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Soft Subalgebras and Ideals of Sheffer Stroke Hilbert Algebras based on $\mathcal{N}$-Structures

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Abstract. In this study, we introduce the concepts of $\mathcal{N}$-ideals of types $(\in, \in)$ and $(\in, \in \vee \mathfrak{q})$, along with soft $\mathcal{N}_{\in}$-sets, soft $\mathcal{N}_{\mathfrak{q}}$-sets, and soft $\mathcal{N}_{\in\vee\mathfrak{q}}$-sets. Additionally, we define soft $\mathcal{N}$-subalgebras and $\mathcal{N}$-ideals within the context of Sheffer stroke Hilbert algebras and explore various properties of these structures. The paper also provides characterizations of $\mathcal{N}$-subalgebras of types $(\in, \in)$ and $(\in, \in \vee \mathfrak{q})$, as well as $\mathcal{N}$-ideals for both of these types. Furthermore, we further examine their corresponding soft versions, extending the algebraic framework in the setting of Sheffer stroke Hilbert algebras.

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Key Words and Phrases: Sheffer stroke Hilbert algebra, $\mathcal{N}$-subalgebra of types $(\in, \in)$ and $(\in, \in \vee \mathfrak{q})$, $\mathcal{N}$-ideal of types $(\in, \in)$ and $(\in, \in \vee \mathfrak{q})$.

1. Introduction

The Sheffer operation, also known as the Sheffer stroke or NAND operator, was first introduced by Henry Maurice Sheffer [1]. Its significance arises from its ability to function as a fundamental logical operator capable of constructing an entire logical system on its own. This feature allows any axiom within a logical framework to be reformulated solely using the Sheffer operation, simplifying the manipulation and control of the system's intrinsic properties. Moreover, the axioms of Boolean algebra, which serve as the algebraic foundation for classical propositional logic, can also be completely expressed through the

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Sheffer operation. This highlights the Sheffer operation's pivotal role and versatility in both logical and algebraic settings. The Sheffer stroke has been utilized in various algebraic structures, including Boolean algebra, MV-algebra, BL-algebra, BCK-algebra, BE-algebra, ortholattices, and Hilbert algebra, among others (refer to [2–8]).

Sheffer stroke Hilbert algebras are crucial structures in algebraic logic, providing an essential framework for modeling logical systems. Defined through the Sheffer stroke, these algebras offer a means to represent negation and other key logical operations. As such, they are indispensable in the study of algebraic logic, particularly in areas like Boolean algebra and lattice theory. Recent studies have further demonstrated their versatility—for instance, Rajesh et al. [9] investigated the notions of length and mean fuzzy ideals within this algebraic framework, revealing new insights into their structural depth and fuzzy generalizations. This line of research continues to highlight the richness of Sheffer stroke Hilbert algebras in both crisp and fuzzy logical environments.

In 1999, Molodtsov [10] introduced the soft set theory, offering a novel mathematical tool designed to address uncertainties, free from the limitations of traditional theoretical models.

Soft sets have recently emerged as a promising approach to modeling imprecision and flexibility in mathematical frameworks. They enable the handling of uncertainty within algebraic structures, allowing for a more adaptable study of these systems under conditions that are not precisely defined. Building on soft set theory, soft subalgebras extend these concepts into classical algebraic systems, providing a broader understanding of their properties and behaviors.

This paper explores new ideas within Sheffer stroke Hilbert algebras, such as soft $\mathcal{N}$-sets, soft $\mathcal{N}$-subalgebras, and soft $\mathcal{N}$-ideals, with a focus on types $(\in, \in)$ and $(\in, \in \vee \mathfrak{q})$. These concepts are analyzed through the lens of $\mathcal{N}$-structures, a generalized framework that investigates algebraic systems in more flexible conditions. The study examines the features of $\mathcal{N}$-subalgebras and $\mathcal{N}$-ideals, along with their soft variants, to provide a deeper understanding of their behavior in various contexts.

The main objective of this research is to study soft $\mathcal{N}$-subalgebras and $\mathcal{N}$-ideals within Sheffer stroke Hilbert algebras. Additionally, the paper aims to investigate the interconnections between soft $\mathcal{N}$-sets and $\mathcal{N}$-structures, contributing to the broader field of algebraic logic. Through this analysis, we seek to further our comprehension of the role that Sheffer stroke Hilbert algebras play in both algebraic and logical systems.

2. Preliminaries

In this section, we revisit essential concepts and results concerning Sheffer stroke Hilbert algebras, which will be referenced in the subsequent sections.

Definition 1. *[1] Let $\mathcal{A} := (A, |)$ be a groupoid. Then the operation $|$ is said to be a Sheffer stroke or a Sheffer operation if it satisfies:*

(s1) $(\forall \xi, \zeta \in A)\ (\xi \mid \zeta = \zeta \mid \xi)$, (s2) $(\forall \xi, \zeta \in A)\ ((\xi \mid \xi) \mid (\xi \mid \zeta) = \xi)$,

(s3) $(\forall \xi, \zeta, \tau \in A)\ (\xi \mid ((\zeta \mid \tau) \mid (\zeta \mid \tau)) = ((\xi \mid \zeta) \mid (\xi \mid \zeta)) \mid \tau)$,

(s4) $(\forall \xi, \zeta, \tau \in A)\ ((\xi \mid ((\xi \mid \xi) \mid (\zeta \mid \zeta))) \mid (\xi \mid ((\xi \mid \xi) \mid (\zeta \mid \zeta))) = \xi)$.

To improve the clarity of this manuscript, we introduce the following notation, which will be used consistently throughout the text:

$$\xi \mid (\zeta \mid \zeta) := \xi^{\zeta}.$$

for all elements $\xi, \zeta \in A$.

Definition 2. *[11] A Sheffer stroke Hilbert algebra (SSH-algebra) is a structure $\mathcal{H}_S := (H, |, 0)$ of type $(2, 0)$, in which H is a nonempty set, $|$ is a Sheffer stroke on H, and 0 is the fixed element in H satisfying specific conditions:*

(sH1) $(\forall \xi, \zeta, \tau \in H)\ ((\xi \mid (\zeta^{\tau} \mid \zeta^{\tau})) \mid ((\xi^{\zeta} \mid (\xi^{\tau} \mid \xi^{\tau})) \mid (\xi^{\zeta} \mid (\xi^{\tau} \mid \xi^{\tau}))) = \xi^{\xi})$,

(sH2) $(\forall \xi, \zeta \in H)\ (\xi^{\zeta} = \zeta^{\xi} = \xi^{\xi} \Rightarrow \xi = \zeta)$.

Proposition 1. *[11] Let $\mathcal{H}_S := (H, |, 0)$ be an SSH-algebra. Then the binary relation*

$$(\forall \xi, \zeta \in H)\, (\xi \leq \zeta \Leftrightarrow \xi^{\zeta} = 0)$$

is a partial order on H.

Definition 3. *[11] Let $\mathcal{H}_S := (H, |, 0)$ be an SSH-algebra. A nonempty subset S of H is said to be a subalgebra of H if $\xi^{\zeta} \mid \xi^{\zeta} \in S$ for all $\xi, \zeta \in S$.*

Definition 4. *[11] Let $\mathcal{H}_S := (H, |, 0)$ be an SSH-algebra. A nonempty subset I of H is called an ideal of H if*

(1) $0 \in I$,

(2) $(\forall \xi, \zeta \in H)\ (\xi^{\zeta} \mid \xi^{\zeta} \in I$ *and* $\zeta \in I \Rightarrow \xi \in I)$.

3. Soft subalgebras/ideals of Sheffer stroke Hilbert algebras

Let $\mathcal{H}_S := (H, |, 0)$ represent an SSH-algebra. For a subset Υ of $[-1, 0]$, the pair $(\mathcal{S}, \Upsilon)$ is referred to as a soft $\mathcal{N}$-set over H, where $\mathcal{S}$ is a mapping from Υ to $P(H)$, i.e., $\mathcal{S} : \Upsilon \to P(H)$. Given an $\mathcal{N}$-structure $(H, \mathfrak{f})$ and $\Upsilon = [-1, 0]$, we introduce two mappings:

$$\mathcal{S}_{\in} : \Upsilon \to P(H); \quad \eta \mapsto \{\xi \in H : (H, \xi_{\eta}) \text{ is an } \mathcal{N}_{\in}\text{-subset of } (H, \mathfrak{f})\}$$

and

$$\mathcal{S}_{\mathsf{q}} : \Delta \to P(H); \quad \eta \mapsto \{\xi \in H : (H, \xi_{\eta}) \text{ is an } \mathcal{N}_{\mathsf{q}}\text{-subset of } (H, \mathfrak{f})\}.$$

Then, the pairs $(\mathcal{S}_{\in}, \Upsilon)$ and $(\mathcal{S}_{\mathsf{q}}, \Upsilon)$ are soft $\mathcal{N}$-sets over H. If $\mathcal{S}_{\in}(\eta) \neq \emptyset$ (or $\mathcal{S}\mathsf{q}(\eta) \neq \emptyset$) for some $\eta \in \Upsilon$, we say that $(\mathcal{S}_{\in}, \Upsilon)$ (or $(\mathcal{S}\mathsf{q}, \Upsilon)$) forms a soft $\mathcal{N}_{\in}$-set (or soft $\mathcal{N}\mathsf{q}$-set) over H. A soft $\mathcal{N}\!\in \vee\mathsf{q}$-set over H is defined as the union of a soft $\mathcal{N}_{\in}$-set and a soft $\mathcal{N}\mathsf{q}$-set over H, and is denoted by $(\mathcal{S}_{\in\vee\mathsf{q}}, \Upsilon)$, where

$$\mathcal{S}_{\in\vee\mathsf{q}}(\eta) = \mathcal{S}_{\in}(\eta) \vee \mathcal{S}_{\mathsf{q}}(\eta) \text{ for all } \eta \in \Upsilon.$$

Definition 5. *Let $\mathcal{H}_S := (H, |, 0)$ be an SSH-algebra. A soft $\mathcal{N}$-set $(\mathcal{S}, \Upsilon)$ over H is called a soft $\mathcal{N}$-subalgebra over H if*

$$(\forall \eta \in \Upsilon)\,(\mathcal{S}(\eta) \neq \emptyset \Rightarrow \mathcal{S}(\eta) \text{ is a subalgebra of } H). \tag{1}$$

Theorem 1. *Let $\mathcal{H}_S := (H, |, 0)$ be an SSH-algebra. Given an $\mathcal{N}$-structure $(H, \mathfrak{f})$ and $\Upsilon = [-1, 0)$, the soft $\mathcal{N}_{\in}$-set $(\mathcal{S}_{\in}, \Upsilon)$ is a soft $\mathcal{N}$-subalgebra over H if and only if $(H, \mathfrak{f})$ is an $\mathcal{N}$-subalgebra of type $(\in, \in)$.*

Proof. Assume that $(\mathcal{S}_{\in}, \Upsilon)$ is a soft $\mathcal{N}$-subalgebra over H. If $(H, \mathfrak{f})$ is not an $\mathcal{N}$-subalgebra of type $(\in, \in)$, then there exist $\varrho, \varpi \in H$ and $\mathfrak{t} \in \Upsilon$ such that $(H, \varrho_{\mathfrak{t}})$ and $(H, \varpi_{\mathfrak{t}})$ are $\mathcal{N}_{\in}$-subsets of $(H, \mathfrak{f})$, but $(H, (\varrho^{\varpi} \mid \varrho^{\varpi})_{\mathfrak{t}})$ is not an $\mathcal{N}_{\in}$-subset of $(H, \mathfrak{f})$. Therefore, $\varrho, \varpi \in \mathcal{S}_{\in}(\mathfrak{t})$ and $\varrho^{\varpi} \mid \varrho^{\varpi} \notin \mathcal{S}_{\in}(\mathfrak{t})$, implying that $\mathcal{S}_{\in}(\mathfrak{t})$ is not a subalgebra of H. This leads to a contradiction, which means that $(H, \mathfrak{f})$ must be an $\mathcal{N}$-subalgebra of type $(\in, \in)$.

Conversely, suppose $(H, \mathfrak{f})$ is an $\mathcal{N}$-subalgebra of type $(\in, \in)$. Let $\eta \in \Upsilon$ be such that $\mathcal{S}_{\in}(\eta) \neq \emptyset$. If $\xi, \zeta \in \mathcal{S}_{\in}(\eta)$, then both (H, ξ_η) and (H, ζ_η) are $\mathcal{N}_{\in}$-subsets of $(H, \mathfrak{f})$. Consequently, $(H, (\xi^\zeta \mid \xi^\zeta)_\eta) = (H, (\xi^\zeta \mid \xi^\zeta))_{\bigvee\{\eta,\eta\}})$ is an $\mathcal{N}_{\in}$-subset of $(H, \mathfrak{f})$. This implies that $(\xi^\zeta \mid \xi^\zeta) \in \mathcal{S}_{\in}(\eta)$, showing that $\mathcal{S}_{\in}(\eta)$ is a subalgebra of H for all $\eta \in \Upsilon$. Hence, $(\mathcal{S}_{\in}, \Upsilon)$ is a soft $\mathcal{N}$-subalgebra over H.

Lemma 1. *Let $\mathcal{H}_S := (H, |, 0)$ be an SSH-algebra. An $\mathcal{N}$-structure $(H, \mathfrak{f})$ is an $\mathcal{N}$-subalgebra of type $(\in, \in)$ if and only if*

$$(\forall \xi, \zeta \in H)\,(\mathfrak{f}(\xi^\zeta \mid \xi^\zeta) \leq \bigvee\{\mathfrak{f}(\xi), \mathfrak{f}(\zeta)\}). \tag{2}$$

Proof. Suppose that $(H, \mathfrak{f})$ is an $\mathcal{N}$-subalgebra of type $(\in, \in)$. Then for all $\xi, \zeta \in H$ and for all $\eta \in \Upsilon$, if $\mathfrak{f}(\xi) \leq \eta$ and $\mathfrak{f}(\zeta) \leq \eta$, then $\mathfrak{f}(\xi^\zeta \mid \xi^\zeta) \leq \eta$. This implies that

$$\mathfrak{f}(\xi^\zeta \mid \xi^\zeta) \leq \bigvee\{\mathfrak{f}(\xi), \mathfrak{f}(\zeta)\}.$$

Conversely, suppose that the inequality holds for all $\xi, \zeta \in H$. Let $\eta \in \Upsilon$, and suppose that $\mathfrak{f}(\xi) \leq \eta$ and $\mathfrak{f}(\zeta) \leq \eta$. Then,

$$\bigvee\{\mathfrak{f}(\xi), \mathfrak{f}(\zeta)\} \leq \eta \Rightarrow \mathfrak{f}(\xi^\zeta \mid \xi^\zeta) \leq \eta.$$

Thus, $(H, \mathfrak{f})$ is closed under the Sheffer operation for type $(\in, \in)$, and hence is an $\mathcal{N}$-subalgebra of type $(\in, \in)$.

Theorem 2. *Let $\mathcal{H}_S := (H, |, 0)$ be an SSH-algebra. Given an $\mathcal{N}$-structure $(H, \mathfrak{f})$ and $\Upsilon = [-1, 0)$, the soft $\mathcal{N}_{\mathfrak{q}}$-set $(\mathcal{S}_{\mathfrak{q}}, \Upsilon)$ is a soft $\mathcal{N}$-subalgebra over H if and only if $(H, \mathfrak{f})$ is an $\mathcal{N}$-subalgebra of type $(\in, \in)$.*

Proof. Assume that $(H, \mathfrak{f})$ is an $\mathcal{N}$-subalgebra of type $(\in, \in)$ and let $\eta \in \Upsilon$ be such that $\mathcal{S}_{\mathfrak{q}}(\eta) \neq \emptyset$. If $\xi, \zeta \in \mathcal{S}_{\mathfrak{q}}(\eta)$, then (H, ξ_η) and (H, ζ_η) are $\mathcal{N}_{\mathfrak{q}}$-subsets of $(H, \mathfrak{f})$, and so $\mathfrak{f}(\xi) + \eta + 1 < 0$ and $\mathfrak{f}(\zeta) + \eta + 1 < 0$. It follows from (2) that

$$\mathfrak{f}(\xi^\zeta \mid \xi^\zeta) + \eta + 1 \leq \bigvee\{\mathfrak{f}(\xi), \mathfrak{f}(\zeta)\} + \eta + 1 < 0,$$

and so $(H,(\xi^\zeta \mid \xi^\zeta)_\eta) = (H,(\xi^\zeta \mid \xi^\zeta)_{\bigvee\{\eta,\eta\}})$ is an $\mathcal{N}_{\mathsf{q}}$-subset of $(H,\mathfrak{f})$. Hence, $\xi^\zeta \mid \xi^\zeta \in \mathcal{S}_{\mathsf{q}}(\eta)$, and thus $\mathcal{S}_{\mathsf{q}}(\eta)$ is a subalgebra of H for all $\eta \in \Upsilon$ with $\mathcal{S}_{\mathsf{q}}(\eta) \neq \emptyset$. Therefore, $(\mathcal{S}_{\mathsf{q}}, \Upsilon)$ is a soft $\mathcal{N}$-subalgebra over H.

Conversely, suppose that the soft $\mathcal{N}_{\mathsf{q}}$-set $(\mathcal{S}_{\mathsf{q}}, \Upsilon)$ is a soft $\mathcal{N}$-subalgebra over H, and assume that $\mathfrak{f}(\varrho^\varpi \mid \varrho^\varpi) > \bigvee\{\mathfrak{f}(\varrho), \mathfrak{f}(\varpi)\}$ for some $\varrho, \varpi \in H$. Then there exists $\mathfrak{t} \in \Upsilon$ such that

$$\mathfrak{f}(\varrho^\varpi \mid \varrho^\varpi) + \mathfrak{t} + 1 < 0 \text{ and } \bigvee\{\mathfrak{f}(\varrho), \mathfrak{f}(\varpi)\} + \mathfrak{t} + 1 < 0.$$

It follows that $(H, \varrho_{\mathfrak{t}})$ and $(H, \varpi_{\mathfrak{t}})$ are $\mathcal{N}_{\mathsf{q}}$-subsets of $(H,\mathfrak{f})$ but $(H,(\varrho^\varpi \mid \varrho^\varpi)_{\mathfrak{t}})$ is not an $\mathcal{N}_{\mathsf{q}}$-subset of $(H,\mathfrak{f})$. This is a contradiction, and hence

$$(\forall \xi, \zeta \in H)\,(\mathfrak{f}(\xi^\zeta \mid \xi^\zeta) \leq \bigvee\{\mathfrak{f}(\xi), \mathfrak{f}(\zeta)\}).$$

Therefore, $(H,\mathfrak{f})$ is an $\mathcal{N}$-subalgebra of type $(\in, \in)$ by Lemma 1.

Theorem 3. *Let $\mathcal{H}_S := (H, |, 0)$ be an SSH-algebra. Given an $\mathcal{N}$-structure $(H,\mathfrak{f})$ and the soft $\mathcal{N}_\in$-set $(\mathcal{S}_\in, \Upsilon)$ with $\Upsilon = [-1, -0.5)$, the following are equivalent:*

(1) $(\mathcal{S}_\in, \Upsilon)$ *is a soft $\mathcal{N}$-subalgebra over H.*

(2) $(\forall \xi, \zeta \in H)\,(\bigwedge\{\mathfrak{f}(\varrho^\varpi \mid \varrho^\varpi), -0.5\} \leq \bigvee\{\mathfrak{f}(\xi), \mathfrak{f}(\zeta)\})$.

Proof. Assume that the soft $\mathcal{N}_\in$-set $(\mathcal{S}_\in, \Upsilon)$ is a soft $\mathcal{N}$-subalgebra over H. Then $\mathcal{S}_\in(\eta)$ is a subalgebra of H for all $\eta \in \Upsilon$ with $\mathcal{S}_\in(\eta) \neq \emptyset$. If there exist $\varrho, \varpi \in H$ such that

$$\bigwedge\{\mathfrak{f}(\varrho^\varpi \mid \varrho^\varpi), -0.5\} > \mathfrak{t} = \bigvee\{\mathfrak{f}(\varrho), \mathfrak{f}(\varpi)\},$$

then $\mathfrak{t} \in \Upsilon$ and $(H, \varrho_{\mathfrak{t}})$ and $(H, \varpi_{\mathfrak{t}})$ are $\mathcal{N}_\in$-subsets of $(H,\mathfrak{f})$, that is, $\varrho, \varpi \in \mathcal{S}_\in(\mathfrak{t})$, but $(H,(\varrho^\varpi \mid \varrho^\varpi)_{\mathfrak{t}})$ is not an $\mathcal{N}_\in$-subset of $(H,\mathfrak{f})$, that is, $\varrho^\varpi \mid \varrho^\varpi \notin \mathcal{S}_\in(\mathfrak{t})$, a contradiction. Thus,

$$\bigwedge\{\mathfrak{f}(\xi^\zeta \mid \xi^\zeta), -0.5\} \leq \bigvee\{\mathfrak{f}(\xi), \mathfrak{f}(\zeta)\}$$

for all $\xi, \zeta \in h$.

Conversely, suppose that (2) is valid. Let $\xi, \zeta \in \mathcal{S}_\in(\eta)$ for every $\eta \in \Upsilon$. Then (H, ξ_η) and (H, ζ_η) are $\mathcal{N}_\in$-subsets of $(H,\mathfrak{f})$, and so

$$\bigwedge\{\mathfrak{f}(\xi^\zeta \mid \xi^\zeta), -0.5\} \leq \bigvee\{\mathfrak{f}(\xi), \mathfrak{f}(\zeta)\} \leq \eta < -0.5.$$

It follows that $(H,(\xi^\zeta \mid \xi^\zeta)_\eta)$ is an $\mathcal{N}_\in$-subset of $(H,\mathfrak{f})$, that is, $\xi^\zeta \mid \xi^\zeta \in \mathcal{S}_\in(\eta)$. Thus, $\mathcal{S}_\in(\eta)$ is a subalgebra of H, and therefore, $(\mathcal{S}_\in, \Upsilon)$ is a soft $\mathcal{N}$-subalgebra over H.

Lemma 2. *Let $\mathcal{H}_S := (H, |, 0)$ be an SSH-algebra. An $\mathcal{N}$-structure $(H,\mathfrak{f})$ is an $\mathcal{N}$-subalgebra of type $(\in, \in \vee \mathsf{q})$ if and only if*

$$(\forall \xi, \zeta \in H)\,(\mathfrak{f}(\xi^\zeta \mid \xi^\zeta) \leq \bigvee\{\mathfrak{f}(\xi), \mathfrak{f}(\zeta), -0.5\}). \tag{3}$$

Proof. Suppose that $(H, \mathfrak{f})$ is an $\mathcal{N}$-subalgebra of type $(\in, \in \vee \mathfrak{q})$. Let $\xi, \zeta \in H$. Then, if both $\mathfrak{f}(\xi) \leq \eta$ and $\mathfrak{f}(\zeta) \leq \eta$ for some $\eta \in \Upsilon$, it must follow that

$$\mathfrak{f}(\xi^\zeta \mid \xi^\zeta) \leq \bigvee\{\mathfrak{f}(\xi), \mathfrak{f}(\zeta), -0.5\}.$$

Conversely, suppose the above inequality holds for all $\xi, \zeta \in H$. Let $\eta \in \Upsilon$, and assume that $\mathfrak{f}(\xi) \leq \eta$ and $\mathfrak{f}(\zeta) \leq \eta$. Then,

$$\bigvee\{\mathfrak{f}(\xi), \mathfrak{f}(\zeta), -0.5\} \leq \eta \Rightarrow \mathfrak{f}(\xi^\zeta \mid \xi^\zeta) \leq \eta.$$

This confirms that $(H, \mathfrak{f})$ is an $\mathcal{N}$-subalgebra of type $(\in, \in \vee \mathfrak{q})$.

Theorem 4. *Let $\mathcal{H}_S := (H, |, 0)$ be an SSH-algebra. Given an $\mathcal{N}$-structure $(H, \mathfrak{f})$ and a soft $\mathcal{N}_\in$-set $(\mathcal{S}_\in, \Upsilon)$, the following assertions are equivalent:*

(1) *$(H, \mathfrak{f})$ is an $\mathcal{N}$-subalgebra of type $(\in, \in \vee \mathfrak{q})$.*

(2) *$(\mathcal{S}_\in, \Upsilon)$ is a soft $\mathcal{N}$-subalgebra over H for $\Upsilon = [-0.5, 0)$.*

Proof. Assume that $(H, \mathfrak{f})$ is an $\mathcal{N}$-subalgebra of type $(\in, \in \vee \mathfrak{q})$. Let $\xi, \zeta \in H$ and $\eta \in \Upsilon$ be such that $\xi, \zeta \in \mathcal{S}_\in(\eta)$. Then (H, ξ_η) and (H, ζ_η) are $\mathcal{N}_\in$-subsets of $(H, \mathfrak{f})$. It follows from (3) that

$$(\forall \xi, \zeta \in H)\,(\mathfrak{f}(\xi^\zeta \mid \xi^\zeta) \leq \bigvee\{\mathfrak{f}(\xi), \mathfrak{f}(\zeta), -0.5\} \leq \bigvee\{\eta, -0.5\} = \eta).$$

Then $(H, (\xi^\zeta \mid \xi^\zeta)_\eta)$ is an $\mathcal{N}_\in$-subset of $(H, \mathfrak{f})$. Thus $\xi^\zeta \mid \xi^\zeta \in \mathcal{S}_\in(\eta)$, and so $(\mathcal{S}_\in, \Upsilon)$ is a soft $\mathcal{N}$-subalgebra over H.

Conversely, suppose that the soft $\mathcal{N}_\in$-set $(\mathcal{S}_\in, \Upsilon)$ with $\Upsilon = [-0.5, 0)$ is a soft $\mathcal{N}$-subalgebra over H. Assume that (3) is not valid. Then

$$\mathfrak{f}(\varrho^\varpi \mid \varrho^\varpi) > \mathfrak{t} \geq \bigvee\{\mathfrak{f}(\varrho), \mathfrak{f}(\varpi), -0.5\}$$

for some $\mathfrak{t} \in \Upsilon$ and $\varrho, \varpi \in H$. It follows that $(H, \varrho_\mathfrak{t})$ and $(H, \varpi_\mathfrak{t})$ are $\mathcal{N}_\in$-subsets of $(H, \mathfrak{f})$, and so $\varrho, \varpi \in \mathcal{S}_\in(\mathfrak{t})$. But $f(\varrho^\varpi \mid \varrho^\varpi) > \mathfrak{t}$ induces that $(H, (\varrho^\varpi \mid \varrho^\varpi)_\mathfrak{t})$ is not an $\mathcal{N}_\in$-subset of $(H, \mathfrak{f})$. This is a contradiction, and thus

$$(\forall \xi, \zeta \in H)\,(\mathfrak{f}(\xi^\zeta \mid \xi^\zeta) \leq \bigvee\{\mathfrak{f}(\xi), \mathfrak{f}(\zeta), -0.5\}).$$

Using Lemma 2, we know that $(H, \mathfrak{f})$ is an $\mathcal{N}$-subalgebra of type $(\in, \in \vee \mathfrak{q})$.

Theorem 5. *Let $\mathcal{H}_S := (H, |, 0)$ be an SSH-algebra. Let $(\mathcal{S}_\in, \Upsilon)$ be a soft $\mathcal{N}_\in$-set over H. If $\Upsilon = [-0.5, 0)$, then for any subalgebra L of H, there exists an $\mathcal{N}$-subalgebra $(H, \mathfrak{f})$ of type $(\in, \in \vee \mathfrak{q})$ such that $\mathcal{S}_\in(\eta) = L$ for all $\eta \in \Upsilon$.*

Proof. Take an $\mathcal{N}$-subalgebra $(H, \mathfrak{f})$ in which $\mathfrak{f}$ is given as follows:

$$\mathfrak{f} : H \to [-1, 0]; \quad x \mapsto \begin{cases} \eta \in \Upsilon & \text{if } \xi \in L \\ 0 & \text{otherwise} \end{cases}$$

Obviously, $\mathcal{S}_{\in}(\eta) = L$ for all $\eta \in \Upsilon$. Assume that

$$\mathfrak{f}(\varrho^{\varpi} \mid \varrho^{\varpi}) > \bigvee\{\mathfrak{f}(\varrho), \mathfrak{f}(\varpi), -0.5\}$$

for some $\varrho, \varpi \in H$. Then $\mathfrak{f}(\varrho^{\varpi} \mid \varrho^{\varpi}) = 0$ and $\bigvee\{\mathfrak{f}(\varrho), \mathfrak{f}(\varpi), -0.5\} = \eta$, since $|\mathrm{Im}(\mathfrak{f})| = 2$. It follows that $\mathfrak{f}(\varrho) = \eta = \mathfrak{f}(\varpi)$, so that $\varrho, \varpi \in L$. But $\varrho^{\varpi} \mid \varrho^{\varpi} \notin L$, since $\mathfrak{f}(\varrho^{\varpi} \mid \varrho^{\varpi}) = 0$. This is impossible, and so

$$(\forall \xi, \zeta \in H)\, (\mathfrak{f}(\xi^{\zeta} \mid \xi^{\zeta}) \leq \bigvee\{\mathfrak{f}(\xi), \mathfrak{f}(\zeta), -0.5\}).$$

Therefore, $(H, \mathfrak{f})$ is an $\mathcal{N}$-subalgebra of type $(\in, \in \vee \mathfrak{q})$ by Lemma 2.

Definition 6. *Let $\mathcal{H}_S := (H, |, 0)$ be an SSH-algebra. An $\mathcal{N}$-structure $(H, \mathfrak{f})$ is called an $\mathcal{N}$-ideal of type $(\in, \in)$ (resp., type $(\in, \in_{\mathfrak{q}})$) if the following assertions are valid:*

(1) *If a point $\mathcal{N}$-structure (H, ξ_η) is an $\mathcal{N}_{\in}$-subset of $(H, \mathfrak{f})$, then $(H, 0_\eta)$ is an $\mathcal{N}_{\in}$-subset (resp., $\mathcal{N}_{\in \vee \mathfrak{q}}$-subset) of $(H, \mathfrak{f})$.*

(2) *If two point $\mathcal{N}$-structures $(H, (\xi^{\zeta} \mid \xi^{\zeta})_\eta)$ and (H, ζ_δ) are $\mathcal{N}_{\in}$-subsets of $(H, \mathfrak{f})$, then the point $\mathcal{N}$-structure $(H, \xi_{\eta \vee \delta})$ is an $\mathcal{N}_{\in}$-subset (resp., $\mathcal{N}_{\in \vee \mathfrak{q}}$-subset) of $(H, \mathfrak{f})$.*

Lemma 3. *Let $\mathcal{H}_S := (H, |, 0)$ be an SSH-algebra. Let $(H, \mathfrak{f})$ be an $\mathcal{N}$-structure. Then $\mathfrak{f}(0) \leq \mathfrak{f}(\xi)$ for all $\xi \in H$ if and only if $(H, 0_\eta)$ is an $\mathcal{N}_{\in}$-subset of $(H, \mathfrak{f})$ whenever (H, ξ_η) is an $\mathcal{N}_{\in}$-subset of $(H, \mathfrak{f})$ for all $\xi \in H$ and $\eta \in \Upsilon = [-1, 0)$.*

Proof. Assume that $\mathfrak{f}(0) \leq \mathfrak{f}(\xi)$ for all $\xi \in H$ and let $\eta \in \Upsilon$ be such that (H, ξ_η) is an $\mathcal{N}_{\in}$-subset of $(H, \mathfrak{f})$. Then $\mathfrak{f}(0) \leq \mathfrak{f}(\xi) \leq \eta$, and so $(H, 0_\eta)$ is an $\mathcal{N}_{\in}$-subset of $(H, \mathfrak{f})$.

Conversely, suppose that $(H, 0_\eta)$ is an $\mathcal{N}_{\in}$-subset of $(H, \mathfrak{f})$ whenever (H, ξ_η) is an $\mathcal{N}_{\in}$-subset of $(H, \mathfrak{f})$ for all $\xi \in H$ and $\eta \in \Upsilon = [-1, 0)$. If we take $\eta = \mathfrak{f}(\xi)$ for any $\xi \in H$, then (H, ξ_η) is an $\mathcal{N}_{\in}$-subset of $(H, \mathfrak{f})$, and thus $(H, 0_\eta)$ is an $\mathcal{N}_{\in}$-subset of $(H, \mathfrak{f})$. Hence, $\mathfrak{f}(0) \leq \eta = \mathfrak{f}(\xi)$ for all $\xi \in H$.

Lemma 4. *Let $\mathcal{H}_S := (H, |, 0)$ be an SSH-algebra. Given an $\mathcal{N}$-structure $(H, \mathfrak{f})$, the following are equivalent:*

(1) $\mathfrak{f}(\xi) \leq \bigvee\{\mathfrak{f}(\xi^{\zeta} \mid \xi^{\zeta}), \mathfrak{f}(\zeta)\}$ *for all* $\xi, \zeta \in H$.

(2) *For any $\xi, \zeta \in H$ and $\eta, \delta \in \Upsilon = [-1, 0)$, if $(H, (\xi^{\zeta} \mid \xi^{\zeta})_\eta)$ and (H, ζ_δ) are $\mathcal{N}_{\in}$-subsets of $(H, \mathfrak{f})$, then $(H, \xi_{\eta \vee \delta})$ is an $\mathcal{N}_{\in}$-subset of $(H, \mathfrak{f})$.*

Proof. Assume that $\mathfrak{f}(\xi) \leq \bigvee\{\mathfrak{f}(\xi^\zeta \mid \xi^\zeta), \mathfrak{f}(\zeta)\}$ for all $\xi, \zeta \in H$. Let $\xi, \zeta \in H$ and $\eta, \delta \in \Upsilon$ be such that $(H, (\xi^\zeta \mid \xi^\zeta)_\eta)$ and (H, ζ_δ) are $\mathcal{N}_\in$-subsets of $(H, \mathfrak{f})$. Then $\mathfrak{f}(\xi^\zeta \mid \xi^\zeta) \leq \eta$ and $\mathfrak{f}(\zeta) \leq \delta$, which imply that

$$\mathfrak{f}(\xi) \leq \bigvee\{\mathfrak{f}(\xi^\zeta \mid \xi^\zeta), \mathfrak{f}(\zeta)\} \leq \eta \vee \delta.$$

Hence, $(H, \xi_{\eta\vee\delta})$ is an $\mathcal{N}_\in$-subset of $(H, \mathfrak{f})$.

Conversely, suppose (2) is valid. If we take $\eta = \mathfrak{f}(\xi^\zeta \mid \xi^\zeta)$ and $\delta = \mathfrak{f}(\zeta)$, then $(H, (\xi^\zeta \mid \xi^\zeta)_\eta)$ and (H, ζ_δ) are $\mathcal{N}_\in$-subsets of $(H, \mathfrak{f})$. It follows that $(H, \xi_{\eta\vee\delta})$ is an $\mathcal{N}_\in$-subset of $(H, \mathfrak{f})$ and thus, we have $\mathfrak{f}(\xi) \leq \eta \vee \delta = \bigvee\{\mathfrak{f}(\xi^\zeta \mid \xi^\zeta), \mathfrak{f}(\zeta)\}$.

Combining Lemmas 3 and 4, we have the following theorem.

Theorem 6. *Let $\mathcal{H}_S := (H, |, 0)$ be an SSH-algebra. An $\mathcal{N}$-structure $(H, \mathfrak{f})$ is an $\mathcal{N}$-ideal of type $(\in, \in)$ if and only if*

$$(\forall \xi, \zeta \in H)\, (\mathfrak{f}(0) \leq \mathfrak{f}(\xi) \leq \bigvee\{\mathfrak{f}(\xi^\zeta \mid \xi^\zeta), \mathfrak{f}(\zeta)\}). \tag{4}$$

Definition 7. *Let $\mathcal{H}_S := (H, |, 0)$ be an SSH-algebra. A soft $\mathcal{N}$-set $(\mathcal{S}, \Upsilon)$ over H is called a soft $\mathcal{N}$-ideal over H if*

$$(\forall \eta \in \Upsilon)\, (\mathcal{S}(\eta) \neq \emptyset \Rightarrow \mathcal{S}(\eta) \text{ is an ideal of } H). \tag{5}$$

Theorem 7. *Let $\mathcal{H}_S := (H, |, 0)$ be an SSH-algebra. Given an $\mathcal{N}$-structure $(H, \mathfrak{f})$ and the soft $\mathcal{N}_\in$-set $(\mathcal{S}_\in, \Upsilon)$, the following are equivalent:*

(1) *$(\mathcal{S}_\in, \Upsilon)$ is a soft $\mathcal{N}$-ideal over H for $\Upsilon = [-1, 0)$.*

(2) *$(H, \mathfrak{f})$ is an $\mathcal{N}$-ideal of type $(\in, \in)$.*

Proof. Assume that $(\mathcal{S}_\in, \Upsilon)$ is a soft $\mathcal{N}$-ideal over H for $\Upsilon = [-1, 0)$. If there exists $\varrho \in H$ such that $\mathfrak{f}(0) > \mathfrak{f}(\varrho)$, then we can take $\eta \in \Upsilon$ such that $\mathfrak{f}(0) > \eta \geq \mathfrak{f}(\varrho)$. Thus, $(H, 0_\eta)$ is not an $\mathcal{N}_\in$-subset of $(H, \mathfrak{f})$, and so $0 \notin \mathcal{S}_\in(\eta)$. This is a contradiction, and so $\mathfrak{f}(0) \leq \mathfrak{f}(\xi)$ for all $\xi \in H$.

Suppose that there exist $\varrho, \varpi \in H$ such that $\mathfrak{f}(\varrho) > \bigvee\{\mathfrak{f}(\varrho^\varpi \mid \varrho^\varpi), \mathfrak{f}(\varpi)\}$. Taking $\eta = \bigvee\{\mathfrak{f}(\varrho^\varpi \mid \varrho^\varpi), \mathfrak{f}(\varpi)\}$ implies that $(H, (\varrho^\varpi \mid \varrho^\varpi)_\eta)$ and (H, ϖ_η) are $\mathcal{N}_\in$-subsets of $(H, \mathfrak{f})$. That is, $\varrho^\varpi \mid \varrho^\varpi \in \mathcal{S}_\in(\eta)$ and $\varpi \in \mathcal{S}_\in(\eta)$. Since $\mathcal{S}_\in(\eta)$ is an ideal of H, $\varrho \in \mathcal{S}_\in(\eta)$. Hence, (H, a_η) is an $\mathcal{N}_\in$-subset of $(H, \mathfrak{f})$, and so $\mathfrak{f}(\varrho) \leq \eta$. This is a contradiction, and therefore $\mathfrak{f}(\xi) \leq \bigvee\{\mathfrak{f}(\xi^\zeta \mid \xi^\zeta), \mathfrak{f}(\zeta)\}$ for all $\xi, \zeta \in H$. Hence, $(H, \mathfrak{f})$ is an $\mathcal{N}$-ideal of type $(\in, \in)$.

Conversely, assume that $(H, \mathfrak{f})$ is an $\mathcal{N}$-ideal of type $(\in, \in)$ and let $\eta \in \Upsilon$ be such that $\mathcal{S}_\in(\eta) \neq \emptyset$. Then there exists $\xi \in \mathcal{S}_\in(\eta)$, and so (H, ξ_η) is an $\mathcal{N}_\in$-subset of $(H, \mathfrak{f})$. It follows that $\mathfrak{f}(0) \leq \mathfrak{f}(\xi) \leq \eta$ and so that $(H, 0_\eta)$ is an $\mathcal{N}_\in$-subset of $(H, \mathfrak{f})$, that is, $0 \in \mathcal{S}_\in(\eta)$. Let $\xi^\zeta \mid \xi^\zeta \in \mathcal{S}_\in(\eta)$ and $\zeta \in \mathcal{S}_\in(\eta)$. Then $(H, (\xi^\zeta \mid \xi^\zeta)_\eta)$ and (H, ζ_η) are $\mathcal{N}_\in$-subsets of $(H, \mathfrak{f})$. Thus, $\mathfrak{f}(\xi^\zeta \mid \xi^\zeta) \leq \eta$ and $\mathfrak{f}(\zeta) \leq \eta$. It follows from (4) that $\mathfrak{f}(\xi) \leq \bigvee\{\mathfrak{f}(\xi^\zeta \mid \xi^\zeta), \mathfrak{f}(y)\} \leq \eta$. Hence, $(H, \xi_\eta) = (H, \xi_{\eta\vee\eta})$ is an $\mathcal{N}_\in$-subset of $(H, \mathfrak{f})$, and so $\xi \in \mathcal{S}_\in(\eta)$. Thus, $\mathcal{S}_\in(\eta)$ is an ideal of H for all $\eta \in \Upsilon$, and therefore $(\mathcal{S}_\in, \Upsilon)$ is a soft $\mathcal{N}$-ideal over H for $\Upsilon = [-1, 0)$.

Theorem 8. *Let $\mathcal{H}_S := (H, |, 0)$ be an SSH-algebra. The soft $\mathcal{N}_{\mathfrak{q}}$-set $(\mathcal{S}_{\mathfrak{q}}, \Upsilon)$ is a soft $\mathcal{N}$-ideal over H for $\Upsilon = [-1, 0)$ if and only if the $\mathcal{N}$-structure $(H, \mathfrak{f})$ is an $\mathcal{N}$-ideal of type $(\in, \in)$.*

Proof. Assume that $(H, \mathfrak{f})$ is an $\mathcal{N}$-ideal of type $(\in, \in)$, and let $\eta \in \Upsilon$ be such that $\mathcal{S}_{\mathfrak{q}}(\eta) \neq \emptyset$. This implies there exists $\xi \in \mathcal{S}_{\mathfrak{q}}(\eta)$, so (H, ξ_η) is an $\mathcal{N}_{\mathfrak{q}}$-subset of $(H, \mathfrak{f})$. If $0 \notin \mathcal{S}_{\mathfrak{q}}(\eta)$, then $(H, 0_{\mathfrak{q}})$ is not an $\mathcal{N}_{\mathfrak{q}}$-subset of $(H, \mathfrak{f})$, which implies $\mathfrak{f}(0) + \eta + 1 < 0$. Using (4), we obtain $\mathfrak{f}(\xi) + \eta + 1 \geq \mathfrak{f}(0) + \eta + 1 \geq 0$, leading to a contradiction, thus $0 \in \mathcal{S}_{\mathfrak{q}}(\eta)$.

Next, consider $\xi^\zeta \mid \xi^\zeta \in \mathcal{S}_{\mathfrak{q}}(\eta)$ and $\zeta \in \mathcal{S}_{\mathfrak{q}}(\eta)$. Then $(H, (\xi^\zeta \mid \xi^\zeta)_\eta)$ and (H, ζ_η) are $\mathcal{N}_{\mathfrak{q}}$-subsets of $(H, \mathfrak{f})$. If (H, ξ_η) is not an $\mathcal{N}_{\mathfrak{q}}$-subset of $(H, \mathfrak{f})$, then $\mathfrak{f}(\xi) + \eta + 1 \geq 0$. From (4), we have $\bigvee\{\mathfrak{f}(\xi^\zeta \mid \xi^\zeta), \mathfrak{f}(\zeta)\} + \eta + 1 \geq \mathfrak{f}(\xi) + \eta + 1 \geq 0$. Hence, either $\mathfrak{f}(\xi^\zeta \mid \xi^\zeta) + \eta + 1 \geq 0$ or $\mathfrak{f}(\zeta) + \eta + 1 \geq 0$, leading to a contradiction. Thus, $\xi \in \mathcal{S}_{\mathfrak{q}}(\eta)$. Therefore, $(\mathcal{S}_{\mathfrak{q}}, \Upsilon)$ is a soft $\mathcal{N}$-ideal over H for $\Upsilon = [-1, 0)$.

Conversely, suppose that the soft $\mathcal{N}_{\mathfrak{q}}$-set $(\mathcal{S}_{\mathfrak{q}}, \Upsilon)$ is a soft $\mathcal{N}$-ideal over H for $\Upsilon = [-1, 0)$. If $\mathfrak{f}(0) > \mathfrak{f}(\varrho)$ for some $\varrho \in H$, then there exists $\delta \in \Upsilon$ such that $\mathfrak{f}(0) + \delta + 1 \geq 0$ and $\mathfrak{f}(\varrho) + \delta + 1 \geq 0$. This implies that (H, ϱ_δ) is an $\mathcal{N}_{\mathfrak{q}}$-subset of $(H, \mathfrak{f})$, and $(H, 0)$ is an $\mathcal{N}_{\mathfrak{q}}$-subset of $(H, \mathfrak{f})$, leading to a contradiction. Thus, $\mathfrak{f}(0) \leq \mathfrak{f}(\xi)$ for all $\xi \in H$. Suppose there exist $\varrho, \varpi \in H$ such that $\mathfrak{f}(\varrho) > \bigvee\{f(\varrho^\varpi \mid \varrho^\varpi), \mathfrak{f}(\varpi)\}$. Then for some $\delta \in \Upsilon$, we have $\mathfrak{f}(\varrho) + \delta + 1 \geq 0$ and $\bigvee\{f(\varrho^\varpi \mid \varrho^\varpi), \mathfrak{f}(\varpi)\} + \delta + 1 \geq 0$. This implies that $f(\varrho^\varpi \mid \varrho^\varpi) + \delta + 1 < 0$ and $\mathfrak{f}(\varpi) + \delta + 1 < 0$, meaning that $(H, (\varrho^\varpi \mid \varrho^\varpi)_\delta)$ and (H, ϖ_δ) are $\mathcal{N}_{\mathfrak{q}}$-subsets of $(H, \mathfrak{f})$. Therefore, $\varrho^\varpi \mid \varrho^\varpi \in \mathcal{S}_{\mathfrak{q}}(\delta)$ and $\varpi \in \mathcal{S}_{\mathfrak{q}}(\delta)$. Since $\mathcal{S}_q(\delta)$ is an ideal of H, we conclude that $\varrho \in \mathcal{S}_{\mathfrak{q}}(\delta)$, and thus (H, ϱ_δ) is an $\mathcal{N}_{\mathfrak{q}}$-subset of $(H, \mathfrak{f})$. This is a contradiction, so $\mathfrak{f}(\xi) \leq \bigvee\{f(\xi^\zeta \mid \xi^\zeta), \mathfrak{f}(\zeta)\}$ for all $\xi, \zeta \in H$. Thus, $(H, \mathfrak{f})$ is an $\mathcal{N}$-ideal of type $(\in, \in)$.

Theorem 9. *Let $\mathcal{H}_S := (H, |, 0)$ be an SSH-algebra. Given an $\mathcal{N}$-structure $(H, \mathfrak{f})$ and the soft $\mathcal{N}_\in$-set $(\mathcal{S}_\in, \Upsilon)$, the following are equivalent:*

(1) *$(H, \mathfrak{f})$ is an $\mathcal{N}$-ideal of type $(\in, \in \vee \mathfrak{q})$.*

(2) *$(\mathcal{S}_\in, \Upsilon)$ is a soft $\mathcal{N}$-ideal over H for $\Upsilon = [-0.5, 0)$.*

Proof. Assume that $(H, \mathfrak{f})$ is an $\mathcal{N}$-ideal of type $(\in, \in \vee \mathfrak{q})$. We first show that

$$(\forall \xi \in H)\, (\mathfrak{f}(0) \leq \bigvee\{\mathfrak{f}(\xi), -0.5\}).$$

Suppose that $\mathfrak{f}(0) > \mathfrak{f}(\xi) > -0.5$. Then, there exists $\eta \in (-0.5, 0)$ such that $\mathfrak{f}(0) > \eta \geq \mathfrak{f}(\xi)$, which implies that (H, ξ_η) is an $\mathcal{N}_\in$-subset of $(H, \mathfrak{f})$. However, $(H, 0_\eta)$ is not an $\mathcal{N}_\in$-subset of $(H, \mathfrak{f})$, nor is it an $\mathcal{N}_{\mathfrak{q}}$-subset of $(H, \mathfrak{f})$ because $\mathfrak{f}(0) + \eta + 1 \geq 0$. This is a contradiction. Hence, $\mathfrak{f}(0) \leq \mathfrak{f}(\xi)$ for all $\xi \in H$.

Now, if $\mathfrak{f}(\xi) \leq -0.5$, then $(H, \xi_{-0.5})$ is an $\mathcal{N}_\in$-subset of $(H, \mathfrak{f})$, and thus $(H, 0_{-0.5})$ is an $\mathcal{N}_{\in \vee \mathfrak{q}}$-subset of $(H, \mathfrak{f})$. Therefore, $\mathfrak{f}(0) \leq -0.5$, as if $\mathfrak{f}(0) > -0.5$, we would have $\mathfrak{f}(0) - 0.5 + 1 > 0$, which is a contradiction. Let $\eta \in \Upsilon = [-0.5, 0)$. Then for all $\xi \in \mathcal{S}_\in(\eta)$, $\mathfrak{f}(0) \leq \bigvee\{\mathfrak{f}(\xi), -0.5\}$, and so $\mathfrak{f}(0) \leq \bigvee\{\mathfrak{f}(\xi), -0.5\} \leq \bigvee\{\eta, -0.5\} = \eta$. Thus, $(H, 0_\eta)$ is an $\mathcal{N}_\in$-subset of $(H, \mathfrak{f})$, implying that $0 \in \mathcal{S}_\in(\eta)$. Now, we show that

$$(\forall \xi, \zeta \in H)\, (\mathfrak{f}(\xi) \leq \bigvee\{\mathfrak{f}(\xi^\zeta \mid \xi^\zeta), \mathfrak{f}(\zeta), -0.5\}).$$

If $\bigvee\{\mathfrak{f}(\xi^\zeta \mid \xi^\zeta), \mathfrak{f}(\zeta)\} > -0.5$, then $\mathfrak{f}(\xi) \leq \bigvee\{f(\xi^\zeta \mid \xi^\zeta), \mathfrak{f}(\zeta)\}$. Otherwise, there exists $\delta \in (-0.5, 0)$ such that $\mathfrak{f}(\xi) > \delta \geq \bigvee\{\mathfrak{f}(\xi^\zeta \mid \xi^\zeta), \mathfrak{f}(\zeta)\}$. Then $(H, (\xi^\zeta \mid \xi^\zeta)_\delta)$ and (H, y_δ) are $\mathcal{N}_\in$-subsets of $(H, \mathfrak{f})$, but (H, ξ_δ) is not an $\mathcal{N}_\in$-subset of $(H, \mathfrak{f})$, and neither is it an $\mathcal{N}_\mathfrak{q}$-subset of $(H, \mathfrak{f})$, as $\mathfrak{f}(\xi) + \delta + 1 > 2\delta + 1 > 0$. This leads to a contradiction. Therefore, we must have $\mathfrak{f}(\xi) \leq \bigvee\{\mathfrak{f}(\xi^\zeta \mid \xi^\zeta), \mathfrak{f}(\zeta), -0.5\}$ for all $\xi, \zeta \in H$.

Let $\xi, \zeta \in H$ be such that $\xi^\zeta \mid \xi^\zeta \in \mathcal{S}_\in(\eta)$ and $\zeta \in \mathcal{S}_\in(\eta)$. Then $(H, (\xi^\zeta \mid \xi^\zeta)_\eta)$ and (H, ζ_δ) are $\mathcal{N}_\in$-subsets of $(H, \mathfrak{f})$, and so $\mathfrak{f}(\xi^\zeta \mid \xi^\zeta) \leq \eta$ and $\mathfrak{f}(\zeta) \leq \eta$. Hence,

$$\mathfrak{f}(\xi) \leq \bigvee\{\mathfrak{f}(\xi^\zeta \mid \xi^\zeta), \mathfrak{f}(\zeta), -0.5\} \leq \bigvee\{\eta, -0.5\} = \eta,$$

which implies that (H, ξ_δ) is an $\mathcal{N}_\in$-subset of $(H, \mathfrak{f})$, and therefore $\xi \in \mathcal{S}_\in(\eta)$. Thus, $(\mathcal{S}_\in, \Upsilon)$ is a soft $\mathcal{N}$-ideal over H for $\Upsilon = [-0.5, 0)$.

Conversely, suppose that (2) is valid. If $\mathfrak{f}(0) > \bigvee\{\mathfrak{f}(\varrho), -0.5\}$ for some $\varrho \in H$, then there exists $\eta \in \Upsilon$ such that $\mathfrak{f}(0) > \eta \geq \bigvee\{\mathfrak{f}(\varrho), -0.5\}$. Then $\eta \in \Upsilon$, and (H, ϱ_η) is an $\mathcal{N}_\in$-subset of $(H, \mathfrak{f})$. However, $(H, 0_\eta)$ is not an $\mathcal{N}_\in$-subset of $(H, \mathfrak{f})$, meaning $0 \notin \mathcal{S}_\in(\eta)$. This is a contradiction, and so $\mathfrak{f}(0) \leq \bigvee\{\mathfrak{f}(\xi), -0.5\}$ for all $\xi \in H$. Let $\xi \in H$ and $\eta \in \Upsilon$ be such that (H, ξ_δ) is an $\mathcal{N}_\in$-subset of $(H, \mathfrak{f})$. Then $\mathfrak{f}(\xi) \leq \eta$. Suppose that $(H, 0_\eta)$ is not an $\mathcal{N}_\in$-subset of $(H, \mathfrak{f})$. Then $\mathfrak{f}(0) > \eta$. If $\mathfrak{f}(\xi) > -0.5$, then $\mathfrak{f}(0) \leq \bigvee\{\mathfrak{f}(\xi), -0.5\} = \mathfrak{f}(\xi) \leq \eta$, which is impossible. Thus, $\mathfrak{f}(\xi) \leq -0.5$, and

$$\mathfrak{f}(0) + \eta + 1 < 2\mathfrak{f}(0) + 1 \leq 2\bigvee\{\mathfrak{f}(\xi), -0.5\} + 1 = 0,$$

implying that $(H, 0_\eta)$ is an $\mathcal{N}_\mathfrak{q}$-subset of $(H, \mathfrak{f})$. Therefore, $(H, 0_\eta)$ is an $\mathcal{N}_{\in\vee\mathfrak{q}}$-subset of $(H, \mathfrak{f})$. Assume that there exist $\varrho, \varpi \in H$ such that

$$\mathfrak{f}(\varrho) > \bigvee\{\mathfrak{f}(\varrho^\varpi \mid \varrho^\varpi), \mathfrak{f}(\varpi), -0.5\}.$$

Let $\eta = \bigvee\{\mathfrak{f}(\varrho^\varpi \mid \varrho^\varpi), \mathfrak{f}(\varpi), -0.5\}$. Then $\eta \in \Upsilon$, and $(H, (\varrho^\varpi \mid \varrho^\varpi)_\eta)$ and (H, ϖ_η) are $\mathcal{N}_\in$-subsets of $(H, \mathfrak{f})$. But (H, ϱ_η) is not an $\mathcal{N}_\in$-subset of $(H, \mathfrak{f})$, leading to a contradiction. Therefore,

$$(\forall \xi, \zeta \in H)\,(\mathfrak{f}(\xi) \leq \bigvee\{\mathfrak{f}(\xi^\zeta \mid \xi^\zeta), \mathfrak{f}(\zeta), -0.5\}).$$

Let $\xi, \zeta \in H$ and $\eta, \delta \in \Upsilon$ be such that $(H, (\xi^\zeta \mid \xi^\zeta)_\eta)$ and (H, ζ_δ) are $\mathcal{N}_\in$-subsets of $(H, \mathfrak{f})$, and suppose that $(H, \xi_{\eta\vee\delta})$ is not an $\mathcal{N}_\in$-subset of $(H, \mathfrak{f})$. Then, $\mathfrak{f}(\xi^\zeta \mid \xi^\zeta) \leq \eta$, $\mathfrak{f}(\zeta) \leq \eta$, and $\mathfrak{f}(\xi) > \eta \vee \delta$. If $\bigvee\{\mathfrak{f}(\xi^\zeta \mid \xi^\zeta), \mathfrak{f}(\zeta)\} > -0.5$, then

$$\mathfrak{f}(\xi) \leq \bigvee\{\mathfrak{f}(\xi^\zeta \mid \xi^\zeta), \mathfrak{f}(\zeta), -0.5\} = \bigvee\{\mathfrak{f}(\xi^\zeta \mid \xi^\zeta), \mathfrak{f}(\zeta)\} \leq \eta \vee \delta,$$

which is a contradiction. Thus,

$$\bigvee\{\mathfrak{f}(\xi^\zeta \mid \xi^\zeta), \mathfrak{f}(\zeta)\} \leq -0.5,$$

and so

$$\mathfrak{f}(\xi) + (\eta \vee \delta) + 1 < 2\mathfrak{f}(\xi) + 1 \leq 2\bigvee\{\mathfrak{f}(\xi^\zeta \mid \xi^\zeta), \mathfrak{f}(\zeta), -0.5\} + 1 = 0,$$

which shows that $(H, \xi_{\eta\vee\delta})$ is an $\mathcal{N}_\mathfrak{q}$-subset of $(H, \mathfrak{f})$. Therefore, $(H, \xi_{\eta\vee\delta})$ is an $\mathcal{N}_{\in\vee\mathfrak{q}}$-subset of $(H, \mathfrak{f})$. Consequently, $(H, \mathfrak{f})$ is an $\mathcal{N}$-ideal of type $(\in, \in\vee\mathfrak{q})$.

4. Conclusion

In this study, we have introduced and explored the notions of soft $\mathcal{N}$-subalgebras and soft $\mathcal{N}$-ideals in the context of Sheffer stroke Hilbert algebras, incorporating the framework of $\mathcal{N}$-structures. By defining and characterizing these algebraic constructs, we have established fundamental properties and the interrelationships between different types of soft $\mathcal{N}$-subalgebras and soft $\mathcal{N}$-ideals, particularly under the types $(\in, \in)$ and $(\in, \in \vee \mathsf{q})$. Our results provide a deeper understanding of the algebraic behavior of Sheffer stroke Hilbert algebras under soft set theory, highlighting their flexibility in modeling uncertainty within logical systems. These findings not only contribute to the ongoing development of algebraic logic but also open pathways for further research in generalizing soft algebraic structures in broader mathematical settings. Future work may focus on extending these concepts to other non-classical algebraic frameworks and exploring their potential applications in computational logic and fuzzy systems.

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References

[1] H. M. Sheffer. A set of five independent postulates for Boolean algebras, with application to logical constants. *Trans. Am. Math. Soc.*, 14(4):481–488, 1913.

[2] I. Chajad. Sheffer operation in ortholattices. *Acta Univ. Palacki. Olomuc., Fac. Rerum Nat., Math.*, 44(1):19–23, 2005.

[3] V. Kozarkiewicz and A. Grabowski. Axiomatization of Boolean algebras based on Sheffer stroke. *Formaliz. Math.*, 12(3):355–361, 2004.

[4] T. Oner, T. Katican, and A. Borumand Saeid. BL-algebras defined by an operator. *Honam Math. J.*, 44(2):18–31, 2022.

[5] T. Oner, T. Katican, and A. Borumand Saeid. Class of Sheffer stroke BCK-algebras. *An. Ştiinţ. Univ. "Ovidius" Constanţa, Ser. Mat.*, 30(1):247–269, 2022.

[6] T. Oner and I. Senturk. The Sheffer stroke operation reducts of basic algebras. *Open Math.*, 15(1):926–935, 2017.

[7] I. Senturk. A new on state operators in Sheffer stroke basic algebras. *Soft Comput.*, 25(17):11471–11484, 2021.

[8] I. Senturk. Riečan and Bosbach state operators on Sheffer stroke MTL-algebras. *Bull. Int. Math. Virtual Inst.*, 12(1):181–193, 2022.

[9] N. Rajesh, T. Oner, A. Iampan, and I. Senturk. On length and mean fuzzy ideals of Sheffer stroke Hilbert algebras. *Eur. J. Pure Appl. Math.*, 18(1):5779, 2025.

[10] D. Molodstov. Soft set theory-first results. *Comput. Math. Appl.*, 37(4-5):19–31, 1999.

[11] T. Oner, T. Katican, and A. Borumand Saeid. Relation between Sheffer stroke and Hilbert algebras. *Categ. Gen. Algebr. Struct. Appl.*, 14(1):245–268, 2021.