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On the Diophantine Equations of the Form $x^2 - kxy + ky^2 + 2^n y = 0$

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Abstract. In this article, we determine all values of k for which the equation $x^2 - kxy + ky^2 + 2^n y = 0$ where $n = 3, 4, 5, 6, 7$ has infinitely many positive solutions.

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1. Introduction

The Diophantine equation

$$x^2 + axy + by^2 + cx + dy + e = 0$$

has been analyzed for various integer values of a, b, c, d and e. Specifically, Keskin, O. Karaatli, and Z. Siar [1, 2] identified conditions under which the equation $x^2 - kxy + y^2 \pm 2^n = 0$ possesses an infinite set of positive integer solutions (x, y) for $0 \leq n \leq 10$ and provided all such solutions for this range of n. Moreover, they proposed hypotheses based on the parity of r regarding integer solutions of the equation $x^2 - kxy + y^2 = 2^r$. R. Boumahdi, O. Kihel, and S. Mavecha [3] gave a proof of this conjecture.

For any integer l, let $T(l)$ be the set of positive integers k for which the equation

$$x^2 - kxy + ky^2 + ly = 0$$

has infinitely many positive integer solutions and let $T'(l)$ be the set of positive integers k for which the equation

$$x^2 - kxy + ky^2 + l = 0, \quad \text{where} \quad (x, y) = 1$$

has infinitely many positive integer solutions (x, y).

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In 2012, O. Karaatli and Z. Siar [4] showed that $T(1) = \{5\}$, $T(2) = \{5, 6\}$, $T(4) = \{5, 6, 8\}$, and $T(8) = \{5, 6, 8, 12\}$. In 2017, Mavecha [5] studied $T(2^n)$ for a non-negative integer n and showed that 5 is the only odd integer in the set $T(2^n)$ for all non-negative integers n. In 2021, Alkabouss et al. [6] determined conditions for a positive integer k in $T(l)$. They showed that if $l^2 < k$ then (l, k) is one of $(1, 5)$, $(2, 5)$ and $(2, 6)$.

Later in 2024, S. Prugsapitak and N. Thongngam [7] showed that for a prime p and a positive integer n,

$$T(p^n) = \bigcup_{k=0}^{n} T'(p^k).$$

Their findings offer a practical method for determining $T(3^n)$ for $n = 1, 2, 3$.

In this article, we will find sets $T'(l)$ for $l = 8, 16, 64$ and 128 in order to find $T(8), T(16), T(64)$ and $T(128)$. Our approach differs from O. Karaatli and Z. Siar's method [4], as well as the method presented in [7].

2. Preliminaries

To lay the groundwork for proving our main theorems, this section establishes some essential results.

Definition 1. *[7] For a positive integer l, let $T(l)$ be the set of integers k for which the equation*

$$x^2 - kxy + ky^2 + ly = 0 \tag{1}$$

has infinitely many positive integer solutions and let $T'(l)$ be the set of integers k for which the equation

$$x^2 - kxy + ky^2 + l = 0 \tag{2}$$

has infinitely many positive integer solutions (x, y) where $\gcd(x, y) = 1$.

Theorem 1. *[7] Let p be a prime and n be a positive integer. Then*

$$T(p^n) = \bigcup_{k=0}^{n} T'(p^k).$$

Moreover, $T(p^n) = T(p^{n-1}) \cup T'(p^n)$.

Definition 2. *[8] Let p be an odd prime and a be an integer. The Legendre symbol defined as*

$$\left(\frac{a}{p}\right) = \begin{cases} 1, & \text{if } a \text{ is a quadratic residue modulo } p, \\ -1, & \text{if } a \text{ is a quadratic non-residue modulo } p, \\ 0, & \text{if } p \mid a. \end{cases}$$

Lemma 1. *[8] Let p be an odd prime. Then*

$$\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}} = \begin{cases} 1, & \text{if } p \equiv 1 \pmod{4}, \\ -1, & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Lemma 2. *[8] Let p be an odd prime. Then*

$$\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}} = \begin{cases} 1, & \text{if } p \equiv 1, 7 \pmod{8}, \\ -1, & \text{if } p \equiv 3, 5 \pmod{8}. \end{cases}$$

Definition 3. *[8] Let N be a nonzero integer and D be a positive integer which is not a perfect square. The least positive integer solution (x_1, y_1) of the equation*

$$x^2 - Dy^2 = N$$

is called the fundamental solution.

Lemma 3. *[8] Let N, D be odd positive integers with D non-square. Suppose that the equation*

$$x^2 - Dy^2 = 4, gcd(x, y) = 1$$

is solvable and let $x_0 + y_0\sqrt{D}$ be the fundamental solution. If the equation

$$u^2 - Dv^2 = -4N,$$

where $u, v \in \mathbb{Z}, \gcd(u, v) \mid 2$, is solvable, then $u^2 - Dv^2 = -4N$ has a solution $u_0 + v_0\sqrt{D}$ with the following property:

$$0 < v_0 \leq \frac{y_0\sqrt{N}}{\sqrt{(x_0 - 2)}}, \quad 0 \leq u_0 \leq \sqrt{(x_0 - 2)N}.$$

Lemma 4. *[8] Let N, D be positive integers with D non-square. Suppose that $x_0 + y_0\sqrt{D}$ is the fundamental solution of the Pell equation $x^2 - Dy^2 = 1$ and the equation*

$$u^2 - Dv^2 = -N, gcd(u, v) = 1$$

is solvable. Then $u^2 - Dv^2 = -N$ has a solution $u_0 + v_0\sqrt{D}$ with the following property:

$$0 < v_0 \leq \frac{y_0\sqrt{N}}{\sqrt{2(x_0 - 1)}}, \quad 0 \leq u_0 \leq \sqrt{\frac{1}{2}(x_0 - 2)N}.$$

Lemma 5. *[9] Let $k > 3$. Then the equation $u^2 - (k^2 - 4)v^2 = -4$ has no integer solutions.*

3. Main Results

To begin our analysis, we will consider solutions to certain Pell's equations as follows:

Lemma 6. *Let k be a positive integer. The equation $u^2 - (k^2 - 4)v^2 = -4$ has positive integer solutions u and v if and only if $k = 3$.*

Proof. Suppose that $u^2 - (k^2 - 4)v^2 = -4$ has positive integer solutions u and v. By Lemma 5, the equation $u^2 - (k^2 - 4)v^2 = -4$ has no integer solutions u and v for $k > 3$. If $k = 1$, then $u^2 + 3v^2 = -4$, which is impossible. If $k = 2$, then $u^2 = -4$, which is impossible. This implies that $k = 3$. Conversely, let $k = 3$. Then $u^2 - 5v^2 = -4$. It is easy to see that $(u, v) = (1, 1)$ is a solution of the equation $u^2 - 5v^2 = -4$.

Lemma 7. *Let $l \geq 1$ and $s \geq 2$ be positive integers. If $u^2 - s(s - 1)v^2 = -2^l$ has a solution, then $s \leq 2^l + 1$.*

Proof. Suppose $u^2 - s(s-1)v^2 = -2^l$ has a solution. We consider two cases as follows.
Case 1. $gcd(u, v) = 1$. Since $(2s-1, 2)$ is the fundamental solution of $u^2 - s(s-1)v^2 = 1$, it follows from Lemma 4 that $4(s - 1) \leq 2^{l+2}$. Thus $s \leq 2^l + 1$.
Case 2. $gcd(u, v) \neq 1$. Assume that $gcd(u, v) = d$. Thus $d|u$ and $d|v$. Then there exist integers m and n such that $u = dm$ and $v = dn$ where $gcd(m, n) = 1$. Substituting the values of u and v into the equation $u^2 - s(s - 1)v^2 = -2^l$, we have

$$(dm)^2 - s(s - 1)(dn)^2 = -2^l$$
$$d^2(m^2 - s(s - 1)n^2) = -2^l.$$

Then $d^2|2^l$. We see that $d = 2^k$ for some $0 < k \leq l/2$. Then $m^2 - s(s - 1)n^2 = -2^{l-2k}$ where $\gcd(m, n) = 1$. By Lemma 4, $0 < n \leq \frac{\sqrt{2^{l-2k}}}{\sqrt{s-1}}$. Thus $s \leq 2^{l-2k} + 1$.

Lemma 8. *Let k and $l \geq 2$ be positive integers. If $x^2 - kxy + ky^2 + 2^l = 0$, where $\gcd(x, y) = 1$, then $x \equiv 0 \pmod 2$, $y \equiv 1 \pmod 2$, and $k \equiv 0 \pmod 4$.*

Proof. Suppose x and y satisfy the equation $x^2 - kxy + ky^2 + 2^l = 0$, where $\gcd(x, y) = 1$. This implies that y is odd. Otherwise x and y are both even, which is a contradiction. Hence $x^2 - kx + k$ is even. It is easy to see that x and k are both even. Since $x^2 - kxy \equiv 0 \pmod 4$ and $x^2 - kxy + ky^2 \equiv 0 \pmod 4$, we have $k \equiv 0 \pmod 4$.

The Diophantine equations $x^2 - kxy + ky^2 + 2^l = 0$ and $u^2 - s(s-1)v^2 = -2^{l-2}$, where $k = 4s$, are now related. There are several details on both equations given.

Lemma 9. *For any positive integers k and $l \geq 3$, the Diophantine equation*

$$x^2 - kxy + ky^2 + 2^l = 0$$

has a solution (x, y) where $k = 4s$ and $\gcd(x, y) = 1$ if and only if

$$u^2 - s(s - 1)v^2 = -2^{l-2}$$

has a solution (u, v) where $\gcd(u, v) = 1$.

Moreover, both equations have infinitely many solutions.

Proof. Let $l \geq 3$ be a positive integer. Suppose

$$x^2 - kxy + ky^2 + 2^l = 0$$

has a solution (x, y) where $\gcd(x, y) = 1$. By Lemma 8, we have x is even and $k = 4s$ for some positive integer s. Now, let $x = 2x'$ for some positive integer x'. Thus

$$(2x')^2 - k(2x')y + ky^2 + 2^l = 0.$$

We now have

$$\begin{aligned} x'^2 - 2sx'y + sy^2 + 2^{l-2} &= 0 \\ (x' - sy)^2 - s(s-1)y^2 &= -2^{l-2}. \end{aligned}$$

Let $u = x' - sy$ and $v = y$. Then $u^2 - s(s-1)v^2 = -2^{l-2}$. Since $\gcd(x, y) = 1$, we have $\gcd(u, v) = \gcd(x', y) = 1$.

Now for the converse, suppose

$$u^2 - s(s-1)v^2 = -2^{l-2}$$

where $\gcd(u, v) = 1$. We can see that u must be even. Now let $x = 2x'$ where $x' = u + sv$ and $y = v$. Then

$$\begin{aligned} (x' - sy)^2 - s(s-1)y^2 &= -2^{l-2}. \\ x'^2 - 2sx'y + sy^2 + 2^{l-2} &= 0 \\ 4x'^2 - 8sx'y + 4sy^2 + 2^l &= 0 \\ x^2 - kxy + ky^2 + 2^l &= 0. \end{aligned}$$

Since u is even and v is odd, we have $\gcd(x, y) = \gcd(2x', y) = \gcd(2u + 2sv, v) = \gcd(2u, v) = 1$.

Lemma 10. *For any non-negative integer l, the Diophantine equation $x^2 - 4xy + 4y^2 + 2^l = 0$ has no solution.*

Proof. Suppose that $x^2 - 4xy + 4y^2 + 2^l = 0$. We can see that $(x - 2y)^2 = -2^l$, which is impossible. Hence $x^2 - 4xy + 4y^2 + 2^l = 0$ has no solution.

Lemma 11. *For positive integers l and k, if the Diophantine equation $x^2 - kxy + ky^2 + l = 0$ is solvable, then $k > 4$.*

Proof. Suppose $x^2 - kxy + ky^2 + l = 0$. Then $(2x - ky)^2 + y^2(4k - k^2) = -4l$. This implies that $4k - k^2 < 0$. Since $k > 0$, we have $k > 4$.

Lemma 12. *Let s and l be positive integers and p be an odd prime. If $p \mid s(s-1)$ and either one of the following holds:*

(i) l is odd and $p \equiv 5 \pmod 8$ or $p \equiv 7 \pmod 8$;

(ii) l is even and $p \equiv 3 \pmod 8$ or $p \equiv 7 \pmod 8$;

then $u^2 - s(s-1)v^2 = -2^l$ is not solvable.

Proof. Since $p \mid s(s-1)$, we obtain that $u^2 \equiv -2^l \pmod p$. We now consider the Legendre symbol $\left(\frac{-2^l}{p}\right)$. We have $\left(\frac{-2^l}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{2^l}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{2}{p}\right)^l = (-1)^{\frac{(p-1)}{2}}(-1)^{\frac{(p^2-1)l}{8}}$. If l is odd and $p \equiv 5, 7 \pmod 8$ or l is even and $p \equiv 3, 7 \pmod 8$, then it is easy to see that $\left(\frac{-2^l}{p}\right) = -1$. Hence, $u^2 \equiv -2^l \pmod p$ has no solution. This implies that $u^2 - s(s-1)v^2 = -2^l$ is not solvable.

Lemma 13. *For any integer $l \geq 2$, we have $2^l + 4 \in T'(2^l)$.*

Proof. Given that $(u, v) = (2^{l-2}, 1)$ is a solution to the equation $u^2 - s(s-1)v^2 = -2^{l-2}$ with $s = 2^{l-2} + 1$, Lemma 9 implies that the Diophantine equation $x^2 - kxy + ky^2 + 2^l = 0$, where $k = 2^l + 4$, has infinitely many coprime solutions (x, y). Therefore, $2^l + 4 \in T'(2^l)$.

We are now ready to find the sets $T'(2^n)$ for $3 \leq n \leq 7$. We first mentioned the previous results on $T'(1), T'(2)$ and $T'(4)$.

Theorem 2. $T'(1) = \{5\}$.

Proof. Suppose $x^2 - kxy + ky^2 + 1 = 0$ where $gcd(x, y) = 1$. Then $(2x - ky)^2 - ((k-2)^2 - 4)y^2 = -4$. By Lemma 6, we have $k - 2 = 3$. Hence, $k = 5$ as desired.

Theorem 3. *[4] $T'(2) = \{6\}$ and $T'(4) = \{8\}$.*

Proof. The proof of this theorem can be found in Theorem 3.2 from [4].

We next find $T'(8), T'(16), T'(16), T'(32), T'(64)$ and $T'(128)$ using our method.

Theorem 4. $T'(8) = \{8, 12\}$.

Proof. Suppose $x^2 - kxy + ky^2 + 8 = 0$ where $gcd(x, y) = 1$. By Lemma 9, it suffices to consider the Diophantine equation $u^2 - s(s-1)v^2 = -2$. If $s = 2$, then $u^2 - 2v^2 = -2$. Since $(4, 3)$ is a solution of the equation $u^2 - 2v^2 = -2$. Thus we obtain $k = 8$. If $s \geq 3$, by Lemma 7 and Lemma 9, it suffices to consider the Diophantine equation $u^2 - s(s-1)v^2 = -2$ for $s \leq 3$. This implies that $s = 3$. It is easy to see that $(2, 1)$ is a solution of the equation $u^2 - 6v^2 = -2$. Thus we obtain $k = 12$. Hence $T'(8) = \{8, 12\}$.

Theorem 5. $T'(16) = \{20\}$.

Proof. By Lemma 7 and Lemma 9, it suffices to consider the Diophantine equation $u^2 - s(s-1)v^2 = -2^2$ for $s \leq 5$. By Lemma 12, $u^2 - s(s-1)v^2 = -2^2$ is not solvable for $s = 1, 3$ and 4.

For $s = 2$, it is easy to see that if $u^2 - 2v^2 = -4$, then u is even. Thus $4 \mid 2v^2$ and this implies that v is even. Hence $2 \mid \gcd(u, v)$ and thus $8 \notin T'(16)$.

For $s = 5$, we see that $(4, 1)$ is a solution of $u^2 - 20v^2 = -4$. Thus by Lemma 9, we obtain that $T'(16) = \{20\}$ as desired.

Theorem 6. $T'(32) = \{16, 36\}$.

Proof. By Lemma 7 and Lemma 9, it suffices to consider the Diophantine equation $u^2 - s(s-1)v^2 = -2^3$ for $s \leq 9$. By Lemma 12, $u^2 - s(s-1)v^2 = -2^3$ is not solvable for $s = 1, 5, 6, 7$ and 8.

For $s = 2, 3$, we can see that if $u^2 - s(s-1)v^2 = -8$, then u and v are both even. Thus all solutions of $u^2 - s(s-1)v^2 = -8$ are not relatively prime.

We see that $(2, 1)$ and $(8, 1)$ are solutions of $u^2 - s(s-1)v^2 = -8$ for $s = 4$ and $s = 9$ respectively. Hence by Lemma 9, we obtain that $T'(32) = \{16, 36\}$ as desired.

Theorem 7. $T'(64) = \{20, 68\}$.

Proof. By Lemma 7 and Lemma 9, it suffices to consider the Diophantine equation $u^2 - s(s-1)v^2 = -2^4$ for $s \leq 17$. By Lemma 12, $u^2 - s(s-1)v^2 = -2^4$ is not solvable for $s = 1, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15$ and 16.

For $s = 2$, it is easy to see that if $u^2 - 2v^2 = -16$, then u is even. Thus $4 \mid 2v^2$ and this implies that v is even. Hence, $2 \mid gcd(u, v)$ and $8 \notin T'(64)$.

We see that $(2, 1)$ and $(16, 1)$ are solutions of $u^2 - s(s-1)v^2 = -16$ for $s = 5$ and $s = 17$ respectively. Hence by Lemma 9, we obtain that $T'(64) = \{20, 68\}$ as desired.

Theorem 8. $T'(128) = \{132\}$.

Proof. By Lemma 7 and Lemma 9, it suffices to consider the Diophantine equation $u^2 - s(s-1)v^2 = -2^5$ for $s \leq 33$ So we will show that the above equation has a solution (u, v) where $\gcd(u, v) = 1$ if and only if $s = 33$. By Lemma 12, $u^2 - s(s-1)v^2 = -2^5$ is not solvable for $s = 1, 5, 6, 7, 8, 10, 11, 13, 14, 15, 16, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31$ and 32. For $s = 2, 3, 18, 19$, we can see that if $u^2 - s(s-1)v^2 = -32$, then u is even. Thus $4 \mid 2v^2$ and this implies that v is even. Hence, $2 \mid gcd(u, v)$. Thus all solutions of $u^2 - s(s-1)v^2 = -32$ are not relatively prime.

We next show that all solutions of $u^2 - s(s-1)v^2 = -32$, where $s = 4, 9$ are not relatively prime.

For $s = 4$, we can see that if $u^2 - 12v^2 = -32$, then u is even. Let $u = 2m$ for some positive integer m. We have $4m^2 - 12v^2 = -32$. Thus $m^2 - 3v^2 = -8$. It is easy to see that m and v have the same parities. If m and v are odd, then $0 \equiv -8 \equiv m^2 - 3v^2 \equiv 6 \pmod 8$. This is a contradiction. This implies that m and v are even. So are u and v. Thus all solutions of $u^2 - 12v^2 = -32$ are not relatively prime.

For $s = 9$, we can see that if $u^2 - 72v^2 = -32$, then u is even and $4 \mid u$. Let $u = 4m$ for some positive integer m. We have $16m^2 - 72v^2 = -32$. Hence $2m^2 - 9v^2 = -4$. It is easy to see that v is even. Thus all solutions of $u^2 - 72v^2 = -32$ are not relatively prime. Similarly, we can show that for $s = 12, 17$, all solutions of $u^2 - s(s-1)v^2 = -32$ are not relatively prime.

For $s = 33$, we see that $(32, 1)$ is a solution of $u^2 - s(s-1)v^2 = -32$. Hence by Lemma 9, we obtain that $T'(128) = \{132\}$ as desired.

4. Conclusion

By applying our method, we can efficiently compute $T'(2^n)$ for $n = 3, 4, 5, 6$ and 7. This, in turn, reveals the values of $T(2^n)$ for that range.

n	$T'(2^n)$	$T(2^n)$
0	{5}	{5}
1	{6}	{5,6}
2	{8}	{5,6,8}
3	{8,12}	{5,6,8,12}
4	{20}	{5,6,8,12,20}
5	{16,36}	{5,6,8,12,16,20,36}
6	{20,68}	{5,6,8,12,16,20,36,68}
7	{132}	{5,6,8,12,16,20,36,68,132}

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