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$\theta(\tau_1, \tau_2)$-continuity for Functions

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Abstract. This paper introduces a new class of functions between bitopological spaces, namely $\theta(\tau_1, \tau_2)$-continuous functions. Moreover, several characterizations and some properties concerning $\theta(\tau_1, \tau_2)$-continuous functions are investigated.

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1. Introduction

Stronger and weaker forms of open sets in topological spaces such as semi-open sets, preopen sets, α-open sets, β-open sets, δ-open sets and θ-open sets play an important role in the research of generalizations of continuity. By using these sets many authors introduced and investigated various types of continuity. The notions of (Λ, sp)-open sets, $s(\Lambda, sp)$-open sets, $p(\Lambda, sp)$-open sets, $\alpha(\Lambda, sp)$-open sets and $\beta(\Lambda, sp)$-open sets were studied in [1]. Viriyapong and Boonpok [2] investigated several characterizations of (Λ, sp)-continuous functions by utilizing the notions of (Λ, sp)-open sets and (Λ, sp)-closed sets. Dungthaisong et al. [3] introduced and studied the concept of $g_{(m,n)}$-continuous functions. Duangphui et al. [4] introduced and investigated the notion of almost $(\mu, \mu')^{(m,n)}$-continuous functions. Moreover, some characterizations of almost (Λ, p)-continuous functions, almost strongly $\theta(\Lambda, p)$-continuous functions, weakly (Λ, b)-continuous functions, $\theta(\star)$-precontinuous functions, $(\Lambda, p(\star))$-continuous functions, $\star$-continuous functions, θ-$\mathscr{I}$-continuous functions, almost (g, m)-continuous functions and pairwise weakly M-continuous functions were presented in [5], [6], [7], [8], [9], [10], [11], [12] and [13], respectively. The concept of θ-continuous functions was introduced by Fomin [14]. Noiri [15] studied some properties of θ-continuous functions. Furthermore, the present author [16] investigated

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several characterizations of θ-continuous functions. Arya and Bhamini [17] introduced the notion of θ-semi-continuous functions. Jafari and Noiri [18] investigated several characterizations of θ-semi-continuous functions. Noiri [19] introduced and investigated the concept of θ-precontinuous functions. Baker [20] introduced and studied the notion of weakly θ-precontinuous functions. Noiri and Popa [21] introduced the concept of θ-M-continuous functions as functions from a set satisfying some minimal conditions into a set satisfying some minimal conditions and investigated some characterizations and several properties of θ-M-continuous functions. In particular, Noiri and Popa [21] defined and studied the notion of strongly θ-M-closed graphs. Noiri and Popa [22] introduced the concept of θ-m-continuous functions as functions from a set satisfying some minimal conditions into a topological space and obtained several characterizations of such functions. Long and Herrington [23] investigated some characterizations of strongly θ-continuous functions. Jafari and Noiri [24] introduced and studied the notion of strongly θ-semi-continuous functions. Noiri [25] introduced and investigated the concept of strongly θ-precontinuous functions. Pue-on and Boonpok [26] introduced and studied the concept of $\theta(\Lambda, p)$-continuous functions. Quite recently, Thongmoon and Boonpok [27] introduced and investigated the notion of strongly $\theta(\Lambda, p)$-continuous functions. On the other hand, the present authors introduced and studied the concepts of (τ_1, τ_2)-continuous functions [28], almost (τ_1, τ_2)-continuous functions [29], weakly (τ_1, τ_2)-continuous functions [30] and quasi $\theta(\tau_1, \tau_2)$-continuous functions [31]. In this paper, we introduce the concept of $\theta(\tau_1, \tau_2)$-continuous functions. We also investigate several characterizations of $\theta(\tau_1, \tau_2)$-continuous functions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [32] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [32] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [32] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 1. [32] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2$-Cl(A) and $\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Cl$(A)) = \tau_1\tau_2$-Cl(A).

(2) If $A \subseteq B$, then $\tau_1\tau_2$-Cl$(A) \subseteq \tau_1\tau_2$-Cl$(B)$.

(3) $\tau_1\tau_2$-Cl(A) is $\tau_1\tau_2$-closed.

(4) A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2$-Cl(A).

(5) $\tau_1\tau_2$-$Cl(X - A) = X - \tau_1\tau_2$-$Int(A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$*-open* [33] (resp. $(\tau_1, \tau_2)s$*-open* [34], $(\tau_1, \tau_2)p$*-open* [34], $(\tau_1, \tau_2)\beta$*-open* [34]) if $A = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ (resp. $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$, $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$, $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))))$. The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is called $(\tau_1, \tau_2)r$*-closed* (resp. $(\tau_1, \tau_2)s$*-closed*, $(\tau_1, \tau_2)p$*-closed*, $(\tau_1, \tau_2)\beta$*-closed*). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\alpha(\tau_1, \tau_2)$*-open* [35] if $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A)))$. The complement of an $\alpha(\tau_1, \tau_2)$-open set is said to be $\alpha(\tau_1, \tau_2)$*-closed*. Let A be a subset of a bitopological space (X, τ_1, τ_2). A point $x \in X$ is called a $(\tau_1, \tau_2)\theta$*-cluster point* [33] of A if $\tau_1\tau_2$-Cl$(U) \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set U containing x. The set of all $(\tau_1, \tau_2)\theta$-cluster points of A is called the $(\tau_1, \tau_2)\theta$*-closure* [33] of A and is denoted by $(\tau_1, \tau_2)\theta$-Cl(A). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)\theta$*-closed* [33] if $A = (\tau_1, \tau_2)\theta$-Cl$(A)$. The complement of a $(\tau_1, \tau_2)\theta$-closed set is said to be $(\tau_1, \tau_2)\theta$*-open*. The union of all $(\tau_1, \tau_2)\theta$-open sets contained in A is called the $(\tau_1, \tau_2)\theta$*-interior* [33] of A and is denoted by $(\tau_1, \tau_2)\theta$-Int(A).

Lemma 2. [33] *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) If A is $\tau_2\tau_2$-open in X, then $\tau_1\tau_2$-$Cl(A) = (\tau_1, \tau_2)\theta$-$Cl(A)$.

(2) $(\tau_1, \tau_2)\theta$-$Cl(A)$ is $\tau_1\tau_2$-closed in X.

3. $\theta(\tau_1, \tau_2)$-continuous functions

In this section, we introduce the concept of $\theta(\tau_1, \tau_2)$-continuous functions. Furthermore, several characterizations of $\theta(\tau_1, \tau_2)$-continuous functions are discussed.

Definition 1. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be $\theta(\tau_1, \tau_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(\tau_1\tau_2$-$Cl(U)) \subseteq \sigma_1\sigma_2$-$Cl(V)$. A function*

$$f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$$

is said to be $\theta(\tau_1, \tau_2)$-continuous if f is $\theta(\tau_1, \tau_2)$-continuous at each point x of X.

Theorem 1. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is $\theta(\tau_1, \tau_2)$-continuous at $x \in X$ if and only if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, $x \in (\tau_1, \tau_2)\theta$-$Int(f^{-1}(\sigma_1\sigma_2$-$Cl(V)))$.*

Proof. Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Since f is $\theta(\tau_1, \tau_2)$-continuous at x, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(\tau_1\tau_2$-Cl$(U)) \subseteq \sigma_1\sigma_2$-Cl$(V)$. Then, we have $x \in U \subseteq \tau_1\tau_2$-Cl$(U) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ and hence $x \in (\tau_1, \tau_2)\theta$-Int$(f^{-1}(\sigma_1\sigma_2$-Cl$(V)))$.

Conversely, let V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Then, by the hypothesis we have $x \in (\tau_1, \tau_2)\theta$-Int$(f^{-1}(\sigma_1\sigma_2$-Cl$(V)))$. There exists a $\tau_1\tau_2$-open set U of X such that $x \in U \subseteq \tau_1\tau_2$-Cl$(U) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$; hence $f(\tau_1\tau_2$-Cl$(U)) \subseteq \sigma_1\sigma_2$-Cl$(V)$. This shows that f is $\theta(\tau_1, \tau_2)$-continuous at x.

Theorem 2. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is $\theta(\tau_1, \tau_2)$-continuous if and only if $f^{-1}(V) \subseteq (\tau_1, \tau_2)\theta\text{-}Int(f^{-1}(\sigma_1\sigma_2\text{-}Cl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y.*

Proof. Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in f^{-1}(V)$. Then, $f(x) \in V$. Since f is $\theta(\tau_1, \tau_2)$-continuous at x, by Theorem 1 we have $x \in (\tau_1, \tau_2)\theta\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$ and hence $f^{-1}(V) \subseteq (\tau_1, \tau_2)\theta\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$.

Conversely, let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Then, we have $x \in f^{-1}(V) \subseteq (\tau_1, \tau_2)\theta\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$ and hence

$$x \in (\tau_1, \tau_2)\theta\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))).$$

By Theorem 1, f is $\theta(\tau_1, \tau_2)$-continuous.

Theorem 3. *For a function $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is $\theta(\tau_1, \tau_2)$-continuous;

(2) $(\tau_1, \tau_2)\theta\text{-}Cl(f^{-1}(B)) \subseteq f^{-1}(\sigma_1, \sigma_2)\theta\text{-}Cl(B))$ for every subset B of Y;

(3) $(\tau_1, \tau_2)\theta\text{-}Cl(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(4) $f^{-1}(V) \subseteq (\tau_1, \tau_2)\theta\text{-}Int(f^{-1}(\sigma_1\sigma_2\text{-}Cl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(5) $f((\tau_1, \tau_2)\theta\text{-}Cl(A)) \subseteq (\sigma_1, \sigma_2)\theta\text{-}Cl(f(A))$ for every subset A of X;

(6) $(\tau_1, \tau_2)\theta\text{-}Cl(f^{-1}(\sigma_1\sigma_2\text{-}Int((\sigma_1, \sigma_2)\theta\text{-}Cl(B)))) \subseteq f^{-1}((\sigma_1, \sigma_2)\theta\text{-}Cl(B))$ for every subset B of Y;

(7) $(\tau_1, \tau_2)\theta\text{-}Cl(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(8) $(\tau_1, \tau_2)\theta\text{-}Cl(f^{-1}(\sigma_1\sigma_2\text{-}Int(K))) \subseteq f^{-1}(K)$ for every $(\sigma_1, \sigma_2)r$-closed set K of Y;

(9) $(\tau_1, \tau_2)\theta\text{-}Cl(f^{-1}(\sigma_1\sigma_2\text{-}Int(K))) \subseteq f^{-1}(K)$ for every $\sigma_1\sigma_2$-closed set K of Y.

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. Suppose that $x \notin f^{-1}(\sigma_1, \sigma_2)\theta\text{-Cl}(B))$. Then, we have $x \in f^{-1}(Y - (\sigma_1, \sigma_2)\theta\text{-Cl}(B)) = f^{-1}((\sigma_1, \sigma_2)\theta\text{-Int}(Y - B))$. Therefore, $f(x) \in (\sigma_1, \sigma_2)\theta\text{-Int}(Y - B)$. There exists a $\sigma_1\sigma_2$-open set V of Y such that

$$f(x) \in V \subseteq \sigma_1\sigma_2\text{-Cl}(V) \subseteq Y - B.$$

Since f is $\theta(\tau_1, \tau_2)$-continuous, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(\tau_1\tau_2\text{-Cl}(U)) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$; hence

$$\tau_1\tau_2\text{-Cl}(U) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)) \subseteq f^{-1}(Y - B) = X - f^{-1}(B).$$

Thus, $\tau_1\tau_2\text{-Cl}(U) \cap f^{-1}(B) = \emptyset$ and so $x \notin (\tau_1, \tau_2)\theta\text{-Cl}(f^{-1}(B))$.

(2) ⇒ (3): This is obvious since $\sigma_1\sigma_2$-Cl$(V) = (\sigma_1, \sigma_2)\theta$-Cl$(V)$ for every $\sigma_1\sigma_2$-open set V of Y.

(3) ⇒ (4): Let V be any $\sigma_1\sigma_2$-open set of Y. Thus by (3), we have

$$\begin{aligned} X - (\tau_1, \tau_2)\theta\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))) &= (\tau_1, \tau_2)\theta\text{-Cl}(X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))) \\ &= (\tau_1, \tau_2)\theta\text{-Cl}(f^{-1}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - V)) \\ &= f^{-1}(Y - V) \\ &= X - f^{-1}(V) \end{aligned}$$

and hence $f^{-1}(V) \subseteq (\tau_1, \tau_2)\theta$-Int$(f^{-1}(\sigma_1\sigma_2$-Cl$(V)))$.

(4) ⇒ (1): It follows from Theorem 2.

(2) ⇒ (5): Let A be any subset of X. By (2), we have

$$(\tau_1, \tau_2)\theta\text{-Cl}(A) \subseteq (\tau_1, \tau_2)\theta\text{-Cl}(f^{-1}(f(A))) \subseteq f^{-1}((\sigma_1, \sigma_2)\theta\text{-Cl}(f(A))).$$

Thus, $f((\tau_1, \tau_2)\theta$-Cl$(A)) \subseteq (\sigma_1, \sigma_2)\theta$-Cl$(f(A))$.

(5) ⇒ (2): Let B be any subset of Y. Then by (5), we have

$$f((\tau_1, \tau_2)\theta\text{-Cl}(f^{-1}(B))) \subseteq (\sigma_1, \sigma_2)\theta\text{-Cl}(f(f^{-1}(B))) \subseteq (\sigma_1, \sigma_2)\theta\text{-Cl}(B)$$

and hence $(\tau_1, \tau_2)\theta$-Cl$(f^{-1}(B)) \subseteq f^{-1}((\sigma_1, \sigma_2)\theta$-Cl$(B))$.

(3) ⇒ (6): Let B be any subset of Y. Since $(\sigma_1, \sigma_2)\theta$-Cl$(B)$ is $\sigma_1\sigma_2$-closed in Y, by (3) we have

$$\begin{aligned} (\tau_1, \tau_2)\theta\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B)))) &\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}((\sigma_1, \sigma_2)\theta\text{-Cl}(B)))) \\ &\subseteq f^{-1}((\sigma_1, \sigma_2)\theta\text{-Cl}(B)). \end{aligned}$$

(6) ⇒ (7): This is obvious since $\sigma_1\sigma_2$-Cl$(V) = (\sigma_1, \sigma_2)\theta$-Cl$(V)$ for every $\sigma_1\sigma_2$-open set V of Y.

(7) ⇒ (8): Let K be any $(\sigma_1, \sigma_2)r$-closed set of Y. Thus by (7), we have

$$\begin{aligned} (\tau_1, \tau_2)\theta\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(K))) &= (\tau_1, \tau_2)\theta\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))))) \\ &\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) \\ &= f^{-1}(K). \end{aligned}$$

(8) ⇒ (9): Let K be any $\sigma_1\sigma_2$-closed set of Y. Since $\sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(K))$ is $(\sigma_1, \sigma_2)r$-closed in Y, by (8) we have

$$\begin{aligned} (\tau_1, \tau_2)\theta\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(K))) &= (\tau_1, \tau_2)\theta\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))))) \\ &\subseteq f^{-1}(K). \end{aligned}$$

(9) ⇒ (4): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, $Y - V$ is $\sigma_1\sigma_2$-closed in Y and by (9), $(\tau_1, \tau_2)\theta$-Cl$(f^{-1}(\sigma_1\sigma_2\text{-Int}(Y - V))) \subseteq f^{-1}(Y - V) = X - f^{-1}(V)$. Moreover, we have

$$\begin{aligned}(\tau_1, \tau_2)\theta\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(Y - V))) &= (\tau_1, \tau_2)\theta\text{-Cl}(f^{-1}(Y - \sigma_1\sigma_2\text{-Cl}(V))) \\ &= (\tau_1, \tau_2)\theta\text{-Cl}(X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))) \\ &= X - (\tau_1, \tau_2)\theta\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))).\end{aligned}$$

Thus, $f^{-1}(V) \subseteq (\tau_1, \tau_2)\theta$-Int$(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$.

Theorem 4. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is $\theta(\tau_1, \tau_2)$-continuous if and only if $(\tau_1, \tau_2)\theta$-Cl$(f^{-1}(V)) \subseteq f^{-1}((\sigma_1, \sigma_2)\theta\text{-Cl}(V))$ for every $\sigma_1\sigma_2$-open set V of Y.*

Proof. This is an immediate consequence of Theorem 3.

Theorem 5. *For a function $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is $\theta(\tau_1, \tau_2)$-continuous;

(2) $(\tau_1, \tau_2)\theta$-Cl$(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$ for every $(\sigma_1, \sigma_2)\beta$-open set V of Y;

(3) $(\tau_1, \tau_2)\theta$-Cl$(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$ for every $(\sigma_1, \sigma_2)s$-open set V of Y.

Proof. (1) ⇒ (2): Let V be any $(\sigma_1, \sigma_2)\beta$-open set of Y. Then,

$$V \subseteq \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$$

and $\sigma_1\sigma_2$-Cl$(V) = \sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$. Since $\sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$ is $(\sigma_1, \sigma_2)r$-closed in Y, by Theorem 3 we have

$$(\tau_1, \tau_2)\theta\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)).$$

(2) ⇒ (3): The proof is obvious.

(3) ⇒ (1): Let V be any $(\sigma_1, \sigma_2)\beta$-open set of Y. Since $\sigma_1\sigma_2$-Cl(V) is $(\sigma_1, \sigma_2)s$-open in Y, by (3) we have $(\tau_1, \tau_2)\theta$-Cl$(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$. By Theorem 3, f is $\theta(\tau_1, \tau_2)$-continuous.

Theorem 6. *For a function $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is $\theta(\tau_1, \tau_2)$-continuous;

(2) $(\tau_1, \tau_2)\theta$-Cl$(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;

(3) $(\tau_1, \tau_2)\theta$*-Cl*$(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2$*-Cl*$(V))$ *for every* $(\sigma_1, \sigma_2)p$*-open set* V *of* Y*;*

(4) $f^{-1}(V) \subseteq (\tau_1, \tau_2)\theta$*-Int*$(f^{-1}(\sigma_1\sigma_2$*-Cl*$(V)))$ *for every* $(\sigma_1, \sigma_2)p$*-open set* V *of* Y*.*

Proof. (1) ⇒ (2): It follows from Theorem 5.

(2) ⇒ (3): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Then by (2), we have

$$\begin{aligned}(\tau_1, \tau_2)\theta\text{-Cl}(f^{-1}(V)) &\subseteq (\tau_1, \tau_2)\theta\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))\\ &\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)).\end{aligned}$$

(3) ⇒ (4): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. By (3), we have

$$\begin{aligned}X - (\tau_1, \tau_2)\theta\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))) &= (\tau_1, \tau_2)\theta\text{-Cl}(X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))\\ &= (\tau_1, \tau_2)\theta\text{-Cl}(f^{-1}(Y - \sigma_1\sigma_2\text{-Cl}(V)))\\ &\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - \sigma_1\sigma_2\text{-Cl}(V)))\\ &= X - f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))\\ &\subseteq X - f^{-1}(V)\end{aligned}$$

and hence $f^{-1}(V) \subseteq (\tau_1, \tau_2)\theta\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$.

(4) ⇒ (1): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, V is $(\sigma_1, \sigma_2)p$-open in Y, by (4) we have $f^{-1}(V) \subseteq (\tau_1, \tau_2)\theta\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$. By Theorem 3, f is $\theta(\tau_1, \tau_2)$-continuous.

Definition 2. [36] *A bitopological space* (X, τ_1, τ_2) *is said to be* (τ_1, τ_2)*-regular if for each* $\tau_1\tau_2$*-closed set* F *and each* $x \notin F$*, there exist disjoint* $\tau_1\tau_2$*-open sets* U *and* V *such that* $x \in U$ *and* $F \subseteq V$*.*

Lemma 3. [37] *A bitopological space* (X, τ_1, τ_2) *is* (τ_1, τ_2)*-regular if and only if for each* $x \in X$ *and each* $\tau_1\tau_2$*-open set* U *containing* x*, there exists a* $\tau_1\tau_2$*-open set* V *such that* $x \in V \subseteq \tau_1\tau_2$*-Cl*$(V) \subseteq U$*.*

Lemma 4. [37] *Let* (X, τ_1, τ_2) *be a* (τ_1, τ_2)*-regular space. Then, the following properties hold:*

(1) $\tau_1\tau_2$*-Cl*$(A) = (\tau_1, \tau_2)\theta$*-Cl*$(A)$ *for every subset* A *of* X*.*

(2) Every $\tau_1\tau_2$*-open set is* $(\tau_1, \tau_2)\theta$*-open.*

Definition 3. [28] *A function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is called* (τ_1, τ_2)*-continuous at a point* $x \in X$ *if for each* $\sigma_1\sigma_2$*-open set* V *of* Y *containing* $f(x)$*, there exists a* $\tau_1\tau_2$*-open set* U *of* X *containing* x *such that* $f(U) \subseteq V$*. A function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is called* (τ_1, τ_2)*-continuous if* f *has this property at each point of* X*.*

Definition 4. [30] *A function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is said to be weakly* (τ_1, τ_2)*-continuous at a point* $x \in X$ *if for each* $\tau_1\tau_2$*-open set* V *of* Y *containing* $f(x)$*, there exists a* $\tau_1\tau_2$*-open set* U *of* X *containing* x *such that* $f(U) \subseteq \sigma_1\sigma_2$*-Cl*$(V)$*. A function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is said to be weakly* (τ_1, τ_2)*-continuous if* f *has this property at each point of* X*.*

Lemma 5. [28] *For a function $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is (τ_1, τ_2)-continuous;

(2) $f^{-1}(V)$ is $\tau_1\tau_2$-open in X for every $\sigma_1\sigma_2$-open set V of Y;

(3) $f(\tau_1\tau_2\text{-}Cl(A)) \subseteq \sigma_1\sigma_2\text{-}Cl(f(A))$ for every subset A of X;

(4) $\tau_1\tau_2\text{-}Cl(f^{-1}(B)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;

(5) $f^{-1}(\sigma_1\sigma_2\text{-}Int(B)) \subseteq \tau_1\tau_2\text{-}Int(f^{-1}(B))$ for every subset B of Y;

(6) $f^{-1}(K)$ is $\tau_1\tau_2$-closed in X for every $\sigma_1\sigma_2$-closed set K of Y.

Theorem 7. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, where (Y, σ_1, σ_2) is (σ_1, σ_2)-regular, the following properties are equivalent:*

(1) f is (τ_1, τ_2)-continuous;

(2) f is $\theta(\tau_1, \tau_2)$-continuous;

(3) f is weakly (τ_1, τ_2)-continuous.

Proof. (1) ⇒ (2): Let V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Thus by Lemma 5, $f^{-1}(V)$ is $\tau_1\tau_2$-open in X. Since $\sigma_1\sigma_2\text{-Cl}(V)$ is $\sigma_1\sigma_2$-closed, by Lemma 5 we have $f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$ is $\tau_1\tau_2$-closed. Put $U = f^{-1}(V)$. Then, U is a $\tau_1\tau_2$-open set U of X such that $x \in U \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)) = \tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$. This implies that $\tau_1\tau_2\text{-Cl}(U) \subseteq \tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))) = f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$. Thus,

$$f(\tau_1\tau_2\text{-Cl}(U)) \subseteq \sigma_1\sigma_2\text{-Cl}(V).$$

This shows that f is $\theta(\tau_1, \tau_2)$-continuous.

(2) ⇒ (3): The proof is obvious.

(3) ⇒ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Since (Y, σ_1, σ_2) is (σ_1, σ_2)-regular, by Lemma 3 there exists a $\sigma_1\sigma_2$-open set W of Y such that $f(x) \in W \subseteq \sigma_1\sigma_2\text{-Cl}(W) \subseteq V$. Since f is weakly (τ_1, τ_2)-continuous, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2\text{-Cl}(W) \subseteq V$. Thus, f is (τ_1, τ_2)-continuous.

Theorem 8. *Let (X, τ_1, τ_2) be (τ_1, τ_2)-regular. Then a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is $\theta(\tau_1, \tau_2)$-continuous if and only if f is weakly (τ_1, τ_2)-continuous.*

Proof. We prove only the sufficiency. Suppose that f is weakly (τ_1, τ_2)-continuous. Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Then, there exists a $\tau_1\tau_2$-open set W of X containing x such that $f(W) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. Since (X, τ_1, τ_2) is (τ_1, τ_2)-regular, by Lemma 3 there exists a $\tau_1\tau_2$-open set U of X such that $x \in U \subseteq \tau_1\tau_2\text{-Cl}(U) \subseteq W$. Thus, $f(\tau_1\tau_2\text{-Cl}(U)) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. This shows that f is $\theta(\tau_1, \tau_2)$-continuous.

4. Some results on $\theta(\tau_1, \tau_2)$-continuity

Recall that a bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)-T_2 [38] if for any pair of distinct points x, y in X, there exist disjoint $\tau_1\tau_2$-open sets U and V of X containing x and y, respectively.

Definition 5. [39] *A bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-Urysohn if for each pair of distinct points x and y in X, there exist $\tau_1\tau_2$-open sets U and V such that $x \in U$, $y \in V$ and $\tau_1\tau_2$-Cl$(U) \cap \tau_1\tau_2$-Cl$(V) = \emptyset$.*

Theorem 9. *Let (X, τ_1, τ_2) be a bitopological space. If for any distinct points x and x' in X, there exists a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ such that*

(1) (Y, σ_1, σ_2) is $\sigma_1\sigma_2$-Urysohn,

(2) $f(x) \neq f(x')$, and

(3) f is $\theta(\tau_1, \tau_2)$-continuous at x and x',

then (X, τ_1, τ_2) is $\tau_1\tau_2$-Urysohn.

Proof. Let x, x' be any distinct points of X. Then, by the hypothesis there exists a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ which satisfies three conditions. Now let $y = f(x)$ and $y' = f(x')$. Then, $y \neq y'$. Since (Y, σ_1, σ_2) is $\sigma_1\sigma_2$-Urysohn, there exist $\sigma_1\sigma_2$-open sets V and V' of Y containing y and y', respectively, such that $\sigma_1\sigma_2$-Cl$(V) \cap \sigma_1\sigma_2$-Cl$(V') = \emptyset$. Since f is $\theta(\tau_1, \tau_2)$-continuous at x and x', there exist $\tau_1\tau_2$-open sets U and U' of X containing x and x', respectively, such that $f(\tau_1\tau_2\text{-Cl}(U)) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$ and

$$f(\tau_1\tau_2\text{-Cl}(U')) \subseteq \sigma_1\sigma_2\text{-Cl}(V').$$

This implies that $\tau_1\tau_2$-Cl$(U) \cap \tau_1\tau_2$-Cl$(U') = \emptyset$. Thus, (X, τ_1, τ_2) is $\tau_1\tau_2$-Urysohn.

Definition 6. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to have a strong $\theta(\tau_1, \tau_2)$-closed graph if for each $(x, y) \in (X \times Y) - G(f)$, there exist a $\tau_1\tau_2$-open set U of X containing x and a $\sigma_1\sigma_2$-open set V of Y containing y such that*

$$[\tau_1\tau_2\text{-}Cl(U) \times \sigma_1\sigma_2\text{-}Cl(V)] \cap G(f) = \emptyset.$$

Lemma 6. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ has a strong $\theta(\tau_1, \tau_2)s$-closed graph if and only if for each $(x, y) \in (X \times Y) - G(f)$, there exist a $\tau_1\tau_2$-open set U of X containing x and a $\sigma_1\sigma_2$-open set V of Y containing y such that $f(\tau_1\tau_2\text{-}Cl(U)) \cap \sigma_1\sigma_2\text{-}Cl(V) = \emptyset$.*

Theorem 10. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is $\theta(\tau_1, \tau_2)$-continuous and (Y, σ_1, σ_2) is $\sigma_1\sigma_2$-Urysohn, then $G(f)$ is strong $\theta(\tau_1, \tau_2)$-closed.*

Proof. Suppose that $(x, y) \in (X \times Y) - G(f)$. Then, $y \neq f(x)$. Since (Y, σ_1, σ_2) is $\sigma_1\sigma_2$-Urysohn, there exist $\sigma_1\sigma_2$-open sets V and W of Y containing y and $f(x)$, respectively, such that $\sigma_1\sigma_2$-Cl$(V) \cap \sigma_1\sigma_2$-Cl$(W) = \emptyset$. Since f is $\theta(\tau_1, \tau_2)$-continuous, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(\tau_1\tau_2\text{-Cl}(U)) \subseteq \sigma_1\sigma_2\text{-Cl}(W)$. This implies that $f(\tau_1\tau_2\text{-Cl}(U)) \cap \sigma_1\sigma_2\text{-Cl}(V) = \emptyset$ and by Lemma 6, $G(f)$ is strong $\theta(\tau_1, \tau_2)$-closed.

Theorem 11. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is an injective $\theta(\tau_1, \tau_2)$-continuous function with a strong $\theta(\tau_1, \tau_2)$-closed graph, then (X, τ_1, τ_2) is $\sigma_1\sigma_2$-Urysohn.*

Proof. Let x and y be any distinct points of X. Since f is injective, $f(x) \neq f(y)$. Then, we have $(x, f(y)) \in (X \times Y) - G(f)$. Since $G(f)$ is strong $\theta(\tau_1, \tau_2)$-closed, by Lemma 6 there exist a $\tau_1\tau_2$-open set U of X containing x and a $\sigma_1\sigma_2$-open set V of Y containing $f(y)$ such that such that $f(\tau_1\tau_2\text{-Cl}(U)) \cap \sigma_1\sigma_2\text{-Cl}(V) = \emptyset$. Since f is $\theta(\tau_1, \tau_2)$-continuous, there exists a $\tau_1\tau_2$-open set W of X containing y such that $f(\tau_1\tau_2\text{-Cl}(W)) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. Thus, $f(\tau_1\tau_2\text{-Cl}(U)) \cap f(\tau_1\tau_2\text{-Cl}(W)) = \emptyset$ and hence $\tau_1\tau_2\text{-Cl}(U) \cap \tau_1\tau_2\text{-Cl}(W) = \emptyset$. This shows that (X, τ_1, τ_2) is $\sigma_1\sigma_2$-Urysohn.

Recall that a bitopological space (X, τ_1, τ_2) is said to be *quasi* (τ_1, τ_2)-$\mathscr{H}$-*closed* [40] if every $\tau_1\tau_2$-open cover $\{U_\gamma \mid \gamma \in \Gamma\}$, there exists a finite subset Γ_0 of Γ such that $X = \cup\{\tau_1\tau_2\text{-Cl}(U_\gamma) \mid \gamma \in \Gamma_0\}$. A subset K of a bitopological space (X, τ_1, τ_2) is said to be *quasi* (τ_1, τ_2)-$\mathscr{H}$-*closed relative to* X if for any cover $\{V_\gamma \mid \gamma \in \Gamma\}$ by $\tau_1\tau_2$-open sets of X, there exists a finite subset Γ_0 of Γ such that $K \subseteq \cup\{\tau_1\tau_2\text{-Cl}(V_\gamma) \mid \gamma \in \Gamma_0\}$.

Theorem 12. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is $\theta(\tau_1, \tau_2)$-continuous and K is quasi (τ_1, τ_2)-$\mathscr{H}$-closed relative to X, then $f(K)$ is quasi (σ_1, σ_2)-$\mathscr{H}$-closed relative to Y.*

Proof. Let $\{V_\gamma \mid \gamma \in \Gamma\}$ be a cover of $f(K)$ by $\sigma_1\sigma_2$-open sets of Y. For each $k \in K$, there exists $\gamma(k) \in \Gamma$ such that $f(k) \in V_{\gamma(k)}$. Since f is $\theta(\tau_1, \tau_2)$-continuous, there exists a $\tau_1\tau_2$-open set U_k of X containing k such that $f(\tau_1\tau_2\text{-Cl}(U_k)) \subseteq \sigma_1\sigma_2\text{-Cl}(V_{\gamma(k)})$. Since $\{U_k \mid k \in K\}$ is a cover of K by $\tau_1\tau_2$-open sets in X, there exists a finite subset K_0 of K such that $K \subseteq \cup\{\tau_1\tau_2\text{-Cl}(U_k) \mid k \in K_0\}$. Thus,

$$\begin{aligned} f(K) &\subseteq \cup\{f(\tau_1\tau_2\text{-Cl}(U_k)) \mid k \in K_0\} \\ &\subseteq \cup\{\sigma_1\sigma_2\text{-Cl}(V_{\gamma(k)}) \mid k \in K_0\}. \end{aligned}$$

This shows that $f(K)$ is quasi (σ_1, σ_2)-$\mathscr{H}$-closed relative to Y.

Corollary 1. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a $\theta(\tau_1, \tau_2)$-continuous surjection and (X, τ_1, τ_2) is quasi (τ_1, τ_2)-$\mathscr{H}$-closed, then (Y, σ_1, σ_2) is quasi (σ_1, σ_2)-$\mathscr{H}$-closed.*

Definition 7. *Let A be a subset of a bitopological space (X, τ_1, τ_2). The $(\tau_1, \tau_2)\theta$-frontier of A, $(\tau_1, \tau_2)\theta$-fr(A), is defined by $(\tau_1, \tau_2)\theta\text{-fr}(A) = (\tau_1, \tau_2)\theta\text{-Cl}(A) \cap (\tau_1, \tau_2)\theta\text{-Cl}(X - A)$.*

Theorem 13. *The set of all points $x \in X$ at which a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is not $\theta(\tau_1, \tau_2)$-continuous is identical with the union of the $(\tau_1, \tau_2)\theta$-frontier of the inverse images of the $\sigma_1\sigma_2$-closure of $\sigma_1\sigma_2$-open sets containing $f(x)$.*

Proof. Suppose that f is not $\theta(\tau_1, \tau_2)$-continuous. Then, there exists a $\sigma_1\sigma_2$-open set V of Y containing $f(x)$ such that $f(\tau_1\tau_2\text{-Cl}(U))$ is not contained in $\sigma_1\sigma_2\text{-Cl}(V)$ for every $\tau_1\tau_2$-open set U of X containing x. Then, $\tau_1\tau_2\text{-Cl}(U) \cap (X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))) \neq \emptyset$ for every $\tau_1\tau_2$-open set U of X containing x. Thus, $x \in (\tau_1, \tau_2)\theta\text{-Cl}(X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$. On the other hand, we have $x \in f^{-1}(V) \subseteq (\tau_1, \tau_2)\theta\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$ and hence $x \in (\tau_1, \tau_2)\theta\text{-fr}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$.

Conversely, suppose that f is $\theta(\tau_1, \tau_2)$-continuous at $x \in X$. Let V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Then by Theorem 2 we have

$$x \in f^{-1}(V) \subseteq (\tau_1, \tau_2)\theta\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))).$$

Thus, $x \notin (\tau_1, \tau_2)\theta\text{-fr}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)))$ for every $\sigma_1\sigma_2$-open set V of Y containing $f(x)$. This completes the proof.

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