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NJ-Abelian Rings: an Abelian-Like Approach

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Abstract. This article extends the concept of NJ-semicommutative rings to introduce the broader class of NJ-abelian rings, which are defined by properties involving nilpotent elements and the Jacobson radical. We investigate the unique algebraic properties of NJ-abelian rings and analyze their relationships with various types of rings, including abelian, reduced, J-clean, local, and Dedekind-finite rings. In particular, we show that every NJ-semicommutative ring is NJ-abelian, much like every semicommutative ring is abelian.

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1. Introduction

Though R is an associative ring with an identity, $\boldsymbol{J}(R)$ is the Jacobson radical of R, and $\boldsymbol{N}(R)$ is the set of nilpotent elements of R. A ring R is called *semiprimative* if $\boldsymbol{J}(R) = 0$ and *reduced* if $\boldsymbol{N}(R) = 0$. An idempotent e of a ring R is said to be *left* (resp. *right*) *semicentral* if $(1-e)Re = 0$ (resp. $eR(1-e) = 0$). If an idempotent e is both left and right semicentral, then it is central. A ring R is *abelian* if all of its idempotents are central. A ring R is called *J-abelian* if $ae - ea \in \boldsymbol{J}(R)$ for all $e^2 = e, a \in R$. Abelian rings are easily shown to be J-abelian, but the converse is not true in general (see [1, 2]). Recall [3], a ring R is said to be *semicommutative* if $ab = 0$ implies $aRb = 0$ for any $a, b \in R$. The concept of semicommutative rings has been introduced in other terms in [4–6].

In [7], Subba and Subedi investigated a new class of rings called NJ-semicomutative. These rings generalize the notion of semicommutative rings by exploring the relationship between nilpotent elements and the Jacobson radical. A ring R is said to be *NJ-semicommutative* if $aRb \subseteq \boldsymbol{J}(R)$ wherever $ab \in \boldsymbol{N}(R)$ for all elements $a, b \in R$. This

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allows for a deeper understanding of the structure of these rings, particularly how the properties of nilpotent elements influence the behavior of the ring as a whole. In the same context, we define the NJ-abelian property. By introducing the concept of NJ-abelian rings, we aim to define a class of rings that incorporates both nilpotent elements and the Jacobson radical, potentially exploring the interactions between these two sets and the properties that result from them.

Investigating the structure of NJ-abelian rings may reveal intriguing characteristics and properties that either align with or contrast with those found in NJ-semicommutative rings. You might explore questions such as

- How do NJ-abelian rings relate to traditional abelian rings?
- What conditions define a ring as NJ-abelian?
- Are there any specific examples or counterexamples that illustrate the behavior of these rings?

In this article, the following notations are used for a ring R: $\boldsymbol{I}(R)$ for the set of idempotents of R, $\boldsymbol{N}_2(R)$ for the set of all *square-zero* elements of R (i.e., the nilpotent elements of index 2 or 1), $\boldsymbol{U}(R)$ for the set of units of R, and $\boldsymbol{M}_n(R)$ (resp. $\boldsymbol{T}_n(R)$) for the ring of all matrices (resp. upper triangular matrices) over R.

2. Basic Results

We will talk about the NJ-abelian concept, come up with some basic results, and show how it is related to other concepts like the J-abelian, NJ-semicommutative, and J-reduced conditions.

Definition 1. *A ring R (not necessarily with identity) is called **NJ-abelian** if $e^2 = e, a \in R$, and $ae \in \boldsymbol{N}(R)$, then $aRe \subseteq \boldsymbol{J}(R)$.*

The next proposition gives an equivalent condition for J-abelianity of rings, which will be used to show that every NJ-abelian ring is J-abelian.

Proposition 1. *A ring R is J-abelian if and only if $ef \in \boldsymbol{N}(R)$ for any $e, f \in \boldsymbol{I}(R)$ implies $eRf \subseteq \boldsymbol{J}(R)$.*

Proof. ($\Leftarrow$) Let e be an idempotent of R. Then $e(1-e) = 0 \in \boldsymbol{N}(R)$, and hence $eR(1-e) \subseteq \boldsymbol{J}(R)$. Similarly, $eR(1-e) \subseteq \boldsymbol{J}(R)$. So, for every $r \in R$, we have $er - re = er(1-e) + (1-e)(-r)e \in eR(1-e) + (1-e)Re \subseteq \boldsymbol{J}(R)$ and R is J-abelian.

($\Rightarrow$) Let $ef \in \boldsymbol{N}(R)$ for some idempotents e and f of R of nilpotency index n. Define the idempotent $g = 1-f+(1-f)erf$ for arbitrary r in R. So, $(1-f)erf = fg-gf \in \boldsymbol{J}(R)$ and $erf-eferf = e(erf-ferf) = e(1-f)erf \in \boldsymbol{J}(R)$. Applying the J-abelian condition, we get $fer - fefer \in \boldsymbol{J}(R)$. Beginning with the inclusion and sequentially multiplying the element by ef from left to right, we obtain

$$\big(fer-(fe)^2r\big)+\big((fe)^2r-(fe)^3r\big)+\big((fe)^3r-(fe)^4r\big)+\cdots\big((fe)^{n-1}r-(fe)^nr\big)\in \boldsymbol{J}(R)$$

and $fer \in \boldsymbol{J}(R)$ for every $r \in R$. From the J-abelianity of R, we get $eRf \subseteq \boldsymbol{J}(R)$.

Corollary 1. *Every NJ-abelian ring R is J-abelian.*

Here is a J-abelian ring that is not NJ-abelian.

Example 1. *Let $R = \mathbb{Z}_4[x]$ and $S = R/\langle x \rangle$. So, $\boldsymbol{J}(S) = \langle 2x \rangle$ and $\boldsymbol{N}(S) = \langle 2, x \rangle$. We have $1x \in \boldsymbol{N}(S)$ while $1Sx \nsubseteq \boldsymbol{J}(S)$. Thus, S is not NJ-abelian even though S is J-abelian.*

The next proposition shows that the definition of the NJ-abelian property is left-right symmetric.

Proposition 2. *Any ring R can satisfy the following equivalent conditions:*

(i) *R is NJ-abelian;*

(ii) *$ae \in \boldsymbol{N}(R)$ implies then $eRa \subseteq \boldsymbol{J}(R)$, where $e^2 = e, a \in R$;*

(iii) *$ea \in \boldsymbol{N}(R)$ implies then $eRa \subseteq \boldsymbol{J}(R)$, where $e^2 = e, a \in R$;*

(iv) *$ea \in \boldsymbol{N}(R)$ implies then $aRe \subseteq \boldsymbol{J}(R)$, where $e^2 = e, a \in R$.*

Proof. (i)⇒(ii): If $ae \in \boldsymbol{N}(R)$, then $aRe \subseteq \boldsymbol{J}(R)$ from the NJ-abelianity of R. For every $r \in R$, we have $era = e(ra) - (ra)e + rae \in \boldsymbol{J}(R)$, since R is J-abelian from Proposition 1, and hence $eRa \subseteq \boldsymbol{J}(R)$.

(ii)⇒(iii) is direct since ae is nilpotent if and only if ea is.

(iii)⇒(iv): As in Proposition 1, one can prove that R is J-abelian. Now, $are = (ar)e - e(ar) + ear \in \boldsymbol{J}(R)$, for every $e^2 = e, r \in R$ by the hypotheses; that is, $aRe \subseteq \boldsymbol{J}(R)$.

(iv)⇒(i) is direct again.

Now, we aim to get a sufficient condition for the J-abelian ring to be NJ-abelian. According to [8], a ring R is called *J-reduced* if $\boldsymbol{N}(R) \subseteq \boldsymbol{J}(R)$. In [9], a J-reduced ring is called an ***NJ** ring*. Obviously, every NJ-abelian ring R is J-reduced since all rings are with identity. The converse is not necessarily true, as the ring S in Example 1. Remind that every reduced ring is abelian. Hence, every reduced ring is NJ-abelian.

The next proposition shows that J-reducedness is a sufficient and necessary condition for the J-abelian ring to become NJ-abelian.

Proposition 3. *A ring R is NJ-abelian if and only if it is J-reduced and J-abelian.*

Proof. The necessity is obvious. For sufficiency, assume that R is a J-reduced and J-abelian ring. Let $ae \in \boldsymbol{N}(R)$ for some $e^2 = e, a \in R$. So, $ea \in \boldsymbol{J}(R)$ from the J-reducedness of R. For every $r \in R$, we have $era = (e(ra) - (ra)e) + r(ae) \in \boldsymbol{J}(R)$ since R is J-abelian. Thus $eRa \subseteq \boldsymbol{J}(R)$ and R is NJ-abelian.

Nevertheless, it's worth noting that NJ-abelian and abelian properties are distinct. The next examples show that the classes of NJ-abelian rings and abelian rings are independent.

Example 2. *Consider the ring*

$$R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a \equiv d \bmod 2, b \equiv c \bmod 2, a, b, c, d \in \mathbb{Z} \right\}.$$

R has no nontrivial idempotents; consequently, R is abelian. The element $\alpha = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ *in R satisfies $\alpha 1 \in \boldsymbol{N}(R)$ while* $\alpha R 1 = \begin{bmatrix} 2\mathbb{Z} & 2\mathbb{Z} \\ 0 & 0 \end{bmatrix} \nsubseteq \boldsymbol{J}(R) = 0$. *Therefore, R is not NJ-abelian.*

Example 3. *For any field F, the ring $\boldsymbol{T}_2(F)$ is not abelian, where the set of nontrivial idempotents of R is* $\left\{ \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & a \\ 0 & 1 \end{bmatrix} \mid a \in F \right\}$ *is not central. For the idempotents of the form* $\begin{bmatrix} 0 & a \\ 0 & 1 \end{bmatrix}$, *if $\alpha e \in \boldsymbol{N}(R)$ for some $\alpha \in R$, then* $\alpha = \begin{bmatrix} b & c \\ 0 & 0 \end{bmatrix}$ *for arbitrary $b, c \in F$. So,* $\alpha R e \in \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix} = \boldsymbol{J}(R)$. *Therefore, R is NJ-abelian.*

The two examples above demonstrate that the abelian and NJ-abelian properties of rings are independent of each other. However, this does not imply that the abelian and NJ-abelian classes are completely disjoint. For instance, the ring $\mathbb{Z}$ is both abelian and NJ-abelian. Additionally, there exists a ring that is neither abelian nor NJ-abelian (see the next example).

Example 4. *Consider the ring $R = F + F\mathfrak{j}$, where F is a field, with $2^{-1} \in F$, and $\mathfrak{j}^2 = 1$. R is a commutative reduced ring, and its set of idempotents is $\{0, 1, \frac{1}{2}(1+\mathfrak{j}), \frac{1}{2}(1-\mathfrak{j})\}$. Define the automorphism $\sigma : R \to R$ as $\sigma(a + b\mathfrak{j}) = a - b\mathfrak{j}$, for every $a, b \in F$. Let $S = R[x; \sigma]$ be the skew polynomial ring with an indeterminate x over R. The element $\alpha = (1 + j)x$ in S satisfies $\alpha 1 \in \boldsymbol{N}(S)$ while $\alpha S 1 \neq 0$. Therefore, S is not NJ-abelian since R is semiprimitive. Also, the nontrivial idempotents of S have the form $e + (1 \pm \mathfrak{j}) f x$, where $e^2 = e, f \in R$. So R is not abelian.*

The next proposition gives a sufficient condition to make an NJ-abelian ring abelian.

Proposition 4. *If R is a semiprimitive NJ-abelian ring, then R is abelian.*

Proof. For every idempotent e of a ring R, we have $0 = e(1 - e) \in \boldsymbol{N}(R)$. So, $eR(1 - e) \subseteq \boldsymbol{J}(R) = 0$. Hence, e is right semicentral. Similarly, we demonstrate that e is also left semicentral. Therefore, e is central, and hence R is abelian.

According to the definitions, every NJ-semicommutative ring is NJ-abelian. However, the converse does not necessarily hold, as illustrated by the following example.

Example 5. *Let K be a countable field. By [10, Lemma 3.7], there exists a nonzero nil algebra A over K such that $A[x]$ has a zero upper nil radical. The ring $R = (K + A)[x]$ is not NJ-semicommutative, as shown in [7, Example 4]. However, R is J-reduced, with $\boldsymbol{N}(R) = J(R) = A[x]$, and has only trivial idempotents. Therefore, R is NJ-abelian.*

Remind that a ring R is said to be *Dedekind-finite* if $ab = 1$ implies $ba = 1$ for every $a, b \in R$. The next proposition shows that every NJ-abelian ring is Dedekind-finite.

Proposition 5. *Every NJ-Abelian ring is Dedekind-finite.*

Proof. Let R be an NJ-Abelian ring with $ab = 1$ for some $a, b \in R$. So that $(1-ba)^2 = 1 - 2ba + baba = 1 - ba$ and $1 - ba$ is an idempotent of R. We have $(1-ba)b = 0 \in \boldsymbol{N}(R)$ and consequently $1 - ba = (1-ba)ab \in (1-ba)Rb \subseteq \boldsymbol{J}(R)$, from the NJ-Abelianity of R. Therefore, we have $1 - ba = 0$ and $ba = 1$, indicating that R is Dedekind-finite.

In [11], an element a of a ring R is called *left minimal* if Ra is a minimal left ideal of R. Write $\boldsymbol{ME}_l(R)$ to denote the set of all left minimal idempotents of R. A ring R is called *left min-Abel* if each left minimal idempotent is left semicentral. The next proposition states that a ring R is left min-Abel whenever it is NJ-Abelian.

Proposition 6. *Every NJ-abelian ring is left-min Abel.*

Proof. Assume $e \in \boldsymbol{ME}_l(R)$ and $r \in R$. Define the element $x = re - ere$; hence, $Rx \subseteq R$. But Re is a minimal left ideal of R. So $Rx = Re$ or $x = 0$. Indeed, $xe \in \boldsymbol{N}(R)$ and $xRe \subseteq \boldsymbol{J}(R)$, from the NJ-abelianity of R. If $Rx = Re$, then $e = e^2 \in RxRe \subseteq \boldsymbol{J}(R)$ and $e = 0$; it is a contradiction. So, $x = 0$ and $(1-e)Re = 0$. Thus e is a left semicentral idempotent of R, and hence R left-min Abel.

Remind that a ring R is called *J-clean* if for every $a \in R$, there exists an idempotent $e \in R$ and $b \in \boldsymbol{J}(R)$ such that $a = e + b$; that is, $R = \boldsymbol{I}(R) + \boldsymbol{J}(R)$. We show that every J-clean ring is NJ-abelian in the following proposition.

Proposition 7. *Every J-clean ring R is NJ-abelian.*

Proof. Let R be a J-clean ring and a be a nilpotent element of R with index of nilpotency n. So, $a = e + b$ for some $e \in \boldsymbol{I}(R)$ and $0 = (e+b)^n \in e + \boldsymbol{J}(R)$ and $e \in \boldsymbol{J}(R)$. Therefore, $e = 0$ and $a \in \boldsymbol{J}(R)$; that R is J-reduced. Hence, R is NJ-abelian based on the results of [1, Lemma 2.4.] and Proposition 3.

From the proposition above, we can get some corollaries that have been proved before in another work.

Corollary 2 ([1], Lemma 2.4)**.** *Every J-clean ring is J-abelian.*

By Proposition 5, we also have the next corollary.

Corollary 3 ([1], Theorem 2.10)**.** *Every J-clean ring is Dedekind-finite.*

Recall from [12] that the set of all elements of R that are nilpotent in $R/J(R)$ is denoted by $\boldsymbol{J}^{\#}(R)$; that is, $\boldsymbol{J}^{\#}(R) = \{a \in R \mid a^n \in \boldsymbol{J}(R)\}$. It is obvious that both $\boldsymbol{J}(R)$ and $\boldsymbol{N}(R)$ are contained in $\boldsymbol{J}^{\#}(R)$. In [13], a ring R is called *feckly reduced* if $R/\boldsymbol{J}(R)$ is a reduced ring. The following proposition provides a more general result of [1, Proposition 2.6] under the same assumptions and shows that every feckly reduced ring is NJ-abelian.

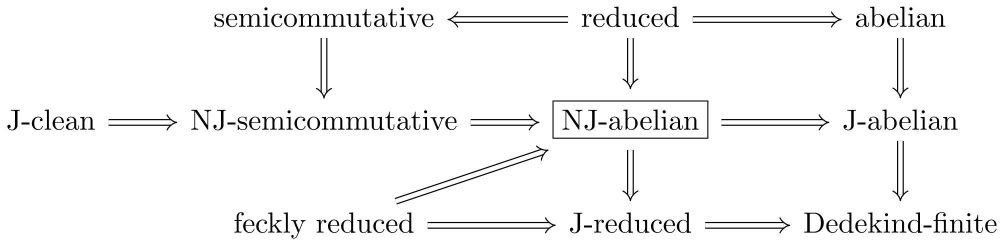
Proposition 8. *If R is a feckly reduced ring, then R is NJ-abelian.*

Proof. From [13, Proposition 2.6], R satisfies $\boldsymbol{J}^{\#}(R)$. Now, let $ea \in \boldsymbol{N}(R)$ where $e^2 = e, a \in R$. So, for every $r \in R$, we have $(e-1)are$ is nilpotent, and consequently $ea, (e-1)are \in \boldsymbol{J}^{\#}(R) = \boldsymbol{J}(R)$. So, $are = (e-1)are + eare \in \boldsymbol{J}(R)$, for every $r \in R$, Thus, $aRe \subseteq \boldsymbol{J}(R)$, and hence R is NJ-abelian.

Remind that a ring R is said to be *local* if it has only one maximal left (or right) ideal; equivalently, $R/\boldsymbol{J}(R)$ is a division ring. From Proposition 8 and [12, Lemma 1], we get directly the following result.

Corollary 4. *Every local ring is NJ-abelian.*

Here are interesting nontrivial implications in the class of rings with respect to the class of NJ-abelian rings.



3. Extending of NJ-abelianity

Note that the class of NJ-abelian rings is closed under direct products but not under closed subrings. Remember that the Nagata extension of a commutative ring R by an R-module M and an endomorphism σ of R is the ring of direct sum of the abelian groups R and M, with componentwise addition and multiplication defined as $(r_1, m_1)(r_2, m_2) = (r_1 r_2, \sigma(r_1)m_2 + m_1 r_2)$ for all $r_1, r_2 \in R$ and $m_1, m_2 \in M$. The next examples give an NJ-abelian subring of a ring that is not $\boldsymbol{NJ}$.

Example 6. *Let $R = \mathbb{Z} \oplus \mathbb{Z}$ and σ be an automorphism on R defined as $\sigma(a,b) = (b,a)$ for every $(a,b) \in R$. The Nagata extension of R by S and σ, denoted as S, is semiprimitive since R is reduced. Moreover, the idempotent $((1,0),(0,1))$ and element $((0,1),(0,1))$ of S satisfy $ea \in \boldsymbol{N}(S)$ while $eSa = ((0,0),(0,\mathbb{Z})) \nsubseteq \boldsymbol{J}(S) = 0$. Therefore, S is not NJ-abelian, while R is an NJ-abelian subring of S.*

Here is an NJ-abelian ring that has a non-$\boldsymbol{NJ}$ subring.

Example 7. *From [14, Example 4.8], let $R = F\langle x, y\rangle$ be a free algebra over a field F generated by the noncommuting indeterminates x and y. Consider the subring $S = R/\langle y^2\rangle$ of R. According to [9], S is not J-reduced, and consequently it is not NJ-reduced since S has identity. However, R is an NJ-abelian ring.*

Now, we give some results for subrings that are NJ-abelian due to the NJ-abelianity of their rings.

Proposition 9. *Let* $\{R_i\}_{i\in\Lambda}$ *be a class of rings for some index set* Λ*. Then* $\prod_{i\in\Lambda} R_i$ *is NJ-abelian if and only if* R_i *is NJ-abelian for every* $i \in \Lambda$*.*

Proof. The proof is routine.

Corollary 5. *Let* R *be a ring and* e *be a central idempotent of* R*. Then* eR *and* $(1-e)R$ *are NJ-abelian if and only if* R *is NJ-abelian.*

Proposition 10. *A ring* R *is NJ-abelian if and only if every corner of* R *is NJ-abelian.*

Proof. The sufficiency is trivial. For the necessity, let $f^2 = f, a \in eRe \in R$, for some idempotent e of R such that $af \in \boldsymbol{N}(eRe) \subseteq \boldsymbol{N}(R)$. So, $a(eRe)f = aRf \in \boldsymbol{J}(R)$ since R is NJ-abelian. But $\boldsymbol{J}(eRe) = e\boldsymbol{J}(R)e$. So, $a(eRe)f = eaRfe \subseteq e\boldsymbol{J}(R)e = \boldsymbol{J}(eRe)$ and eRe is NJ-abelian.

The next example shows that even if every corner eRe of a ring R is NJ-abelian for all nonidentity idempotents e, R is not necessarily NJ-abelian.

Example 8. *Let* $R = \boldsymbol{M}_2(\mathbb{Z})$ *be the ring of all* 2×2 *matrices over the ring of integers* $\mathbb{Z}$*. Every nontrivial idempotent* e *of* R *satisfies* $eRe = \mathbb{Z}$*, which is NJ-abelian, but* R *is not NJ-abelian since the elements* $\alpha = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ *and* $\epsilon^2 = \epsilon = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ *satisfy* $\alpha\epsilon \in \boldsymbol{N}(R)$ *but* $\alpha R\epsilon = \begin{bmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{bmatrix}$.

The following proposition shows that if a subring of an NJ-abelian ring is an ideal, then it is also NJ-abelian.

Proposition 11. *Every ideal of an NJ-abelian ring is NJ-abelian (as a ring without identity).*

Proof. Let R be an NJ-abelian ring and I be an ideal of R. Assume that $ea \in \boldsymbol{N}(I)$ where $e^2, a \in I$. But $\boldsymbol{N}(I) \subseteq \boldsymbol{N}(R)$ and R is NJ-abelian; hence, $aRe \subseteq \boldsymbol{J}(R)$. So, $aIe \subseteq aRe \subseteq I \cap \boldsymbol{J}(R) = \boldsymbol{J}(R)$ and I is NJ-abelian.

Proposition 12. *Let* R *be a ring such that* $R[x]$ *is NJ-abelian. Then* R *is an NJ-abelian ring.*

Proof. Let $e^2 = e, a \in R$ such that $ea \in \boldsymbol{N}(R) \subseteq \boldsymbol{N}(R[x])$. From the NJ-abelianity of $R[x]$, we have $eRa \subseteq eR[x]a \in \boldsymbol{J}(R[x])$, and $1 - eras$ is invertible in $R[x]$ for all $r, s \in R$. But $1 - eras \in R$, and hence $1 - eras \in \boldsymbol{U}(R)$ for all $r, s \in R$. Thus R is NJ-abelian.

It is natural to conjecture that R is an NJ-abelian ring if for any nonzero proper ideal I of R, R/I and I are both NJ-abelian rings, where I is considered a ring without identity. However, the following example provides a negative answer to this conjecture.

Example 9. *For a ring R, let $S = \boldsymbol{M}_2(R)$, and I is the ideal generated by the commutators of S. Then S/I is commutative, and therefore it is NJ-abelian. The elements* $e = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ *and* $a = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ *of S satisfy $e^2 = e$ and $ea \in \boldsymbol{N}(R)$. However,* $eSa = \begin{bmatrix} R & R \\ 0 & 0 \end{bmatrix} \nsubseteq \boldsymbol{J}(R)$. *Thus, S is not NJ-abelian.*

However, if we take stronger independent conditions, such as "I is nil" and "$I = \boldsymbol{J}(R)$", then we may have an affirmative answer, as in the following.

Proposition 13. *For a ring R, if $R/\boldsymbol{J}(R)$ is NJ-abelian, then R is NJ-abelian.*

Proof. If $e, f \in \boldsymbol{I}(R)$ and $ef \in \boldsymbol{N}(R)$, then $\bar{e}\bar{f} \in \boldsymbol{N}(R/\boldsymbol{J}(R))$. But $R/\boldsymbol{J}(R)$ is NJ-abelian and $\bar{e}\bar{r}\bar{f} \in \boldsymbol{N}(R/\boldsymbol{J}(R)) = \overline{0}$, for every $r \in R$. Thus, $aRb \subseteq \boldsymbol{J}(R)$, and R is NJ-abelian.

Since every reduced ring is NJ-abelian, we have the following corollary.

Corollary 6. *If $R/\boldsymbol{J}(R)$ is reduced, then R is NJ-abelian.*

The converse of the previous proposition is not necessarily true, as shown in the next example.

Example 10. *Let S be the localization of $\mathbb{Z}$ at $3\mathbb{Z}$ and R the set of quaternions over the ring S. According to [7], $\boldsymbol{J}(R) = 3R$ and $R/\boldsymbol{J}(R) = \boldsymbol{M}_2(\mathbb{Z}_3)$. Also, R is an NJ-semicommutative ring and consequently NJ-abelian. On the other hand, the idempotents* $e = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ *and* $f = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$ *of $R/\boldsymbol{J}(R)$ satisfy $ef \in \boldsymbol{N}(R/\boldsymbol{J}(R))$ while* $e \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} f = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \notin \boldsymbol{J}(R/\boldsymbol{J}(R))$. *Thus, $R/\boldsymbol{J}(R)$ is not NJ-abelian.*

Proposition 14. *Let I be a nil ideal of R such that R/I is an NJ-abelian ring. Then R is NJ-abelian.*

Proof. Suppose that R/I is NJ-abelian and $e = e^2, a \in R$ such that $ae \in \boldsymbol{N}(R)$. Then $\overline{ae} \in \boldsymbol{N}(R/I)$ and consequently $\overline{a}(R/I)\overline{e} \subseteq \boldsymbol{J}(R/I)$. Therefore, $\overline{1 - ares} \in \boldsymbol{U}(R/I)$, for every $r, s \in R$. It means that $1 - (1 - ares)x \in I \subseteq \boldsymbol{N}(R)$, for some $x \in R$. Thus $(1 - ares)x$ is a unit in R, and hence $1 - ares$ has a right inverse for every $r, s \in R$. Therefore, $aRe \subseteq \boldsymbol{J}(R)$ and R are NJ-abelian.

4. Matrix extensions of NJ-abelian rings

In this section, we study the NJ-abelian property for some ring extensions and their subrings. First, we show that the matrix ring $\boldsymbol{M}_n(R)$ over a ring R is not NJ-abelian for any ring R and $n \geq 2$.

Proposition 15. *For any ring R and integer $n \geq 2$, $M_n(R)$ is not NJ-abelian.*

Proof. As shown in Example 9, $\boldsymbol{M}_2(R)$ is not NJ-abelian for any ring R. From Proposition 10, every corner of an NJ-abelian ring is NJ-abelian. By induction, $\boldsymbol{M}_n(R)$ is not NJ-abelian for every $n \geq 2$ since $\boldsymbol{M}_n(R)$ is a corner of $\boldsymbol{M}_{n+1}(R)$ for every $n \geq 1$.

While it is impossible to get a matrix ring satisfying the NJ-abelian condition, as shown in the example, we will explore the extent to which this property holds in some of its subrings or certain matrix contexts.

For rings R and S, let M and N be (R, S)-bimodule and (S, R)-bimodule, respectively. The set of all matrices of the form $\begin{bmatrix} r & m \\ n & s \end{bmatrix}$, where $r \in R$, $s \in S$, $m \in M$, and $n \in N$. Using the standard matrix addition, we can identify this as an abelian group. To define a matrix multiplication for these elements, we need to define the products of mn and nm in R and S, respectively, for every $m \in M$ and $n \in N$. We assume that there are two bimodule homomorphisms $\phi : (M, N) \to R$ and $\psi(N, M) \to S$. Simplify $mn = \phi(m, n)$ and $nm = \psi(n, m)$ for all $m \in M$ and $n \in N$. These maps satisfy the associativity conditions that are required to make the set with usual matrix addition and context multiplication an associative ring with identity, notated by $\begin{bmatrix} R & M \\ N & S \end{bmatrix}$. This ring is called the *Morita context* (R, M, N, S, ϕ, ψ), or a *formal matrix* ring (of order 2), or a *ring of generalized matrices*. The readers are referred to [15–19] as well as the references there for detailed information on the study in Morita contexts. Recall [20], a Morita context $T = \begin{bmatrix} R & M \\ N & S \end{bmatrix}$ is called *trivial* if $MN = 0$ and $NM = 0$. The following lemma describes the idempotents, nilpotent elements, and Jacobson radicals in a trivial Morita context.

Lemma 1. *For two rings, R and S, let $T = \begin{bmatrix} R & M \\ N & S \end{bmatrix}$ be a trivial Morita context. Then we have the following descriptions:*

(i) $\boldsymbol{I}(T) \subseteq \begin{bmatrix} \boldsymbol{I}(R) & M \\ N & \boldsymbol{I}(S) \end{bmatrix}$;

(ii) $\boldsymbol{N}(T) = \begin{bmatrix} \boldsymbol{N}(R) & M \\ N & \boldsymbol{N}(S) \end{bmatrix}$;

(iii) $\boldsymbol{J}(T) = \begin{bmatrix} \boldsymbol{J}(R) & M \\ N & \boldsymbol{J}(S) \end{bmatrix}$.

Proof. The proofs of (i) and (ii) are straightforward, while (iii) is obtained directly from [21, Lemma 3.1].

Proposition 16. *Suppose that $T = \begin{bmatrix} R & M \\ N & R \end{bmatrix}$ is a trivial Morita context. Then R is NJ-abelian if and only if M and N are NJ-abelian.*

Proof. The sufficiency is straightforward from Proposition 10. For the necessity, assume that both M and N are NJ-abelian. Let $\epsilon = \begin{bmatrix} e & m \\ n & f \end{bmatrix}$ be an idempotent of T and $\alpha = \begin{bmatrix} a & x \\ y & b \end{bmatrix}$ be an arbitrary element of T. So, $e \in \boldsymbol{I}(R)$ and $f \in \boldsymbol{I}(S)$, from the previous lemma. If $\epsilon\alpha \in \boldsymbol{N}(T)$, then $ea \in \boldsymbol{N}(R)$ and $fb \in \boldsymbol{N}(S)$. So, $eRa \subseteq \boldsymbol{J}(R)$ and $fRb \subseteq \boldsymbol{J}(S)$ from the NJ-abelianity of R and S. Now, $\epsilon T\alpha = \begin{bmatrix} eRa & eRx \\ fNy & fSb \end{bmatrix} \subseteq \begin{bmatrix} \boldsymbol{J}(R) & M \\ N & \boldsymbol{J}(S) \end{bmatrix} = \boldsymbol{J}(T)$ and T is an NJ-abelian ring.

Notice that formal triangular matrix rings are obvious examples of trivial Morita contexts. So, we have the following corollary.

Corollary 7. *Let M represent an (R, S)-bimodule for the rings R and S. Then $T = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$ is NJ-abelian if and only if both R and S are NJ-abelian.*

In [22], if R is a ring and M is an (R, R)-bimodule, then the direct sum of abelian groups R and M with standard addition and multiplication defined as $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$, for all $r_1, r_2 \in R$, $m_1, m_2 \in M$, is a ring with the identity $(1, 0)$. This ring is the *trivial extension* of R by M, denoted by $T(R, M)$. Notice that the trivial extension of R by M is that the formal triangular matrix ring $\begin{bmatrix} R & M \\ 0 & R \end{bmatrix}$. So, we have the following corollary.

Corollary 8. *Let M represent an (R, R)-bimodule associated with a ring R. Then $T(R, M)$ is NJ-abelian if and only if R is NJ-abelian.*

Proposition 17. *For any ring R, the following conditions are equivalent.*

(i) *R is NJ-abelian;*

(ii) *$\boldsymbol{T}_n(R)$ is NJ-abelian for some $n > 1$;*

(iii) *$\boldsymbol{T}_n(R)$ is NJ-abelian for every $n > 1$.*

Proof. (i)⇒(ii): If R is an NJ-abelian, then $\boldsymbol{T}_2(R)$ is a formal triangular matrix, and consequently it is NJ-abelian from Corollary 7. (ii)⇒(iii): Applying the mathematical induction on n, assume that $\boldsymbol{T}_n(R)$ is NJ-abelian for some $n > 1$; then R is NJ-abelian by Proposition 10. Notice that $\boldsymbol{T}_{n+1}(R)$ is the formal triangular matrix $\begin{bmatrix} \boldsymbol{T}_n(R) & R_n \\ 0 & R \end{bmatrix}$, where R_n is the $(\boldsymbol{T}_n(R), R)$-bimodule of n-by-1 matrices over R. Therefore $\boldsymbol{T}_{n+1}(R)$ is NJ-abelian by Corollary 7.

For a ring R, define a subring $\boldsymbol{S}_n(R)$ of $\boldsymbol{T}_n(R)$ as

$$\boldsymbol{S}_n(R) = \left\{ \begin{bmatrix} a & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ 0 & a & a_{23} & a_{24} & \cdots & a_{2n} \\ 0 & 0 & a & a_{34} & \cdots & a_{3n} \\ 0 & 0 & 0 & a & \cdots & a_{4n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a \end{bmatrix} | a, a_{ij} \in R \right\},$$

where $n \geq 2$ is a positive integer. Also, we have a subring of $\boldsymbol{S}_n(R)$ defined below:

$$\boldsymbol{V}_n(R) = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & \cdots & a_n \\ 0 & a_1 & a_2 & a_3 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & a_2 & \cdots & a_{n-2} \\ 0 & 0 & 0 & a_1 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_1 \end{bmatrix} | a_i \in R \right\},$$

where $n \geq 2$ is a positive integer. The next proposition shows that the NJ-abelianity of a ring R, $S_n(R)$, and $V_n(R)$ are equivalent for every n.

Proposition 18. *For any ring R, the following conditions are equivalent.*

(i) *R is NJ-abelian;*

(ii) *$\boldsymbol{S}_n(R)$ is NJ-abelian for some $n > 1$;*

(iii) *$\boldsymbol{S}_n(R)$ is NJ-abelian for every $n > 1$;*

(iv) *$\boldsymbol{V}_n(R)$ is NJ-abelian for some $n > 1$;*

(v) *$\boldsymbol{V}_n(R)$ is NJ-abelian for every $n > 1$.*

Proof. (i)⇒(iii): For every $n \geq 2$, consider the ideal $I_n(R)$ of $\boldsymbol{S}_n(R)$ consisting of all elements of $\boldsymbol{S}_n(R)$ with zero diagonal entries. Notice that I is a nil ideal and $R \cong \boldsymbol{S}_n(R)/I_n(R)$. Thus, $\boldsymbol{S}_n(R)$ is NJ-abelian from Proposition 14.

(iii)⇒(i) is direct from Proposition 10.

(i)⇔(v) may be proved by the same technique of proving (i)⇔(iii).

(i)⇔(ii)⇔(iv) is obtained directly from Corollary 8 since $T(R, R) = \boldsymbol{S}_2(R) = \boldsymbol{V}_2(R)$.

Corollary 9. *A ring R is NJ-abelian if and only if $R[x]/\langle x^n \rangle$ is NJ-abelian for any positive integer n.*

Conclusion

In this article, we introduced and studied a new class of rings, called NJ-abelian rings, defined by a condition that connects idempotents and nilpotent elements through the Jacobson radical. We showed that this class properly extends NJ-semicommutative rings, analogous to how abelian rings extend semicommutative rings.

We proved that every NJ-abelian ring is J-abelian and that the NJ-abelian condition is symmetric with respect to left and right multiplication. Several equivalent characterizations were established, notably that a ring is NJ-abelian if and only if it is both J-reduced and J-abelian. Moreover, we demonstrated that many classical ring classes, such as J-clean rings, local rings, and feckly reduced rings, are NJ-abelian, and we clarified their implications with suitable counterexamples.

In addition, we examined the behavior of the NJ-abelian property under various ring constructions and extensions. We showed that NJ-abelianity is not generally preserved in full matrix rings or certain direct sums and ideals, whereas it can hold in specific subrings like upper triangular matrix rings and some of their substructures.

The results presented here contribute to a deeper understanding of the interaction between nilpotent elements, idempotents, and the Jacobson radical. They open possible avenues for further exploration, such as the behavior of NJ-abelianity in skew polynomial rings, endomorphism rings, or more general Morita contexts.

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