EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 2, Article Number 6017
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

On Strongly $\theta(\tau_1, \tau_2)$-continuous Functions

Prapart Pue-on[1], Supunnee Sompong[2], Chawalit Boonpok[1,*]

[1] *Mathematics and Applied Mathematics Research Unit, Department of Mathematics, Faculty of Science, Mahasarakham University, Maha Sarakham, 44150, Thailand*
[2] *Department of Mathematics and Statistics, Faculty of Science and Technology, Sakon Nakhon Rajbhat University, Sakon Nakhon, 47000, Thailand*

Abstract. This paper deals with the concept of strongly $\theta(\tau_1, \tau_2)$-continuous functions. Furthermore, some characterizations and several properties concerning strongly $\theta(\tau_1, \tau_2)$-continuous functions are considered.

2020 Mathematics Subject Classifications: 54C08; 54E55
Key Words and Phrases: $(\tau_1, \tau_2)\theta$-open set, strongly $\theta(\tau_1, \tau_2)$-continuous function

1. Introduction

In 1941, Fomin [1] introduced the concept of θ-continuous functions. Noiri [2] studied some properties of θ-continuous functions. Popa [3] investigated several characterizations of θ-continuous functions. Arya and Bhamini [4] introduced the notion of θ-semi-continuous functions. Jafari and Noiri [5] investigated several characterizations of θ-semi-continuous functions. Noiri [6] introduced and investigated the concept of θ-precontinuous functions. In 1981, Long and Herrington [7] investigated some characterizations of strongly θ-continuous functions. Jafari and Noiri [8] introduced and investigated the concept of strongly θ-semi-continuous functions. Noiri [9] introduced and studied the notion of strongly θ-precontinuous functions. Noiri and Popa [10] introduced and investigated the concept of strongly θ-β-continuous functions. On the other hand, Di Maio and Noiri [11] introduced the concept of strongly irresolute functions. Pal and Bhattacharyya [12] introduced and investigated the notion of strongly preirresolute functions. Noiri [13] investigated several characterizations of strongly β-irresolute functions. Jafari and Noiri [14] studied some characterizations of strongly sober θ-continuous functions. These classes of functions have characterizations similar to the class of strongly θ-continuous functions. In 2005, Noiri and Popa [15] introduced a new class of functions called strongly θ-M-continuous functions as functions defined between sets satisfying

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i2.6017

Email addresses: prapart.p@msu.ac.th (P. Pue-on),
s_sompong@snru.ac.th (S. Sompong), chawalit.b@msu.ac.th (C. Boonpok)

some minimal conditions and obtained several characterizations and some properties of such functions. Furthermore, the present authors [15] defined and studied the notions of strongly θ-M-closed graphs and m-closed spaces. Thongmoon and Boonpok [16] introduced and studied the notion of strongly $\theta(\Lambda, p)$-continuous functions. Quite recently, the present authors [17] introduced and investigated the notion of almost strongly $\theta(\Lambda, p)$-continuous functions. Moreover, several characterizations of (τ_1, τ_2)-continuous functions, almost (τ_1, τ_2)-continuous functions, weakly (τ_1, τ_2)-continuous functions, almost weakly (τ_1, τ_2)-continuous functions, faintly (τ_1, τ_2)-continuous functions, weakly quasi (τ_1, τ_2)-continuous functions, almost quasi (τ_1, τ_2)-continuous functions, $\delta(\tau_1, \tau_2)$-continuous functions and quasi $\theta(\tau_1, \tau_2)$-continuous functions were established in [18], [19], [20], [21], [22], [23], [24], [25] and [26], respectively. In this paper, we introduce the concept of strongly $\theta(\tau_1, \tau_2)$-continuous functions. We also investigate several characterizations of strongly $\theta(\tau_1, \tau_2)$-continuous functions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [27] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [27] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [27] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 1. [27] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2$-Cl(A) and $\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Cl$(A)) = \tau_1\tau_2$-Cl(A).

(2) If $A \subseteq B$, then $\tau_1\tau_2$-Cl$(A) \subseteq \tau_1\tau_2$-Cl$(B)$.

(3) $\tau_1\tau_2$-Cl(A) is $\tau_1\tau_2$-closed.

(4) A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2$-Cl(A).

(5) $\tau_1\tau_2$-Cl$(X - A) = X - \tau_1\tau_2$-Int$(A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$-*open* [28] (resp. $(\tau_1, \tau_2)s$-*open* [29], $(\tau_1, \tau_2)p$-*open* [29], $(\tau_1, \tau_2)\beta$-*open* [29]) if $A = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ (resp. $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$, $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$, $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))))$. The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is called $(\tau_1, \tau_2)r$-*closed* (resp. $(\tau_1, \tau_2)s$-*closed*, $(\tau_1, \tau_2)p$-*closed*, $(\tau_1, \tau_2)\beta$-*closed*). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\alpha(\tau_1, \tau_2)$-*open* [30] if $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A)))$. The complement of an $\alpha(\tau_1, \tau_2)$-open set is said to be

$\alpha(\tau_1,\tau_2)$-*closed*. For a subset A of a bitopological space (X,τ_1,τ_2), a point $x\in X$ is called a $(\tau_1,\tau_2)\theta$-*cluster point* [28] of A if $\tau_1\tau_2$-Cl$(U)\cap A\neq\emptyset$ for every $\tau_1\tau_2$-open set U containing x. The set of all $(\tau_1,\tau_2)\theta$-cluster points of A is called the $(\tau_1,\tau_2)\theta$-*closure* [28] of A and is denoted by $(\tau_1,\tau_2)\theta$-Cl(A). A subset A of a bitopological space (X,τ_1,τ_2) is said to be $(\tau_1,\tau_2)\theta$-*closed* [28] if $A=(\tau_1,\tau_2)\theta$-Cl(A). The complement of a $(\tau_1,\tau_2)\theta$-closed set is said to be $(\tau_1,\tau_2)\theta$-*open*. The union of all $(\tau_1,\tau_2)\theta$-open sets contained in A is called the $(\tau_1,\tau_2)\theta$-*interior* [28] of A and is denoted by $(\tau_1,\tau_2)\theta$-Int(A).

Lemma 2. [28] *For a subset A of a bitopological space (X,τ_1,τ_2), the following properties hold:*

(1) If A is $\tau_2\tau_2$-open in X, then $\tau_1\tau_2$-Cl$(A)=(\tau_1,\tau_2)\theta$-Cl$(A)$.

(2) $(\tau_1,\tau_2)\theta$-Cl(A) is $\tau_1\tau_2$-closed in X.

3. On strongly $\theta(\tau_1,\tau_2)$-continuous functions

In this section, we introduce the concept of strongly $\theta(\tau_1,\tau_2)$-continuous functions. Moreover, some characterizations of strongly $\theta(\tau_1,\tau_2)$-continuous functions are discussed.

Definition 1. *A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is said to be strongly $\theta(\tau_1,\tau_2)$-continuous at a point $x\in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(\tau_1\tau_2$-Cl$(U))\subseteq V$. A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is said to be strongly $\theta(\tau_1,\tau_2)$-continuous if f is strongly $\theta(\tau_1,\tau_2)$-continuous at each point x of X.*

Theorem 1. *A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is strongly $\theta(\tau_1,\tau_2)$-continuous at $x\in X$ if and only if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$,*

$$x\in(\tau_1,\tau_2)\theta\text{-}Int(f^{-1}(V)).$$

Proof. Suppose that f is strongly $\theta(\tau_1,\tau_2)$-continuous at $x\in X$. Let V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Then, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(\tau_1\tau_2$-Cl$(U))\subseteq V$. Thus, $\tau_1\tau_2$-Cl$(U)\subseteq f^{-1}(V)$ and hence $x\in(\tau_1,\tau_2)\theta$-Int$(f^{-1}(V))$.

Conversely, let V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Then, by the hypothesis we have $x\in(\tau_1,\tau_2)\theta$-Int$(f^{-1}(V))$. There exists a $\tau_1\tau_2$-open set U of X such that

$$x\in U\subseteq\tau_1\tau_2\text{-Cl}(U)\subseteq f^{-1}(V);$$

hence $f(\tau_1\tau_2$-Cl$(U))\subseteq V$. This shows that f is strongly $\theta(\tau_1,\tau_2)$-continuous at $x\in X$.

Theorem 2. *For a function $(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$, the following properties are equivalent:*

(1) f is strongly $\theta(\tau_1,\tau_2)$-continuous;

(2) $f^{-1}(V)$ is $(\tau_1,\tau_2)\theta$-open in X for every $\sigma_1\sigma_2$-open set V of Y;

(3) $f^{-1}(F)$ *is* $(\tau_1, \tau_2)\theta$*-closed in* X *for every* $\sigma_1\sigma_2$*-closed set* F *of* Y*;*

(4) $f((\tau_1, \tau_2)\theta\text{-}Cl(A)) \subseteq \sigma_1\sigma_2\text{-}Cl(f(A))$ *for every subset* A *of* X*;*

(5) $(\tau_1, \tau_2)\theta\text{-}Cl(f^{-1}(B)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(B))$ *for every subset* B *of* Y*.*

Proof. (1) ⇒ (2): Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in f^{-1}(V)$. Then, $f(x) \in V$. Since f is strongly $\theta(\tau_1, \tau_2)$-continuous, by Theorem 1 we have $x \in (\tau_1, \tau_2)\theta\text{-Int}(f^{-1}(V))$. Thus, $f^{-1}(V) \subseteq (\tau_1, \tau_2)\theta\text{-Int}(f^{-1}(V))$ and hence $f^{-1}(V) = (\tau_1, \tau_2)\theta\text{-Int}(f^{-1}(V))$. This shows that $f^{-1}(V)$ is $(\tau_1, \tau_2)\theta$-open in X.

(2) ⇒ (3): The proof is obvious.

(3) ⇒ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. By (3), $f^{-1}(Y - V)$ is $(\tau_1, \tau_2)\theta$-closed and so $f^{-1}(V)$ is $(\tau_1, \tau_2)\theta$-open. Then, there exists a $\tau_1\tau_2$-open set U of X such that $x \in U \subseteq \tau_1\tau_2\text{-Cl}(U) \subseteq f^{-1}(V)$. Thus, $f(\tau_1\tau_2\text{-Cl}(U)) \subseteq V$. This shows that f is strongly $\theta(\tau_1, \tau_2)$-continuous.

(1) ⇒ (4): Let A be any subset of X. Let $x \in (\tau_1, \tau_2)\theta\text{-Cl}(A)$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Since f is strongly $\theta(\tau_1, \tau_2)$-continuous, there exists a $\tau_1\tau_2$-open set U of X such that $f(\tau_1\tau_2\text{-Cl}(U)) \subseteq V$. Since $x \in (\tau_1, \tau_2)\theta\text{-Cl}(A)$, we have $\tau_1\tau_2\text{-Cl}(U) \cap A \neq \emptyset$. It follows that $\emptyset \neq f(\tau_1\tau_2\text{-Cl}(U)) \cap f(A) \subseteq V \cap f(A)$. Thus, $f(x) \in \sigma_1\sigma_2\text{-Cl}(f(A))$.

(4) ⇒ (5): Let B be any subset of Y. By (4), we have

$$f((\tau_1, \tau_2)\theta\text{-Cl}(f^{-1}(B))) \subseteq \sigma_1\sigma_2\text{-Cl}(f(f^{-1}(B))) \subseteq \sigma_1\sigma_2\text{-Cl}(B)$$

and hence $(\tau_1, \tau_2)\theta\text{-Cl}(f^{-1}(B)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(B))$.

(5) ⇒ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Since $V \cap (Y - V) = \emptyset$, $f(x) \notin \sigma_1\sigma_2\text{-Cl}(Y - V)$ and so $x \notin f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - V))$. By (5), we have $x \notin (\tau_1, \tau_2)\theta\text{-Cl}(f^{-1}(Y - V)) = X - (\tau_1, \tau_2)\theta\text{-Int}(f^{-1}(V))$. Thus, $x \in (\tau_1, \tau_2)\theta\text{-Int}(f^{-1}(V))$. Then, there exists a $\tau_1\tau_2$-open set U of X such that $\tau_1\tau_2\text{-Cl}(U) \subseteq f^{-1}(V)$; hence

$$f(\tau_1\tau_2\text{-Cl}(U)) \subseteq V.$$

This shows that f is strongly $\theta(\tau_1, \tau_2)$-continuous.

Definition 2. [18] *A function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is called* (τ_1, τ_2)*-continuous at a point* $x \in X$ *if for each* $\sigma_1\sigma_2$*-open set* V *of* Y *containing* $f(x)$*, there exists a* $\tau_1\tau_2$*-open set* U *of* X *containing* x *such that* $f(U) \subseteq V$*. A function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is called* (τ_1, τ_2)*-continuous if* f *has this property at each point of* X*.*

Lemma 3. [18] *For a function* $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) f *is* (τ_1, τ_2)*-continuous;*

(2) $f^{-1}(V)$ *is* $\tau_1\tau_2$*-open in* X *for every* $\sigma_1\sigma_2$*-open set* V *of* Y*;*

(3) $f(\tau_1\tau_2\text{-}Cl(A)) \subseteq \sigma_1\sigma_2\text{-}Cl(f(A))$ *for every subset* A *of* X*;*

(4) $\tau_1\tau_2$-$Cl(f^{-1}(B)) \subseteq f^{-1}(\sigma_1\sigma_2$-$Cl(B))$ *for every subset* B *of* Y*;*

(5) $f^{-1}(\sigma_1\sigma_2$-$Int(B)) \subseteq \tau_1\tau_2$-$Int(f^{-1}(B))$ *for every subset* B *of* Y*;*

(6) $f^{-1}(K)$ *is* $\tau_1\tau_2$*-closed in* X *for every* $\sigma_1\sigma_2$*-closed set* K *of* Y*.*

Definition 3. [20] *A function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is said to be weakly* (τ_1, τ_2)*-continuous at a point* $x \in X$ *if for each* $\tau_1\tau_2$*-open set* V *of* Y *containing* $f(x)$*, there exists a* $\tau_1\tau_2$*-open set* U *of* X *containing* x *such that* $f(U) \subseteq \sigma_1\sigma_2$-$Cl(V)$*. A function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is said to be weakly* (τ_1, τ_2)*-continuous if* f *has this property at each point of* X*.*

Lemma 4. [20] *For a function* $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) f *is weakly* (τ_1, τ_2)*-continuous;*

(2) $f(\tau_1\tau_2$-$Cl(A)) \subseteq (\sigma_1, \sigma_2)\theta$-$Cl(f(A))$ *for every subset* A *of* X*;*

(3) $\tau_1\tau_2$-$Cl(f^{-1}(B)) \subseteq f^{-1}((\sigma_1, \sigma_2)\theta$-$Cl(B))$ *for every subset* B *of* Y*.*

Definition 4. [31] *A function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is called faintly* (τ_1, τ_2)*-continuous at a point* $x \in X$ *if for each* $(\sigma_1, \sigma_2)\theta$*-open set* V *of* Y *containing* $f(x)$*, there exists a* $\tau_1\tau_2$*-open set* U *of* X *containing* x *such that* $f(U) \subseteq V$*. A function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is called faintly* (τ_1, τ_2)*-continuous if* f *has this property at every point of* X*.*

Lemma 5. [31] *For a function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$*, the following properties are equivalent:*

(1) f *is faintly* (τ_1, τ_2)*-continuous;*

(2) $f^{-1}(V)$ *is* $\tau_1\tau_2$*-open in* X *for each* $(\sigma_1, \sigma_2)\theta$*-open set* V *of* Y*;*

(3) $f^{-1}(K)$ *is* $\tau_1\tau_2$*-closed in* X *for each* $(\sigma_1, \sigma_2)\theta$*-closed set* K *of* Y*.*

Recall that a bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)*-regular* [32] if for each $\tau_1\tau_2$-closed set F and each $x \notin F$, there exist disjoint $\tau_1\tau_2$-open sets U and V such that $x \in U$ and $F \subseteq V$.

Lemma 6. [33] *A bitopological space* (X, τ_1, τ_2) *is* (τ_1, τ_2)*-regular if and only if for each* $x \in X$ *and each* $\tau_1\tau_2$*-open set* U *containing* x*, there exists a* $\tau_1\tau_2$*-open set* V *such that* $x \in V \subseteq \tau_1\tau_2$-$Cl(V) \subseteq U$*.*

Lemma 7. [33] *Let* (X, τ_1, τ_2) *be a* (τ_1, τ_2)*-regular space. Then, the following properties hold:*

(1) $\tau_1\tau_2$-$Cl(A) = (\tau_1, \tau_2)\theta$-$Cl(A)$ *for every subset* A *of* X*.*

(2) Every $\tau_1\tau_2$*-open set is* $(\tau_1, \tau_2)\theta$*-open.*

Theorem 3. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, where (Y, σ_1, σ_2) is (σ_1, σ_2)-regular, the following properties are equivalent:*

(1) f is (τ_1, τ_2)-continuous;

(2) f is weakly (τ_1, τ_2)-continuous;

(3) f is faintly (τ_1, τ_2)-continuous;

(4) f is strongly $\theta(\tau_1, \tau_2)$-continuous.

Proof. (1) $\Rightarrow$ (2): The proof is obvious.

(2) $\Rightarrow$ (3): Let K be any $\theta(\sigma_1, \sigma_2)$-closed set of Y. By Lemma 4, we have

$$\tau_1\tau_2\text{-Cl}(f^{-1}(K)) \subseteq f^{-1}((\sigma_1, \sigma_2)\theta\text{-Cl}(K)) = f^{-1}(K)$$

and hence $f^{-1}(K)$ is $\tau_1\tau_2$-closed in X. Thus by Lemma 5, f is faintly (τ_1, τ_2)-continuous.

(3) $\Rightarrow$ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Since (Y, σ_1, σ_2) is (σ_1, σ_2)-regular, by Lemma 7 we have V is a $\theta(\tau_1, \tau_2)$-open set of Y. Since f is faintly (τ_1, τ_2)-continuous, by Lemma 5 we have $f^{-1}(V)$ is $\tau_1\tau_2$-open in X. Then by Lemma 3, f is (τ_1, τ_2)-continuous.

(1) $\Rightarrow$ (4): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Since (Y, σ_1, σ_2) is (σ_1, σ_2)-regular, by Lemma 6 there exists a $\sigma_1\sigma_2$-open set W of Y such that $f(x) \in W \subseteq \sigma_1\sigma_2\text{-Cl}(W) \subseteq V$. Since f is (τ_1, τ_2)-continuous, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq V$. Now, we shall show that

$$f(\tau_1\tau_2\text{-Cl}(U)) \subseteq \sigma_1\sigma_2\text{-Cl}(W).$$

Suppose that $y \notin \sigma_1\sigma_2\text{-Cl}(W)$. Then, there exists a $\sigma_1\sigma_2$-open set G of Y containing y such that $G \cap W = \emptyset$. Since f is (τ_1, τ_2)-continuous, by Lemma 3 we have $f^{-1}(G)$ is $\tau_1\tau_2$-open in X and $f^{-1}(G) \cap U = \emptyset$, which implies that $f^{-1}(G) \cap \tau_1\tau_2\text{-Cl}(U) = \emptyset$. If

$$f^{-1}(G) \cap \tau_1\tau_2\text{-Cl}(U) \neq \emptyset,$$

then $\tau_1\tau_2\text{-Int}(f^{-1}(G)) \cap \tau_1\tau_2\text{-Cl}(U) \neq \emptyset$. Let $z \in \tau_1\tau_2\text{-Int}(f^{-1}(G)) \cap \tau_1\tau_2\text{-Cl}(U)$. Then, $z \in \tau_1\tau_2\text{-Int}(f^{-1}(G))$ and $z \in \tau_1\tau_2\text{-Cl}(U)$. There exists a $\tau_1\tau_2$-open set U_0 of X containing x such that $U_0 \subseteq f^{-1}(G)$. Since $z \in \tau_1\tau_2\text{-Cl}(U)$, we have $U_0 \cap U \neq \emptyset$ and so $f^{-1}(G) \cap U \neq \emptyset$. This is a contradiction. Therefore, $f^{-1}(G) \cap \tau_1\tau_2\text{-Cl}(U) = \emptyset$ which implies that

$$G \cap f(\tau_1\tau_2\text{-Cl}(U)) = \emptyset.$$

Thus, $y \notin f(\tau_1\tau_2\text{-Cl}(U))$ and hence $f(\tau_1\tau_2\text{-Cl}(U)) \subseteq \sigma_1\sigma_2\text{-Cl}(W) \subseteq V$. This shows that f is strongly $\theta(\tau_1, \tau_2)$-continuous.

(4) $\Rightarrow$ (1): The proof is obvious.

Theorem 4. *Let (X, τ_1, τ_2) be (τ_1, τ_2)-regular. Then, a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is strongly $\theta(\tau_1, \tau_2)$-continuous if and only if f is (τ_1, τ_2)-continuous.*

Proof. We prove only the sufficiency. Suppose that f is (τ_1, τ_2)-continuous. Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Then, there exists a $\tau_1\tau_2$-open set G of X containing x such that $f(G) \subseteq V$. Since (X, τ_1, τ_2) is (τ_1, τ_2)-regular, by Lemma 6 there exists a $\tau_1\tau_2$-open set U of X such that $x \in U \subseteq \tau_1\tau_2\text{-Cl}(U) \subseteq G$. Thus, $f(\tau_1\tau_2\text{-Cl}(U)) \subseteq V$. This shows that f is strongly $\theta(\tau_1, \tau_2)$-continuous.

4. Some results on strong $\theta(\tau_1, \tau_2)$-continuity

Recall that a bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)-T_0 [34] if for any pair of distinct points in X, there exists a $\tau_1\tau_2$-open set of X containing one of the points but not the other.

Definition 5. [35] *A bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)-T_2 if for any pair of distinct points x, y in X, there exist disjoint $\tau_1\tau_2$-open sets U and V of X containing x and y, respectively.*

Theorem 5. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a strongly $\theta(\tau_1, \tau_2)$-continuous injection and (Y, σ_1, σ_2) is (σ_1, σ_2)-T_0, then (X, τ_1, τ_2) is (τ_1, τ_2)-T_2.*

Proof. Suppose that (Y, σ_1, σ_2) is (σ_1, σ_2)-T_0. Let x and y be any distinct points of X. Since f is injective, $f(x) \neq f(y)$. Since (Y, σ_1, σ_2) is (σ_1, σ_2)-T_0, there exists a $\sigma_1\sigma_2$-open set V of Y which either contains $f(x)$ and not $f(y)$ or contains $f(y)$ and not $f(x)$. If the first case holds, then there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(\tau_1\tau_2\text{-Cl}(U)) \subseteq V$. Thus, $f(y) \notin f(\tau_1\tau_2\text{-Cl}(U))$ and hence

$$y \in X - \tau_1\tau_2\text{-Cl}(U) = \tau_1\tau_2\text{-Int}(X - U).$$

Then, there exists a $\tau_1\tau_2$-open set W of X such that $y \in W \subseteq X - U$. Therefore, $U \cap W = \emptyset$. This shows that (X, τ_1, τ_2) is (τ_1, τ_2)-T_2.

Definition 6. [36] *A bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-Urysohn if for each pair of distinct points x and y in X, there exist $\tau_1\tau_2$-open sets U and V such that $x \in U$, $y \in V$ and $\tau_1\tau_2\text{-Cl}(U) \cap \tau_1\tau_2\text{-Cl}(V) = \emptyset$.*

Theorem 6. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a strongly $\theta(\tau_1, \tau_2)$-continuous injection and (Y, σ_1, σ_2) is (σ_1, σ_2)-T_2, then (X, τ_1, τ_2) is $\tau_1\tau_2$-Urysohn.*

Proof. Suppose that (Y, σ_1, σ_2) is (σ_1, σ_2)-T_2. Let x and y be any distinct points of X. Since f is injective, $f(x) \neq f(y)$. Since (Y, σ_1, σ_2) is (σ_1, σ_2)-T_2, there exist $\sigma_1\sigma_2$-open sets V and W of Y containing $f(x)$ and $f(y)$, respectively, such that $V \cap W = \emptyset$. Since f is strongly $\theta(\tau_1, \tau_2)$-continuous, there exist $\tau_1\tau_2$-open sets U and G of X containing x and y, respectively, such that $f(\tau_1\tau_2\text{-Cl}(U)) \subseteq V$ and $f(\tau_1\tau_2\text{-Cl}(G)) \subseteq W$. It follows that $\tau_1\tau_2\text{-Cl}(U) \cap \tau_1\tau_2\text{-Cl}(G) = \emptyset$. Thus, (X, τ_1, τ_2) is $\tau_1\tau_2$-Urysohn.

Definition 7. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to have a strongly $\theta(\tau_1, \tau_2)$-closed graph with respect to X if for each $(x, y) \in (X \times Y) - G(f)$, there exist a $\tau_1\tau_2$-open set U of X containing x and a $\sigma_1\sigma_2$-open set V of Y containing y such that*

$$[\tau_1\tau_2\text{-}Cl(U) \times V] \cap G(f) = \emptyset.$$

Lemma 8. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ has a strongly $\theta(\tau_1, \tau_2)$-closed graph with respect to X if and only if for each $(x, y) \in (X \times Y) - G(f)$, there exist a $\tau_1\tau_2$-open set U of X containing x and a $\sigma_1\sigma_2$-open set V of Y containing y such that*

$$f(\tau_1\tau_2\text{-}Cl(U)) \cap V = \emptyset.$$

Theorem 7. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is strongly $\theta(\tau_1, \tau_2)$-continuous and (Y, σ_1, σ_2) is (σ_1, σ_2)-T_2, then $G(f)$ is strongly $\theta(\tau_1, \tau_2)$-closed graph with respect to X.*

Proof. Let $(x, y) \in (X \times Y) - G(f)$. Then, $y \neq f(x)$. Since (Y, σ_1, σ_2) is (σ_1, σ_2)-T_2, there exist $\sigma_1\sigma_2$-open sets V and W of Y containing $f(x)$ and $f(y)$, respectively, such that $V \cap W = \emptyset$. Since f is strongly $\theta(\tau_1, \tau_2)$-continuous, there exist a $\tau_1\tau_2$-open set U of X containing x such that $f(\tau_1\tau_2\text{-Cl}(U)) \subseteq W$. This implies that $f(\tau_1\tau_2\text{-Cl}(U)) \cap V = \emptyset$ and by Lemma 8, $G(f)$ is strongly $\theta(\tau_1, \tau_2)$-closed graph with respect to X.

Definition 8. [37] *Let A be a subset of a bitopological space (X, τ_1, τ_2). The $(\tau_1, \tau_2)\theta$-frontier of A, $(\tau_1, \tau_2)\theta$-fr(A), is defined by*

$$(\tau_1, \tau_2)\theta\text{-}fr(A) = (\tau_1, \tau_2)\theta\text{-}Cl(A) \cap (\tau_1, \tau_2)\theta\text{-}Cl(X - A).$$

Theorem 8. *The set of all points $x \in X$ at which a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is not strongly $\theta(\tau_1, \tau_2)$-continuous is identical with the union of the $(\tau_1, \tau_2)\theta$-frontier of the inverse images of $\sigma_1\sigma_2$-open sets containing $f(x)$.*

Proof. Suppose that f is not strongly $\theta(\tau_1, \tau_2)$-continuous. Then, there exists a $\sigma_1\sigma_2$-open set V of Y containing $f(x)$ such that $f(\tau_1\tau_2\text{-Cl}(U))$ is not contained in V for every $\tau_1\tau_2$-open set U of X containing x. Then, $\tau_1\tau_2\text{-Cl}(U) \cap (X - f^{-1}(V)) \neq \emptyset$ for every $\tau_1\tau_2$-open set U of X containing x. Thus, $x \in (\tau_1, \tau_2)\theta\text{-Cl}(X - f^{-1}(V))$. On the other hand, we have $x \in f^{-1}(V) \subseteq (\tau_1, \tau_2)\theta\text{-Cl}(f^{-1}(V))$ and hence $x \in (\tau_1, \tau_2)\theta\text{-fr}(f^{-1}(V))$.

Conversely, suppose that f is strongly $\theta(\tau_1, \tau_2)$-continuous at $x \in X$. Let V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. By Theorem 1, $x \in (\tau_1, \tau_2)\theta\text{-Int}(f^{-1}(V))$. Thus, $x \notin (\tau_1, \tau_2)\theta\text{-fr}(f^{-1}(V))$ for every $\sigma_1\sigma_2$-open set V of Y containing $f(x)$. This completes the proof.

Recall that a bitopological space (X, τ_1, τ_2) is said to be *quasi* (τ_1, τ_2)-$\mathscr{H}$*-closed* [38] if every $\tau_1\tau_2$-open cover $\{U_\gamma \mid \gamma \in \Gamma\}$, there exists a finite subset Γ_0 of Γ such that $X = \cup\{\tau_1\tau_2\text{-Cl}(U_\gamma) \mid \gamma \in \Gamma_0\}$. A subset K of a bitopological space (X, τ_1, τ_2) is said to be *quasi* (τ_1, τ_2)-$\mathscr{H}$*-closed relative to* (X, τ_1, τ_2) if for any cover $\{V_\gamma \mid \gamma \in \Gamma\}$ by $\tau_1\tau_2$-open sets of X, there exists a finite subset Γ_0 of Γ such that $K \subseteq \cup\{\tau_1\tau_2\text{-Cl}(V_\gamma) \mid \gamma \in \Gamma_0\}$. A subset K of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$*-compact relative to* (X, τ_1, τ_2) if for any cover $\{V_\gamma \mid \gamma \in \Gamma\}$ by $\tau_1\tau_2$-open sets of X, there exists a finite subset Γ_0 of Γ such that $K \subseteq \cup\{V_\gamma \mid \gamma \in \Gamma_0\}$. If X is $\tau_1\tau_2$-compact relative to (X, τ_1, τ_2), then (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-*compact* [27].

Theorem 9. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is strongly $\theta(\tau_1, \tau_2)$-continuous and K is quasi (τ_1, τ_2)-$\mathscr{H}$-closed relative to (X, τ_1, τ_2), then $f(K)$ is $\sigma_1\sigma_2$-compact relative to (Y, σ_1, σ_2).*

Proof. Let K be quasi (τ_1, τ_2)-$\mathscr{H}$-closed relative to (X, τ_1, τ_2). Let $\{V_\gamma \mid \gamma \in \Gamma\}$ be any cover of $f(K)$ by $\sigma_1\sigma_2$-open sets of Y. For each $x \in K$, there exists $\gamma(x) \in \Gamma$ such that $f(x) \in V_{\gamma(x)}$. Since f is strongly $\theta(\tau_1, \tau_2)$-continuous, there exists a $\tau_1\tau_2$-open set $U(x)$ of X containing x such that $f(\tau_1\tau_2\text{-Cl}(U(x))) \subseteq \sigma_1\sigma_2\text{-Cl}(V_{\gamma(x)})$. The family $\{U(x) \mid x \in K\}$ is a cover of K by $\tau_1\tau_2$-open sets of X. Since K is quasi (τ_1, τ_2)-$\mathscr{H}$-closed relative to (X, τ_1, τ_2), there exists a finite number of points, say, $x_1, x_2, x_3, ..., x_n$ in K such that $K \subseteq \cup\{\tau_1\tau_2\text{-Cl}(U(x_k)) \mid x_k \in K; 1 \leq k \leq n\}$. Thus,

$$\begin{aligned} f(K) &\subseteq \cup\{f(\tau_1\tau_2\text{-Cl}(U(x_k))) \mid x_k \in K; 1 \leq k \leq n\} \\ &\subseteq \cup\{V_{\gamma(x_k)} \mid x_k \in K; 1 \leq k \leq n\}. \end{aligned}$$

This shows that $f(K)$ is $\sigma_1\sigma_2$-compact relative to (Y, σ_1, σ_2).

Acknowledgements

This research project was financially supported by Mahasarakham University.

References

[1] S. Fomin. Extensions of topological spaces. *Doklady Akademii Nauk SSSR*, 32:114–116, 1941.

[2] T. Noiri. Properties of θ-continuous functions. *Atti della Accademia Nazionale dei Lincei, Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Series (8)*, 58:887–891, 1975.

[3] V. Popa. Characterizations of θ-continuous functions. *Studii şi Cercetări Ştiinţifice. Seria Matematică*, 32:113–119, 1980.

[4] S. P. Arya and M. P. Bhamini. Some weaker forms of semi-continuous functions. *Ganita*, 33:124–134, 1982.

[5] S. Jafari and T. Noiri. Properties of θ-semi-continuous functions. *Journal of Institute of Mathematics and Computer Sciences, Mathematics Series*, 13:123–128, 2000.

[6] T. Noiri. On θ-precontinuous functions. *International Journal of Mathematics and Mathematical Sciences*, 28:285–292, 2001.

[7] P. E. Long and L. L. Herrington. Strongly θ-continuous functions. *Journal of the Korean Mathematical Society*, 18:21–28, 1981.

[8] S. Jafari and T. Noiri. Strongly θ-semi-continuous functions. *Indian Journal of Pure and Applied Mathematics*, 29:1195–1201, 1998.

[9] T. Noiri. Strongly θ-precontinuous functions. *Acta Mathematica Hungarica*, 90(4):307–316, 2001.

[10] T. Noiri and V. Popa. Strongly θ-β-continuous functions. *Journal of Pure Mathematics*, 19:31–39, 2002.

[11] G. Di Maio and T. Noiri. Weak and strong forms of irresolute functions. *Rendiconti del Circolo Matematico di Palermo (2), Supplemento*, 18:255–273, 1988.

[12] M. C. Pal and P. Bhattacharyya. Feeble and strong forms of preirresolute functions. *Bulletin of the Malaysian Mathematical Sciences Society*, 19:63–75, 1996.

[13] T. Noiri. Weak and strong forms of β-irresolute functions. *Acta Mathematica Hungarica*, 99(4):315–328, 2003.

[14] S. Jafari and T. Noiri. Strongly sober θ-continuous functions. *Journal of Pure Mathematics*, 16:9–17, 1999.

[15] T. Noiri and V. Popa. A unified theory for strongly θ- continuity for functions. *Acta Mathematica Hungarica*, 106(3):167–186, 2005.

[16] M. Thongmoon and C. Boonpok. Strongly $\theta(\Lambda, p)$-continuous functions. *International Journal of Mathematics and Computer Science*, 19(2):475–479, 2024.

[17] J. Khampakdee and C. Boonpok. Almost strong $\theta(\Lambda, p)$-continuity for functions. *European Journal of Pure and Applied Mathematics*, 17(1):300–309, 2024.

[18] C. Boonpok and N. Srisarakham. (τ_1, τ_2)-continuity for functions. *Asia Pacific Journal of Mathematics*, 11:21, 2024.

[19] C. Boonpok and P. Pue-on. Characterizations of almost (τ_1, τ_2)-continuous functions. *International Journal of Analysis and Applications*, 22:33, 2024.

[20] C. Boonpok and C. Klanarong. On weakly (τ_1, τ_2)-continuous functions. *European Journal of Pure and Applied Mathematics*, 17(1):416–425, 2024.

[21] J. Khampakdee, S. Sompong, and C. Boonpok. Almost weakly (τ_1, τ_2)-continuous functions. *European Journal of Pure and Applied Mathematics*, 18(1):5721, 2025.

[22] N. Srisarakham, A. Sama-Ae, and C. Boonpok. Characterizations of faintly (τ_1, τ_2)-continuous functions. *European Journal of Pure and Applied Mathematics*, 17(4):2753–2762, 2024.

[23] M. Chiangpradit, S. Sompong, and C. Boonpok. Weakly quasi (τ_1, τ_2)-continuous functions. *International Journal of Analysis and Applications*, 22:125, 2024.

[24] B. Kong-ied, S. Sompong, and C. Boonpok. Almost quasi (τ_1, τ_2)-continuous functions. *Asia Pacific Journal of Mathematics*, 11:64, 2024.

[25] C. Prachanpol, C. Boonpok, and C. Viriyapong. $\delta(\tau_1, \tau_2)$-continuous functions. *European Journal of Pure and Applied Mathematics*, 17(4):3730–3742, 2024.

[26] N. Srisarakham, S. Sompong, and C. Boonpok. Quasi $\theta(\tau_1, \tau_2)$-continuous functions. *European Journal of Pure and Applied Mathematics*, 18(1):5722, 2025.

[27] C. Boonpok, C. Viriyapong, and M. Thongmoon. On upper and lower (τ_1, τ_2)-precontinuous multifunctions. *Journal of Mathematics and Computer Science*, 18:282–293, 2018.

[28] N. Viriyapong and C. Boonpok. $(\tau_1, \tau_2)\alpha$-continuity for multifunctions. *Journal of Mathematics*, 2020:6285763, 2020.

[29] C. Boonpok. $(\tau_1, \tau_2)\delta$-semicontinuous multifunctions. *Heliyon*, 6:e05367, 2020.

[30] N. Viriyapong, S. Sompong, and C. Boonpok. (τ_1, τ_2)-extremal disconnectedness in bitopological spaces. *International Journal of Mathematics and Computer Science*, 19(3):855–860, 2024.

[31] P. Pue-on, S. Sompong, and C. Boonpok. Upper and lower faint (τ_1, τ_2)-continuity. *International Journal of Analysis and Applications*, 22:169, 2024.

[32] M. Chiangpradit, S. Sompong, and C. Boonpok. On characterizations of (τ_1, τ_2)-regular spaces. *International Journal of Mathematics and Computer Science*, 19(4):1229–1334, 2024.

[33] C. Klanarong, S. Sompong, and C. Boonpok. (τ_1, τ_2)-continuity and $(\tau_1, \tau_2)\theta$-closed sets. *International Journal of Mathematics and Computer Science*, 19(4):1299–1304, 2024.

[34] M. Chiangpradit, S. Sompong, and C. Boonpok. $\Lambda_{(\tau_1,\tau_2)}$-sets and related topological spaces. *Asia Pacific Journal of Mathematics*, 11:49, 2024.

[35] N. Chutiman, S. Sompong, and C. Boonpok. On some separation axioms in bitopological spaces. *Asia Pacific Journal of Mathematics*, 11:41, 2024.

[36] P. Pue-on, A. Sama-Ae, and C. Boonpok. Characterizations of quasi $\theta(\tau_1, \tau_2)$-continuous multifunctions. *International Journal of Analysis and Applications*, 23:59, 2025.

[37] M. Thongmoon, S. Sompong, and C. Boonpok. $\theta(\tau_1, \tau_2)$-continuity for functions. (submitted).

[38] M. Thongmoon, S. Sompong, and C. Boonpok. Upper and lower weak (τ_1, τ_2)-continuity. *European Journal of Pure and Applied Mathematics*, 17(3):1705–1716, 2024.