : 499--510, 2020.$

[21] Muhammad Adil Khan, Yuming Chu, Tahir Ullah Khan, and Jamroz Khan. Some new inequalities of hermite-hadamard type for s-convex functions with applications. *Open Mathematics*, 15(1):1414–1430, 2017.

[22] Pshtiwan Othman Mohammed. Hermite-hadamard inequalities for riemann-liouville fractional integrals of a convex function with respect to a monotone function. *Mathematical Methods in the Applied Sciences*, 44(3):2314–2324, 2021.

[23] Thabet Abdeljawad, Saima Rashid, Zakia Hammouch, and Yu-Ming Chu. Some new local fractional inequalities associated with generalized (s, m)-convex functions and applications. *Advances in Difference Equations*, 2020(1):406, 2020.

[24] İmdat İşcan. Generalization of different type integral inequalities for s-convex functions via fractional integrals. *Applicable Analysis*, 93(9):1846–1862, 2014.

[25] Fuat Usta, Hüseyin Budak, Mehmet Zeki Sarikaya, and Erhan Set. On generalization of trapezoid type inequalities for s-convex functions with generalized fractional integral operators. *Filomat*, 32(6):2153–2171, 2018.

[26] Saad Ihsan Butt, Saba Yousaf, Ahmet Ocak Akdemir, and Mustafa Ali Dokuyucu. New hadamard-type integral inequalities via a general form of fractional integral operators. *Chaos, Solitons & Fractals*, 148:111025, 2021.

[27] Praveen Agarwal, Mohamed Jleli, and Muharrem Tomar. Certain hermite-hadamard type inequalities via generalized k-fractional integrals. *Journal of inequalities and applications*, 2017:1–10, 2017.

[28] Gheorghe Toader. Some generalizations of the convexity. In *Proceedings of the Colloquium on Approximation and Optimization*, volume 329, page 338. University of Cluj-Napoca Cluj-Napoca, Romania, 1984.

[29] Xiaoli Qiang, Ghulam Farid, Muhammad Yussouf, Khuram Ali Khan, and Atiq Ur Rahman. New generalized fractional versions of hadamard and fejér inequalities for harmonically convex functions. *Journal of Inequalities and Applications*, 2020:1–13, 2020.

[30] Muhamet Emin Özdemir, Çetin Yıldız, Ahmet Ocak Akdemir, and Erhan Set. On some inequalities for s-convex functions and applications. *Journal of Inequalities and Applications*, 2013:1–11, 2013.

[31] Josip E Peajcariaac and Yung Liang Tong. *Convex functions, partial orderings, and statistical applications*. Academic Press, 1992.

[32] İmdat İşcan and Shanhe Wu. Hermite–hadamard type inequalities for harmonically convex functions via fractional integrals. *Applied Mathematics and Computation*, 238:237–244, 2014.

[33] Daniel Alexandru Ion. Some estimates on the hermite-hadamard inequality through quasi-

convex functions. *Annals of the University of Craiova-Mathematics and Computer Science Series*, 34:82–87, 2007.

[34] ED Rainville. Special functions, chelsea publ. *Co., Bronx, New York*, 1971.

[35] Aleksandar Petojevic. A note about the pochhammer symbol. *Mathematica Moravica*, 12(1):37–42, 2008.

[36] Shahid Mubeen, Rana Safdar Ali, Iqra Nayab, Gauhar Rahman, Thabet Abdeljawad, and Kottakkaran Sooppy Nisar. Integral transforms of an extended generalized multi-index bessel function. *AIMS Mathematics*, 5(6):7531–7546, 2020.

[37] Rana Safdar Ali, Shahid Mubeen, Iqra Nayab, Serkan Araci, Gauhar Rahman, and Kottakkaran Sooppy Nisar. Some fractional operators with the generalized bessel–maitland function. *Discrete dynamics in nature and society*, 2020(1):1378457, 2020.

[38] Sabila Ali, Shahid Mubeen, Rana Safdar Ali, Gauhar Rahman, Ahmed Morsy, Kottakkaran Sooppy Nisar, Sunil Dutt Purohit, and M Zakarya. Dynamical significance of generalized fractional integral inequalities via convexity. *Aims Math*, 6(9):9705–9730, 2021.

[39] Shahid Mubeen, Rana Safdar Ali, Yasser Elmasry, Ebenezer Bonyah, Artion Kashuri, Gauhar Rahman, and Çetin Yildiz. On novel fractional integral and differential operators and their properties. *Journal of Mathematics*, 2023(1):4165363, 2023.