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A Class of Reduced Bias Estimators of Distortion Risk Measures under Dependence Serials with Heavy-Tailed Marginals

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Abstract. In this paper, we introduce a class of semi-parametric estimators of the distortion risk premiums for dependent insurance losses with heavy-tailed marginals. Our approach is based on the kernel estimation of the tail index and extreme quantiles under the first and second orders regularly varying assumptions for stationary insured risks with heavy-tailed distribution under dependence serials. Moreover, we illustrate the behaviour of our proposed estimator and give a comparison between this estimator and the classical one in terms of the absolute bias and the root median squared error.

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1. Introduction

Risk measurement or premium calculation principles are used to quantify insurance losses and financial valuations. A number of risk measures have been proposed to manage these risks, and we refer to [1], [2], [3], [4] and the references therein. The most commonly used one is the net premium (mean) of a non-negative loss random variable X over the probability space $(\Omega, \mathbf{A}, P)$, with a tail distribution function $\overline{F} := 1 - F$ and defined as

$$\pi = \mathbb{E}(X) = \int_0^\infty \overline{F}(x)dx.$$

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Premiums are required to be greater than or equal to the mean $\mathbb{E}(X)$ in order to avoid that the insurer loses money on average. One way to achieve this goal consists in considering the following distortion risk premium introduced by [4] as:

$$\pi_g = \int_0^{\infty} g(\overline{F}(x))dx, \tag{1}$$

where g is a concave function defined from $[0,1]$ onto $[0,1]$, such that $g(0) = 0$ and $g(1) = 1$, and called *the distortion function.* The distortion function g is parameterized by a one-dimensional parameter $\beta \geq 1$, called the *distortion parameter* and represents the risk aversion. It controls the amount of the risk loading included in the premium for a given riskiness of the loss variable X.
Let Q be the quantile function corresponding to F and defined by $Q(s) = \inf\{x : F(x) \geq s\}$, for every $s \in [0,1)$. The quantile function Q plays a pivotal role in defining numerous risk measures, and is a well known risk measure itself, called the Value-at-Risk (VaR). By a change of variables, the distortion risk premium $\pi_g(R)$ can be rewritten in terms of the quantile function Q as follows:

$$\pi_g = -\int_0^1 g(s)dQ(1-s). \tag{2}$$

The risk measure π_g, which can also be viewed as a premium calculation principle, has manifested in the econometric literature, particularly in dual theory of choice under risk, and has been introduced into actuarial literature by [4]. A number of risk measures of this form have been discussed by [5].
Important properties of the distortion risk measure, such as coherence and second order stochastic dominance have been well studied see, for example [6], [7] and [5].

Note that the class of concave distortion risk measures is only a subset of the class of coherent risk measures. Many special cases that have arisen in the finance and insurance literature are such:

- The Net Premium principle: $g(x) = x$
- Value-at-Risk (VaR_α): $g(x) = \mathbf{1}(1-\alpha, 1)$, for some $\alpha \in (0,1)$, where $\mathbf{1}(.)$ is the indicator function.
- Tail Value at Risk: $g(x) = \min(x/(1-\alpha), 1)$, for some $\alpha \in (0,1)$.
- Proportional Hazard Transform: $g(x) = x^{1/\varrho}$, for some $\varrho \geq 1$.
- Dual-Power Transform: $g(x) = 1 - (1-x)^{\varrho}$, for some $\varrho \geq 1$.
- Gini principle: $g(x) = (1+\varrho)x - \varrho x^2$, with $0 < \varrho \leq 1$.
- Lookback distortion: $g(x) = x^{\varrho}(1 - \varrho \log(x))$, with $0 < \varrho \leq 1$.
- Beta-distortion risk premium (eg, [5]): $g(x) = \frac{1}{\mathbf{B}(a,b)} \int_0^x s^{a-1}(1-s)^{b-1}ds$,
  where $\mathbf{B}(a,b) = \int_0^1 s^{a-1}(1-s)^{b-1}ds$, $a \leq 1 \leq b$..
- MINMAXVAR2 risk premium (see [8] and references there in):
  where $g(t) = 1 - (1 - x^{\frac{1}{1+\mu}})^{1+\nu}$, $\mu > 0$, $\nu > 0$.

Note that these distortion functions $g(\cdot)$ are equal or can be approximated to a power function $g_\beta(y) = t^{1/\beta},\ \beta \geq 1$, since they are regularly varying at zero with index $1/\beta$, that is: $g(t) = t^{1/\beta}\ell_g(t)$, where $\ell_g(\cdot)$ is a slowly varying function at zero satisfying $\ell_g(\lambda t)/\ell_g(t) \to 1$ as $t \to 0$, for $\lambda > 0$. This condition is used in [8] and [9] to estimate the reinsurance risk premiums in the context of independent extreme risks.

A standard reinsurance product is an excess of loss reinsurance, which means that the reinsurer only compensates the cedant's loss above a certain retention amount $R > 0$. Consider an excess-of-loss reinsurance policy in excess of a high retention level $R > 0$, the distorted reinsurance premium of the total claim amount $\max(X - R, 0)$ is defined as:

$$\pi_{g_\beta}(R) = \int_R^\infty g_\beta\left(\overline{F}(x)\right) dx. \tag{3}$$

As in (2), the distortion risk premium $\pi_{g_\beta}(R)$ can be rewritten in terms of Value-at- Risk Q as follows:

$$\pi_{g_\beta}(R) = -\int_0^{\overline{F}(R)} g_\beta(s) dQ(1-s). \tag{4}$$

In the reinsurance context, the purpose of estimating the premium $\pi_g(R)$ is to estimate, for each insured, the expected under the distorted probability of the excess claim amounts for a given period. This evaluation is often done using statistical methods. For more details, see [10].
Thus, reinsurance companies must calculate the premiums to cover these excess claims, which are usually very high. The extreme value theory (EVT) has become one of the leading theories in the development of statistical models for high insurance losses. We refer to [11], for general accounts on extreme-value theory.
Many authors studied the estimation of the premium of these high excess losses by using classical EVT models, mainly based on the independent and identically distributed (i.i.d) assumption of the insured risks with large tails. One can mention among others, [12], [13], [8], [14], [15], [16], [9], etc.

The reinsurance is also a risk mitigating tool, constituting an important instrument in the management of risk of an insurance company where dependencies and the heavy-tailed nature should be taken into account. When transferring risk, the cedent seeks a trade-off between profit and safety, which is on the nature of the insured risk and on the reinsurance premium calculation principle.
The heavy-tailed nature of insurance claims requires that special attention paid to be analyzing the tail distributions of a claims amounts. Such distributions are mainly characterized by their index which make the possibility to indicate the size and the frequency of some extreme phenomena within the framework of a given probability distribution (See eg, [9]). The extreme value theory (EVT) offers satisfactory statistical results such heavy tailed distributions. Semiparametric estimators of reinsurance premiums for independent

and identically distributed (i.i.d) from a heavy tailed losses have been largely studied in the literature. However, only recently, dependencies among risks have been considered (see, [17]) Also, in the dependence context of financial extreme losses with heavy-tailed marginals, [18], [19] and [20] investigated the estimation of the Value-at-Risk for extreme losses (high quantile).

In high excess reinsurance losses, [17] introduced a semiparametric estimator of the risk premiums from heavy tailed dependent insured risks over an optimal retention level. This semiparametric estimator surfers from a bias problem due to the fact it depends on a classical estimator of the Value-at-Risk estimator (the Weissman's estimator (see, [21]), which have the same problem.

The aim of this paper is to generalize the estimator of reinsurance risk premiums proposed in [17]. As it exhibits a potential bias, we introduce its bias reduction approach under under dependent insured risks with heavy tailed marginals. Our consideration is based on the bias reduction approach proposed by [19] in the estimation of the Value-at Risk under dependence serials.
The rest of the paper is organized as follows. In Section 2, we propose a statistical estimation of the distortion risk premiums under dependent serials. In Section 3, we establish the asymptotic properties of the proposed estimator. Then in Section 4, we match our theoretical results with a simulation assessment in order to highlight the efficiency of our methods. Finally, Sections 5 and 6 are respectively devoted to the conclusion and the proofs of our main results.

2. Estimating the distortion risk premiums

2.1. Extreme value theory under dependence serials

Extreme value statistics are based on the fact that under rather mild conditions for large samples, a class of distribution functions can be considered to fit the distribution of the largest observation in a sample. From this limit theorem, it follows that the tail behavior of a distribution function can be characterized mainly by a single shape parameter, called the tail index or extreme value index. Based on the sign of extreme value index, the domain of attraction of the extreme value distribution can be divided into three subclasses.

To this end, let's consider $X_i,\ i \in \mathbb{N}$ a copies from a non negative stationary insured risk X defined over some probability space $(\Omega, \mathcal{A}, \mathbb{P})$, with ncommon marginal distribution function (df) $F(x) = \mathbb{P}(X \leq x)$. Indeed, under mild conditions on the dependence structure, If the $X_i,\ i \in \mathbb{N}$ are weakly dependent then, the corresponding main result of the extreme value theory (See, [22], Section 3.7), under some mild dependence conditions, is based on the following weak convergence of the distribution function from the standardized maximum of n observations $(X_1, .., X_n)$, $n > 1$:

$$\mathcal{L}\left(a_n^{-1}\left(\max_{1\leq i\leq n} X_i - b_n\right)\right) \to G_\gamma^\theta \quad weakly, \tag{5}$$

for some $\theta \in [0,1]$, where $a_n > 0$, $b_n \in \mathbb{R}$ are standardized sequences and

$$G_\gamma(x) = \exp\left(-(1+\gamma x)_+^{-1/\gamma}\right),$$

with $y_+ = \max(y, 0)$ and $G_\gamma(x) = \exp(e^{-x})$, for $\gamma = 0$. Here, the real-valued parameter γ is referred to as the extreme value index of F, which in turn is said to belong to the maximum domain of attraction of G_γ, denoted by $F \in \mathcal{DM}(G_\gamma)$.

Throughout this paper, we assume that the non negative stationary insured risks $X_i,\ i \in \mathbb{N}$ satisfies the following β-mixing dependence structure condition:

$$\beta(m) := \sup_{p\geq 1} \mathbb{E}\left\{\sup_{C\in\mathcal{B}_{p+m+1}^\infty} |\mathbb{P}(C|\mathcal{B}_1^p) - \mathbb{P}(C)|\right\} \to 0, \tag{6}$$

as $m \to \infty$, where $\mathcal{B}_i^j$ denotes the σ-algebra generated by $(X_i, ..., X_j)$. Without loss of generality, $\beta(m)$ measures the total variation distance between the unconditional distribution of the future of the time series and the conditional distribution of the future given the past of the time series when both are detached by m time points.
Also; it is assumed that the common marginal distribution function F of the β-mixing insured risks $X_i, i \in \mathbb{N}$, is heavy-tailed (belonging to the the Fréchet domain of attraction that is $F \in \mathcal{M}(G_\gamma),\ \ \gamma > 0$). This is equivalent to the fact that its associated tail distribution function $1 - F$ is regularly varying at infinity with index $-1/\gamma < 0$. More precisely, that is

$$\overline{F}(x) := 1 - F(x) = x^{-1/\gamma}\ell_F(x),\ x > 0, \tag{7}$$

where ℓ_F is a slowly varying function at infinity, *i.e* for all $x > 0$, $\ell_F(tx)/\ell_F(t) \to 1$, as $t \to \infty$. The relation (7) is also equivalent to $U(z) = Q(1 - z^{-1}) = z^\gamma \ell_U(z),\ z > 1$, where $\ell_U(tz)/\ell_U(t) \to 1$, as $t \to \infty$, for all $z > 1$, where $Q(1 - s) = \inf\{x,\ \ F(x) \geq s\}$ is the quantile function associated to F, namely called the Value-at-Risk. The class of heavy-tailed distributions includes distributions such as Pareto, Burr, Student, Lévy-stable, and log-gamma which are known to be appropriate models in *Extreme value theory* for fitting large insurance claims, large fluctuations of prices, log-returns,etc. (see, e.g., [23], [24], [12], [13], [8], [14], [25], [26], [15], [9], etc.).

From (7), one can easily see that for all $x > 0$ and $z > 1$:

$$\lim_{t\to\infty} \frac{\overline{F}(tx)}{\overline{F}(t)} = x^{-1/\gamma} \quad \Leftrightarrow \quad \lim_{t\to\infty} \frac{U(tz)}{U(t)} = z^\gamma. \tag{8}$$

The relation in (8) is namely called the first order regularly varying condition. The parameter γ is the tail index (or the extreme value index) and governs the tail behavior, with larger values indicating heavier tails. Its estimation has received a great attention in the extreme value literature, especially in the case of i.i.d. random variables (cf. [11]). Although only few papers consider that for the case of time series with serial dependence features. We can mention among others, [27], [28]. And very recently [18], [19] and [20].

Next, we note that:

- When $\gamma > 1$, the first moment $\mathbb{E}(X_1)$ of the dependence insured losses is not defined. Thus, their associated reinsurance distorted risk premiums $\pi_{g_\beta}(R)$ is also not defined and it is not possible to do its statistical estimation.

- When $0 < \gamma \leq 1/2$ (the lower half of the unit interval), then the second moments of the of the dependence insured losses, $\mathbb{E}[X_1^{2+\epsilon}] < \infty$, for some $\epsilon > 0$, and so a nonparametric estimator for the reinsurance distorted risk premiums $\pi_{g_\beta}(R)$ can be obtain by substituting in (3) the unknown distribution function F with its empirical component F_n defined as $F_n(x) = n^{-1}\sum_{i=1}^n \mathbf{1}(X_i \leq x)$, this estimator is asymptotically normal.

- When $1/2 < \gamma < 1$ (the upper half of the unit interval), then the second moment is infinite, and so the asymptotic normality of the nonparametric estimator of $\pi_{g_\beta}(R)$. is violated.

The last situation motivate the need of a specific estimator of the reinsurance distorted risk premiums for dependence insured loses with heavy-tailed distributions and infinite second moments, that is with index in he upper half of the unit interval $(1/2 < \gamma < 1)$.

By making use the extreme value theory which offers satisfactory statistical results for such distributions, we need to estimate the tail index γ and establish a class of semi-parametric estimators of the distorted risk premium $p_g(R)$ in the case of dependence insured risks. The most popular positive tail index estimators in the framework of extreme value theory is the original Hill's estimator [29], defined as:

$$\widehat{\gamma}_k^{(H)} := \frac{1}{k}\sum_{i=1}^{k} \log X_{n-i+1,n} - \log X_{n-k,n}, \tag{9}$$

where $X_{1,n} \leq \cdots \leq X_{n,n}$ stands for the order statistics and $k = k(n)$ represents an intermediate sequence, that is, a sequence such that $k \to \infty$ and $k/n \to 0$, as $n \to \infty$. Using the tail quantile function $X_{n-[kt],n}$, $0 < t < n/k$, [19] proposed the following Kernel-type estimator of the tail index γ under β-mixing series:

$$\widehat{\gamma}_k^{(K)} = \int_0^1 \left(\log X_{n-[kt],n} - \log X_{n-k,n}\right) d(tK(t)), \tag{10}$$

where K is a kernel function integrated to one. Note that in the particular case where $K = \underline{K} := \mathbb{I}_{(0,1)}$, the estimator $\widehat{\gamma}_k^{\underline{K}}$ corresponds to the well-known Hill's estimator (9) of positive tail index γ. Also, it is easy to see that the kernel estimator $\widehat{\gamma}_k^{(K)}$ can be rewritten as follows:

$$\widehat{\gamma}_k^{(K)} := \frac{1}{k}\sum_{i=1}^{k}\left\{\frac{i}{k}K\left(\frac{i}{k}\right) - \frac{i-1}{k}K\left(\frac{i-1}{k}\right)\right\}\left(\log X_{n-i+1,n} - \log X_{n-k,n}\right).$$

2.2. Estimation of distortion risk under depedence serials

To better understand the heavier of tail distribution of insured risks, which is governed by the unknown tail index, many authors used the extreme value methodology and investigated semiparametric estimators of the distortion risk premiums in the case of β-mixing random variables, we use the class of kernel estimator defined in (11) and investigate semiparametric estimators of the distortion risk premiums with optimal retention levels.
Note that, the optimal retention level corresponds to the Value-at-Risk $Q(1-k/n)$, which is the minimum amount that a company must have to cover the risk X with an uncertain ruin probability. Since $\overline{F}(Q(1-k/n)) = k/n$, from (4), the optimal risk distorted reinsurance premiums is defined as:

$$\pi_{\beta,n} := \pi_{g_\beta}(Q(1-k/n)) = -\int_0^{k/n} g_\beta(s)dQ(1-s). \tag{11}$$

The distorted reinsurance premium $\pi_{\beta,n}$ is unknown since it depends on the unknown high quantile $Q(1-s)$, $s \to 0$. A Weissman-type estimator [21] of high quantiles for heavy-tailed distributions, based on the class of kernel estimators in (10), is defined as:

$$\widehat{Q}_k^{(K)}(1-s) = (ns/k)^{-\widehat{\gamma}_k^{(K)}} X_{n-k,n},\ s \to 0, \tag{12}$$

where $\widehat{\gamma}_k^{(K)}$ is the above class of kernel estimator for the extreme value index γ and the quantity $X_{n-k,n}$ is a moderate quantile and assigned to be the empirical estimator of the optimal retention level $Q(1-k/n)$.

Substituting in (11) the extreme quantile $Q(1-s)$, $s \to 0$ with its Weisman's type estimator $Q_k^{(K)}(1-s)$, we introduce the following class of semiparametric estimator for $\pi_{\beta,n}$:

$$\widehat{\pi}_{\beta,k,n}^{(K)} = -\int_0^{k/n} g_\beta(s)d\widehat{Q}_k^{(K)}(1-s), \tag{13}$$

which leads to:

$$\widehat{\pi}_{\beta,k,n}^{(K)} = \frac{\widehat{\gamma}_k^{(K)}}{1/\beta - \widehat{\gamma}_k^{(K)}}\, g_\beta(k/n)X_{n-k,n}, \tag{14}$$

provided that $\mathbb{P}(\widehat{\gamma}_k^{(K)} > 1/\beta) = o(1)$, for large values of n. In the particular case where $K = \underline{K} := \mathbb{I}_{(0,1)}$, $\widehat{\pi}_{\beta,k,n}^{(K)}$ corresponds exactly to the distortion risk premiums estimator introduced in [17] under dependence serials. From the second order regularly varying condition (C_{SO}) and the regularity conditions on the β-mixing coefficients (C_R), [17] established, the asymptotic normality of the estimator $\widehat{\pi}_{\beta,k,n}^{(K)}$.

3. Main Results

3.1. Asymptotic distribution of class of estimators $\widehat{\pi}^{(K)}_{\beta,k,n}$

In this section, we investigate the asymptotic distribution of the class of distortion premiums estimators $\widehat{\pi}^{(K)}_{\beta,k,n}$ introduced in (14). Clearly, this class of estimator is directly related to the kernel estimator $\widehat{\gamma}^{(K)}_{k}$ of the tail index γ.
Note that in extreme value theory (EVT), to prove the asymptotic distribution of the tail index estimators such as the Hill's estimator or the kernel-type one, we need a second order condition which specifies the rate of convergence for the left-hand side of the equations in (8) to their limits (See, eg. [30], [11] and [31]). This condition can be formulated in different ways as shown below. We will use the formulation later-on.

Second order regularly varying condition (C_{SO}). Suppose that there exists a positive or negative function A with $\lim_{t\to\infty} A(t) = 0$ and a real number $\rho \leq 0$ such that

$$\lim_{t\to\infty} \frac{1}{A(t)} \left(\frac{U(tx)}{U(t)} - x^{\gamma} \right) = x^{\gamma} \frac{x^{\rho}-1}{\rho}, \ \forall\, x > 0. \tag{15}$$

The rate of the convergence for the function A to 0 is essential since it helps to exhibit the bias term of the tail index estimators.

The asymptotic normality of the original Hill's estimator has been established for β-mixing sequences in [27] and [28]. Also, from the assumption that the intermediate sequence k is such that $k^{1/2}A(n/k) \to \lambda \in \mathbb{R}$, as $n \to \infty$ and assuming the following regularity conditions on the β-mixing coefficients:
Regularity conditions (C_R). There exist $\epsilon > 0$, a bivariate function r and a sequence ℓ_n such that, as $n \to \infty$,
(a) $\dfrac{\beta(\ell)}{\ell} n + \ell \dfrac{\log^2 k}{\sqrt{k}} \to 0$;
(b) $\dfrac{n}{\ell\, k} Cov \left(\sum_{i=1}^{\ell} \mathbb{I}_{\{X_i > F^{\leftarrow}(1-kx/n)\}}, \sum_{i=1}^{\ell} \mathbb{I}_{\{X_i > F^{\leftarrow}(1-ky/n)\}} \right) \to r(x,y), \ \forall\, 0 \leq x, y \leq 1+\epsilon$;
(c) For some constant C:

$$\frac{n}{\ell\, k} \mathbb{E} \left[\left(\sum_{i=1}^{\ell} \mathbb{I}_{\{F^{\leftarrow}(1-ky/n) < X_i \leq F^{\leftarrow}(1-kx/n)\}} \right)^4 \right] \leq C(y-x), \quad \forall\ 0 \leq x < y \leq 1+\epsilon \text{ and } n \in \mathbb{N}.$$

[19] showed, for a given kernel satisfying the following assumptions:
Condition $(\mathbb{K})$. Let K be a function defined on $(0, 1]$ such that
(i) $K(s) \geq 0$, whenever, $0 < s \leq 1$ and $K(1) = 0$; **(ii)** $K(\cdot)$ is differentiable, non increasing and right continuous on $(0, 1]$; **(iii)** K and K' are bounded; **(iv)** $\int_0^1 K(u)du = 1$;

(v) $\int_0^1 u^{-1/2}K(u)du < 1$, that:

$$\widehat{\gamma}_{n,k}^{(K)} \stackrel{d}{=} \gamma + A(n/k)\int_0^1 t^{-\rho}K(t)dt + k^{-1/2}\gamma \int_0^1 \left(t^{-1}W(t) - W(1)\right) d(tK(t)) + o_{\mathbb{P}}(k^{-1/2}), \tag{16}$$

where $(W(t))_{t\in[0,1]}$ is a Gaussian process with covariance function $r(.,.)$ given in (C_R) and is defined under a Skorohod construction.
In particular if $k^{1/2}A(n/k) \to \lambda \in \mathbb{R}$, as $n \to \infty$, we also have from Theorem 1 in [19]:

$$\sqrt{k}(\widehat{\gamma}_k^{(K)} - \gamma) \xrightarrow{d} \mathcal{N}\left(\lambda \mathcal{AB}_K(\rho), \mathcal{AV}_K(\gamma)\right), \tag{17}$$

where $\mathcal{AB}_K(\rho) = \int_0^1 t^{-\rho}K(t)dt$ and

$$\mathcal{AV}_K(\gamma) = \gamma^2 \iint_{[0,1]^2} \left\{\frac{r(t,s)}{ts} - \frac{r(t,1)}{t} - \frac{r(1,s)}{s} + r(1,1)\right\}(d(tK(t)))(d(sK(s))).$$

The conditions in $(\mathbb{K})$ are not restrictive but are satisfied by the usual weight functions used in the literature, including the power kernel $K(s) = (1+\tau)s^{\tau}\mathbb{I}_{\{0<s<1\}}$, $\tau \geq 0$, and the log-weight function $K(s) = (-\log s)^{\kappa}/\Gamma(\kappa+1)\mathbb{I}_{\{0<s<1\}}$, $\{\kappa \geq 1\}$. In particular, we note that the classical Hill's estimator in (9) can be viewed as a particular case of our power kernel-type estimator corresponding to $\tau = 0$ and $K(s) := \underline{K}(s) = \mathbb{I}_{\{0<s<1\}}$.

The following theorem establishes the asymptotic expansion of our class of distorted risk premiums estimators $\widehat{\pi}_{\beta,k,n}^{(K)}$ in terms of Gaussian process.

Theorem 1. *Let $(X_1, X_2, \dots)$ be a stationary β-mixing time series with a continuous common marginal distribution function F and assume that (C_{SO}) and (C_R) hold. Let K be a kernel function satisfying the condition $(\mathbb{K})$. If $k = k(n)$ is an intermediate sequence such that $k \to \infty$, $k/n \to 0$ and $\sqrt{k}A(n/k) = O(1)$, as $n \to \infty$, then for $1/2 < \gamma < 1$ and $\beta < 1/\gamma$, we have:*

$$\frac{\sqrt{k}\left(\widehat{\pi}_{\beta,k,n}^{(K)} - \pi_{\beta,n}\right)}{g_\beta(k/n)Q(1-k/n)} \xrightarrow{d} \lambda m_K(\gamma,\rho,\beta) + \frac{\beta\gamma^2}{1-\beta\gamma}W(1) + \frac{\gamma\beta}{(1-\beta\gamma)^2}\int_0^1 \left(t^{-1}W(t) - W(1)\right)d(tK(t)),$$

as $n \to \infty$, where

$$m_K(\gamma,\rho,\beta) = \frac{\beta}{(1-\beta\gamma)^2}\int_0^1 t^{-\rho}K(t)dt - \frac{\beta}{(1-\beta\gamma-\beta\rho)(1-\beta\gamma)}$$

and $(W(t))_{t\in[0,1]}$ is a Gaussian process with covariance function $r(.,.)$ defined in (C_R).

From Theorem 1, we deduce in the following corollary the asymptotic normality of the kernel estimator $\widehat{\pi}_{\beta,k,n}^{(K)}$.

Corollary 1. *Under the assum ptions of Theorem 1, we have:*

$$\frac{\sqrt{k}(\widehat{\pi}^{(K)}_{\beta,k,n} - \pi_{\beta,n})}{g_\beta(k/n)Q(1-k/n)} \xrightarrow{d} \mathcal{N}\Big(\lambda\, m_K(\gamma,\rho,\beta), \sigma^2_K(\gamma,\rho,\beta)\Big),$$

where

$$\sigma^2_K(\gamma,\rho,\beta) = \frac{(\beta\gamma)^2}{(1-\beta\,\gamma)^4}\iint_{[0,1]^2}\Big(\frac{r(t,s)}{ts} - \frac{r(t,1)}{t} - \frac{r(1,s)}{s} + r(1,1)\Big)d(sK(s))d(tK(t))$$
$$+ \frac{\beta^2\gamma^4}{(1-\beta\,\gamma)^2}\, r(1,1) + \frac{2\beta^2\gamma^3}{(1-\beta\,\gamma)^3}\int_0^1\Big(\frac{r(t,1)}{t} - r(1,1)\Big)d(tK(t)).$$

Thus, the Corollary1 generalizes Theorem 1 in [17] in the case $\lambda \neq 0$ when we use a general kernel function function K.

Clearly, from Corollary 1, the estimator $\widehat{\pi}^{(K)}_{\beta,k,n}$ suffer from a high bias due to the fact that its depends to kernel estimator $\widehat{\gamma}^{(K)}_k$, which from (17) as the same problem and the bias heavily depends on the intermediate sequence, making the choice of k difficult in practice. In the next section, we introduce a bias reduction estimator of the distortion risk premiums under dependence insured risks.

3.2. Reduced bias estimator for $\pi_{\beta,n}$

In this section, we propose to substitute in the estimation of the distortion risk premium $\pi_{\beta,n}$ introduced in (13), the class of Weissman's type estimator $\widehat{Q}^{(K)}_k$ defined in (12) with the asymptotically unbiased estimator of the extreme quantile under β mixing times series studied in[19].

Next, since $U(z) = Q(1-z^{-1})$, by using the second order condition (C_{SO}), the following approximation holds:

$$Q(1-s) \approx \Big(\frac{ns}{k}\Big)^{-\gamma} Q(1-k/n)\left\{1 - \frac{A(n/k)}{\rho}\left[1 - \Big(\frac{ns}{k}\Big)^{-\rho}\right]\right\},\ s \to 0, \tag{18}$$

where γ, $A(\cdot)$ and ρ are unknown. The first part $\left(\frac{n}{k}s\right)^{-\gamma} Q(1-k/n)$ in the right side of (18) is exactly estimated by the Weissman's type estimator $\widehat{Q}^{(K)}_k(1-s)$ and defined in (12). Clearly, the estimator $\widehat{Q}^{(K)}_k$ exhibits a potential bias because it depends on the Kernel type estimator $\widehat{\gamma}^{(K)}_k$ of the tail index γ, which from (17) has such problem. The expression $1 - \rho^{-1}A(n/k)[1 - \left(\frac{ns}{k}\right)^{-\rho}]$ can be viewed as a correcting term since $a(n/k)$ tends to 0. This leads to the necessity to find good estimators for γ, $a(n/k_n)$ and ρ.

According to [19], to introduce an asymptotically unbiased estimator for γ, one can consider two kernel functions K_1 and K_2 satisfying $(\mathbb{K})$ and to define a mixture of them in the form $K_{\Delta}(s) = \Delta K_1(s) + (1-\Delta) K_2(s)$, for $\Delta \in \mathbb{R}$. Clearly K_{Δ} also satisfies the condition ($\mathbb{K}$) and hence by the result given in (16), the asymptotic bias $\lambda \int_0^1 s^{-\rho} K_{\Delta}(s) ds$ of $\widehat{\gamma}_k^{(K_{\Delta})}$ is such that

$$\lambda \int_0^1 s^{-\rho} K_{\Delta}(s) ds = \lambda \Delta \int_0^1 s^{-\rho} K_1(s) ds + \lambda (1-\Delta) \int_0^1 s^{-\rho} K_2(s) ds.$$

Equating the right-hand side of the above equation to zero leads to the value of eliminating the asymptotic bias

$$\Delta^* = \frac{\int_0^1 s^{-\rho} K_2(s) ds}{\int_0^1 s^{-\rho} \Big\{ K_2(s) - K_1(s) \Big\} ds}, \tag{19}$$

provided $\int_0^1 s^{-\rho} \Big\{ K_2(s) - K_1(s) \Big\} ds \neq 0$. Clearly, the tail index estimator $\widehat{\gamma}_{n,k}^{(K_{\Delta^*})}$ is shown to be asymptotically unbiased in the sense that the mean of its limiting distribution is zero, whatever the value of λ. More precisely, we have from (17):

$$k^{1/2} \left(\widehat{\gamma}_k^{(K_{\Delta^*})} - \gamma \right) \xrightarrow{d} \mathcal{N}\Big(0, \mathcal{AV}_{K_{\Delta^*}}(\gamma)\Big). \tag{20}$$

Among this class of unbiased estimators $\widehat{\gamma}_{n,k}^{(K_{\Delta^*})}$, [19] found an estimator with minimum variance. According to these authors, the minimum of the asymptotic variance $\mathcal{AV}_{K_{\Delta^*}}(\gamma)$ is obtained at the "optimal" function given by:

$$K_{\Delta^*_{\text{opt}}}(s) = \left(\frac{1-\rho}{\rho} \right)^2 - \frac{(1-\rho)(1-2\rho)}{\rho^2} s^{-\rho}, \text{ for } \quad s \in (0,1), \tag{21}$$

and $K_{\Delta^*_{\text{opt}}}(s) = 0$ otherwise. Note that this unction can be viewed as a mixture between two power kernels: $K_1(s) := \underline{K}(s) = \mathbb{I}_{(0<s<1)}$ and $K_2(s) := K_{2,\rho}(s) := (1-\rho) s^{-\rho} \mathbb{I}_{(0<s<1)}$ and $\Delta^* = (1-\rho)^2/\rho^2$ is as in (19). In that case, the minimal variance $\gamma^2 \int_0^1 K^2_{\Delta^*_{\text{opt}}}(s) ds$ equals to $\gamma^2 (1-\rho)^2/\rho^2$.

From a practical point of view, the unbiased tail index estimator with minimum variance $\widehat{\gamma}_{n,k}^{(K_{\Delta^*_{\text{opt}}})}$ cannot be obtained directly, since it depends on the unknown parameters and expressions: γ, ρ, $a(n/k)$ and $K_{\Delta^*_{\text{opt}}}$ are unknown. To solve this issue, we propose to replace ρ by $\widehat{\rho}$, where $\widehat{\rho}$ is either a canonical negative value $\widehat{\rho} = \rho = \rho_0$ or an external estimator $\widehat{\rho} = \widehat{\rho}_{k_\rho}$, consistent in probability to ρ, with $k_\rho := k_\rho(n)$ an intermediate sequence of integers greater than k, satisfying $k_\rho \to \infty$ and $k_\rho/n \to 0$, as $n \to \infty$. Finally, as in (10), the resulting asymptotic unbiased estimator of the tail index is given as follows:

$$\widehat{\gamma}_k^{(K_{\widehat{\Delta}^*_{opt}})} = \frac{1}{k} \sum_{j=1}^{k} \left\{ \frac{j}{k} K_{\widehat{\Delta}^*_{opt}} \left(\frac{j}{k} \right) - \frac{j-1}{k} K_{\widehat{\Delta}^*_{opt}} \left(\frac{j-1}{k} \right) \right\} \log \left(\frac{X_{n-j+1,n}}{X_{n-k,n}} \right),$$

where $K_{\widehat{\Delta}^*_{opt}}$ is defined as $K_{\Delta^*_{opt}}$ in (21) with ρ replaced by $\widehat{\rho}$.

Next, for the estimation of the rate $A(\cdot)$, we use the result in (16) from which we have, as $n \to \infty$,

$$\widehat{\gamma}_k^{(\underline{K})} - \widehat{\gamma}_k^{(K_{2,\rho})} = -A(n/k)\frac{\rho^2}{(1-\rho)(1-2\rho)} + o_{\mathbb{P}}(1).$$

Thus, we can approximate

$$A(n/k)\frac{\rho^2}{(1-\rho)(1-2\rho)} \qquad \text{by} \qquad -\left\{\widehat{\gamma}_{n,k}^{(\underline{K})} - \widehat{\gamma}_{n,k}^{(K_{2,\rho})}\right\},$$

which mean that $A(n/k)$ can be estimated by;

$$\widehat{A}_{n,k}(\widehat{\rho}\,) := -\frac{(1-\widehat{\rho})(1-2\widehat{\rho})}{\widehat{\rho}^{\,2}}\left\{\widehat{\gamma}_{n,k}^{(\underline{K})} - \widehat{\gamma}_{n,k}^{(K_{2,\widehat{\rho}})}\right\}.$$

Finally, using the relation in (18), we arrive at the following unbiased estimator of the extreme quantile $Q(1-s)$, $s \to 0$:

$$\widehat{Q}_{k,\widehat{\rho}}^{(K_{\widehat{\Delta}^*_{opt}})}(1-s) = \left(\frac{ns}{k}\right)^{-\widehat{\gamma}_k^{(K_{\widehat{\Delta}^*_{opt}})}} X_{n-k,n}\left\{1 - \frac{\widehat{A}_{n,k}}{\rho}\left[1 - \left(\frac{ns}{k}\right)^{-\widehat{\rho}}\right]\right\}. \tag{22}$$

Under the second order condition (C_{SO}) and the Regularity assumption (C_R), [19] established the asymptotic normality of $\widehat{\gamma}_k^{(K_{\widehat{\Delta}^*_{opt}})}$. More precisely, if $k^{1/2}A(n/k) \to \lambda \in \mathbb{R}$, as $n \to \infty$, we have:

$$\widehat{\gamma}_k^{(K_{\widehat{\Delta}^*_{opt}})} \overset{d}{=} \gamma + k^{-1/2}\gamma \int_0^1 \left(t^{-1}W(t) - W(1)\right) d(tK_{\Delta^*_{opt}}(t)) + o_{\mathbb{P}}(k^{-1/2}). \tag{23}$$

This leads to

$$\sqrt{k}(\widehat{\gamma}_k^{(K_{\widehat{\Delta}^*_{opt}})} - \gamma) \overset{d}{\to} \mathcal{N}\left(0, \mathcal{AV}_{K_{\Delta^*_{opt}}}(\gamma)\right). \tag{24}$$

In the spirit of (13), substituting the extreme quantile $Q(1-s)$ with $\widehat{Q}_{k,\widehat{\rho}}^{(K_{\widehat{\Delta}^*_{opt}})}(1-s)$, we obtain the following unbiased estimator for the distorted risk measure $\pi_{\beta,n}$:

$$\widetilde{\pi}_{\beta,k,n\widehat{\rho}}^{(K_{\widehat{\Delta}^*_{opt}})} := \left\{\frac{\widehat{\gamma}_{n,k}^{(K_{\widehat{\Delta}^*_{\text{opt}}})}}{\frac{1}{\beta} - \widehat{\gamma}_{n,k}^{(K_{\widehat{\Delta}^*_{\text{opt}}})}} + \frac{\widehat{A}_{n,k}(\widehat{\rho}\,)}{\left(\frac{1}{\beta} - \widehat{\gamma}_{n,k}^{(K_{\widehat{\Delta}^*_{\text{opt}}})}\right)\left(1 - \beta\widehat{\gamma}_{n,k}^{(K_{\widehat{\Delta}^*_{\text{opt}}})} - \beta\widehat{\rho}\right)}\right\} g_\beta(k/n)\ X_{n-k,n}. \tag{25}$$

A possible choice for $\widehat{\rho}_{k_\rho}$ is the most performed estimator among those studied in the i.i.d. case (see, e.g, [32], [33]) and used in the β-mixing case by [20] and [19]:

$$\widehat{\rho}_{k_\rho} = \frac{6S_{k_\rho}^{(2)} - 4 + \left(3S_{k_\rho}^{(2)} - 2\right)^{1/2}}{4S_{k_\rho}^{(2)} - 3}, \quad \text{provided} \quad S_{k_\rho}^{(2)} \in \left(\frac{2}{3}, \frac{3}{4}\right), \tag{26}$$

where

$$S_{k_\rho}^{(2)} = \frac{3}{4}\frac{\left[M_{k_\rho}^{(4)} - 24\left(M_{k_\rho}^{(1)}\right)^4\right]\left[M_{k_\rho}^{(2)} - 2\left(M_{k_\rho}^{(1)}\right)^2\right]}{\left[M_{k_\rho}^{(3)} - 6\left(M_{k_\rho}^{(1)}\right)^3\right]^2}$$

and

$$M_{k_\rho}^{(r)} := \frac{1}{k_\rho}\sum_{j=1}^{k_\rho}\left(\log\frac{X_{n-j+1,n}}{X_{n-k_\rho,n}}\right)^r,\ r > 0.$$

The consistency and the asymptotic normality of $\widehat{\rho}_{k_\rho}$ have been established in [20] in the case of β-mixing serials under the second order condition C_{SO} and the assumptions $k_\rho \to \infty$, $k_\rho/n \to 0$ and $k_\rho^{1/2}A(n/k_\rho) \to \infty$, as $n \to \infty$.
Our next goal is to establish, under suitable assumptions, the asymptotic normality of $\widetilde{\pi}_{\beta,k,n,\widehat{\rho}}^{(K_{\widehat{\Delta}^*_{opt}})}$. This is done in the following theorem.

Theorem 2. *Under the assumptions of Theorem 1, if $\widehat{\rho}$ is either a canonical negative value $\widehat{\rho} = \rho = \rho_0$ or an external estimator $\widehat{\rho} = \widehat{\rho}_{k_\rho}$, consistent in probability to ρ, with $k_\rho := k_\rho(n)$, an intermediate sequence of integers greater than k, satisfying $k_\rho \to \infty$ and $k_\rho/n \to 0$, as $n \to \infty$, then we have:*

$$\frac{\sqrt{k}\left(\widetilde{\pi}_{\beta,k,n\widehat{\rho}}^{(K_{\widehat{\Delta}^*_{opt}})} - \pi_{\beta,n}\right)}{g_\beta(k/n)Q(1-k/n)} \overset{d}{\to} \mathcal{N}\left(0, \widetilde{\mathcal{AV}}(\gamma,\rho,\alpha)\right),$$

where

$$\begin{aligned}\widetilde{\mathcal{AV}}(\gamma,\rho,\alpha) &= (a_0)^2 r(1,1)\\
&+ (a_1)^2\int\int_{[0,1]^2}\left[\frac{r(t,s)}{ts} - \frac{r(t,1)}{t} - \frac{r(1,s)}{s} + r(1,1)\right]d\Big\{sK_{\hat{\Delta}^*_{opt}}(s)\Big\}d\Big\{tK_{\hat{\Delta}^*_{opt}}(t)\Big\}\\
&+ (a_3)^2\int\int_{[0,1]^2}\left[\frac{r(t,s)}{ts} - \frac{r(t,1)}{t} - \frac{r(1,s)}{s} + r(1,1)\right]d\Big\{s\,(1-K_{2,\rho}(s))\Big\}d\Big\{t\,(1-K_{2,\rho}(t))\Big\}\\
&+ 2a_0\,a_1\int_0^1\left[\frac{r(t,1)}{t} - r[1,1]\right]d\Big\{tK_{\hat{\Delta}^*_{opt}}(t)\Big\}\\
&+ 2a_0\,a_3\int_0^1\left[\frac{r(t,1)}{t} - r[1,1]\right]d\Big\{t\,(1-K_{2,\rho}(t))\Big\}\\
&+ 2a_1\,a_3\int\int_{[0,1]^2}\left[\frac{r(t,s)}{ts} - \frac{r(t,1)}{t} - \frac{r(1,s)}{s} + r(1,1)\right]d\Big\{sK_{\hat{\Delta}^*_{opt}}(s)\Big\}d\Big\{t\,(1-K_{2,\rho}(t))\Big\},\end{aligned}$$

with

$$a_0 = \frac{\beta\gamma^2}{1-\beta\gamma},\quad a_1 = \frac{\beta\gamma}{(1-\beta\gamma)^2}\quad \text{and}\quad a_3 = -\frac{\beta\gamma(1-\rho)(1-2\rho)}{\rho^2(1-\beta\gamma)(1-\beta\,\gamma-\beta\,\rho)}.$$

4. Simulation Study

In this section, the class of biased estimator $\widehat{\pi}_{\beta,k,n}^{(K)}$ and the reduced-bias estimator $\widetilde{\pi}_{\beta,k,n,\widehat{\rho}}^{(K_{\widehat{\Delta}^*_{opt}})}$ of the distortion risk measure $\pi_{\beta,n}$ with optimal retention level $Q(1-k/n)$ are compared in a simulation study. To this end, we consider the following classical stationary models, which satisfy the regularity (C_R) assumptions:

• (**Autoregressive (AR) model**): Consider first the stationary solution of the AR(1) equation:

$$X_i = \theta X_{i-1} + Z_i, \ i = 1, ..., n, \tag{27}$$

for some $\theta \in (0,1)$ and i.i.d. random variables Z_i. The distribution function of the innovations Z_i is denoted by F_Z. Assume that F_Z admits a positive Lebesgue density which is L_1 Lipschiz-continuous; see [27] eq. (42). Suppose that as $x \to \infty$, $1 - F_Z(x) \sim px^{-1/\gamma}\ell(x)$ and $F_Z(-x) \sim qx^{-1/\gamma}\ell(x)$, for some slowly varying function ℓ and $p = 1 - q \in (0,1)$. Then from Sect. 3.2 of [27], we get that $1 - F(x) \sim d_\theta(1 - F_Z(x))$, as $x \longrightarrow \infty$, where $d_\theta = (1-\theta^{1/\gamma})^{-1}$. Furthermore, the regularity conditions hold with/ $r(x,y) = x \wedge y + \sum_{m=1}^{\infty} (c_m(x,y) + c_m(y,x))$, where $c_m(x,y) = x \wedge y\theta^{m/\gamma}$.

• (**Moving average (MA) model**): Consider the stationary solution of MA(1) equation:

$$X_i = \theta Z_{i-1} + Z_i, \ i = 1, ..., n; \tag{28}$$

where the innovation Z_i satisfies the same conditions as in the above AR(1) model. And from Sect. 3.2 of [27], we obtain $1 - F(x) \sim d_\theta(1 - F_Z(x))$ as $x \to \infty$, where $d_\theta = 1 + \theta^{1/\gamma}$. One can also compute the covariance structure as : $r(x,y) = x \wedge y + (1+\theta^{1/\gamma})^{-1}(x \wedge y\theta^{1/\gamma} + y \wedge x\theta^{1/\gamma})$.

Now, we proceed by generating the data for the three (03) models. This involves an independent model and the two models mentioned above. We first generate the i.i.d innovations $(Z_1, ..., Z_n)$, such that:

$$F_Z(z) = \begin{cases} (1-q)(1-\widetilde{F}(-z)) & \text{if} \quad z < 0, \\ 1 - q + q\widetilde{F}(z)) & \text{if} \quad z > 0, \end{cases}$$

where $\widetilde{F}$ stands for the Fréchet distribution function $\widetilde{F}(z) = \exp((-z)^{-1/\gamma})$ for $z > 0$, and $p = 0.75$. Then F_Z belongs to the domain of attraction with extreme value index $\gamma > 0$. In the following table, we generate the three (03) time series models under simulation with their tail distribution, which are needed to compute the true distortion risk premiums:

For each generating model, we simulate $N = 1000$ samples with size $n = 1000$. To evaluate of the true value of distortion risk premiums $\pi_{\beta,n}$, we use the approximation of the tail distribution $\overline{F}$ given in Table 1. Also, we apply to each sample both estimators $\widehat{\pi}_{\beta,k,n}^{(K)}$ and $\widetilde{\pi}_{\beta,k,n,\widehat{\rho}}^{(K_{\widehat{\Delta}^*_{opt}})}$, for different integers of top order statistic $k = 1, ..., m$, where m is the number of positive values of the simulated

Description of Models	Independence $X_i = Z_i$	AR(1) $X_i = \theta X_{i-1} + Z_i$	MA(1) $X_i = \theta Z_{i-1} + Z_i$
Tail-distribution	$\overline{F}(x) = \overline{F}_Z(x)$	$\overline{F}(x) \sim (1-\theta^{1/\gamma})^{-1}\overline{F}_Z(x)$	$\overline{F}(x) \sim (1+\theta^{1/\gamma})\overline{F}_Z(x)$
coefficients	$\theta = 0$ and $\gamma = 0.6$	$\theta = 0.3$ and $\gamma = 0.6$	$\theta = 0.3$ and $\gamma = 0.6$

Table 1: Description of models under simulation with their associated tail distribution functions.

samples.

For computation and the comparison of the estimators, we adopt the following steps:

• The class of estimators $\widehat{\pi}^{(K)}_{\beta,k,n}$ is computed with the tail index estimators $\widehat{\gamma}^{(K)}_k$, $k = 1, ..., m_n$ and two different kernel function K satisfying the assumption ($\mathbb{K}$). The first kernel is the power function , defined as $K(s) = (1+\tau)s^\tau \mathbb{I}_{\{0<s<1\}}$, with τ taken in our study in $\{0, 1\}$. In the case where $\tau = 0$, we denote $K := K_1 = \underline{K}$ and $\widehat{\pi}^{(\underline{K})}_{\beta,k,n}$ corresponds to the classical distortion risk principle estimator studied in [17] under the β-mixing insured risks and which is associated to the Hill's estimator $\widehat{\gamma}^{(\underline{K})}_k$. For $\tau = 1$, the corresponding kernel is exactly the above mentioned $K := K_{2,\bar{\rho}}$, with $\bar{\rho} = -1$. the second kernel function is log-weight function $K(s) := K_{L,\kappa}(s) = (-\log s)^\kappa / \Gamma(\kappa+1)\mathbb{I}_{\{0<s<1\}}$, $\kappa \geq 1$ chosen as equals to one in this simulation stydy. Three biased estimators are then considered for this class of distorted risks estimates: $\widehat{\pi}^{(\underline{K})}_{\beta,k,n}$, $\widehat{\pi}^{(K_{2,\overline{\rho}})}_{\beta,k,n}$ for $\overline{\rho} = -1$ and $\widehat{\pi}^{(K_{L,\kappa})}_{\beta,k,n}$ for $\kappa = 1$.

• The asymptotic unbiased estimator $\widetilde{\pi}^{(K_{\widehat{\Delta}^*_{opt}})}_{\beta,k,n,\widehat{\rho}}$ is computed with the tail index estimators $\widehat{\gamma}_k^{(K_{\widehat{\Delta}^*_{opt}})}$, for $k = 1, ..., m_n$ and $\widehat{\rho} := \widehat{\rho}_{k^*_\rho}$ defined in (26), where k^*_ρ is selected as follows:

$$k^*_\rho := \sup\left\{k_\rho : \ k_\rho \leq \min\left(n-1, \frac{2n}{\log\log n}\right) \quad \text{and} \quad \widehat{\rho}_{k_\rho} \quad \text{exists}\right\}.$$

• Next, we compare on the one hand the performance of the mentioned reinsurance premium estimators by computing the absolute value of the mean together with the root mean squared errors (RMSE) based on the N samples, and defined as the following form:

$$\text{ABias}(\pi, k) := \left|\sum_{i-1}^{N} \frac{\widehat{\pi}^{(i)}}{\pi} - 1\right|$$

and

$$\text{RMSE}(\pi, k) := \sqrt{\sum_{i-1}^{N} \left(\frac{\widehat{\pi}^{(i)}}{\pi} - 1\right)^2},$$

where $\pi := \pi_{\beta,n}$ is the true value of the distortion risk premium with optimal retention level $Q(1-k/n)$, and $\widehat{\pi}^{(i)}$ is the i-th value $(i = 1, ..., N)$ of an estimator of $\widehat{\pi}^{(i)}$ evaluated as mentioned above different number of top order statistics $k = 1, ..., m_n$ with different aversion parameters $\beta \in \{1, \ 1.1\}$.

The results are displayed on the graphs in Figure 1 and Figure 2. Regarding the estimation of the distortion risk premiums, we observe from the graphs on the left-side of Figure 1 and Figure 2 that our goal in reducing the bias is well illustrated on finite sample behavior, when using large values of top order statistics k. In addition, on the graphs on the right-side of Figure 1 and Figure 2, the RMSE of our reduced bias estimator $\widetilde{\pi}_{\beta,k,n,\widehat{\rho}}^{(K_{\widehat{\Delta}^*_{opt}})}$, stays at a lower level than that of the class of biased estimators $\widehat{\pi}_{\beta,k,n}^{(\underline{K})}$, $\widehat{\pi}_{\beta,k,n}^{(K_{2,\overline{\rho}})}$ and $\widehat{\pi}_{\beta,k,n}^{(K_{L,\kappa})}$, for large values of k, whatever the value of the aversion index β. We also observe that the reduction in RMSE is higher for dependent series than for independent series. We conclude that the simulation studies show that under bias reduction procedure, the estimators for the distortion risk premiums remain stable for a wider range of k values even if the observations of insured risks exhibit serial dependence. Thus, the bias reduction method under dependence serials helps to tackle the two major critiques for applying extreme value statistics to time series in insurance data.

As mentioned above, it is also crucial to compare the estimators at their optimal number k of top extreme risks. To this end, we use the algorithm of [15], Page 137, which gives an automatic choice of k for any estimator $\widehat{\gamma}_k^{\bullet}$ of the tail index γ. According to these authors, an automatic choice of top extremes as the value k^* that minimizes

$$\frac{1}{k}\sum_{j=1}^{k} j^{\delta}\left|\widehat{\gamma}_j^{\bullet} - \text{median}\left(\widehat{\gamma}_1^{\bullet}, ..., \widehat{\gamma}_k^{\bullet}\right)\right|, \tag{29}$$

where $1 \leq k \leq m$ and $0 \leq \delta < 1/2$. By the way, choosing $\delta = 1/4$, we compute the optimal values k^* as in (29) for each tail index estimator used in the computation of their associated distortion risk premium estimators $\widehat{\pi}_{\beta,k,n}$ and $\widetilde{\pi}_{\beta,k,n,\widehat{\rho}}$.

In Table 2, we present the results of the estimated values of the above mentioned distortions risk premium estimators. Since their asymptotic variances depend on some unknown parameters, we opt to use a block bootstrapping method to construct a 95% confidence interval for the reinsurance premiums.

The block bootstrapping follows the routine boot of the package **boot** in R software. By repeating such bootstrapping procedure $T = 10,000$ times, we obtain T bootstrapped estimates for each distortion risk premium estimator. The sample standard deviation across the T estimates gives an estimate of the standard deviation of the underlying estimators for a given $k \in \{1, ..., m\}$. We construct the 95% confidence interval using the point estimate and the estimated standard deviation. This procedure is applied to all values of k of each estimator. The point estimates of the distortion risk premium at its optimal value k^* as well as the lower bound (Lb), upper bound (Ub) and the cover of the confidence intervals are given in Table 2.

After the inspection of the table, two conclusions can be drawn regardless of the situation. First, we notice that the absolute bias of the reduced bias estimator $\widetilde{\pi}_{\beta,k,n,\widehat{\rho}}^{(K_{\widehat{\Delta}^*_{opt}})}$ is lower than the class of biased estimators $\widehat{\pi}_{\beta,k,n}^{(\underline{K})}$, $\widehat{\pi}_{\beta,k,n}^{(K_{2,\overline{\rho}})}$ and $\widehat{\pi}_{\beta,k,n}^{(K_{L,\kappa})}$,. Second, the reduced bias estimator $\widetilde{\pi}_{\beta,k,n,\widehat{\rho}}^{(K_{\widehat{\Delta}^*_{opt}})}$ is more efficient than the class of biased estimators regardless to the the root median squared errors and the cover values. That illustrates well our conclusions drawn from the graphical analysis.

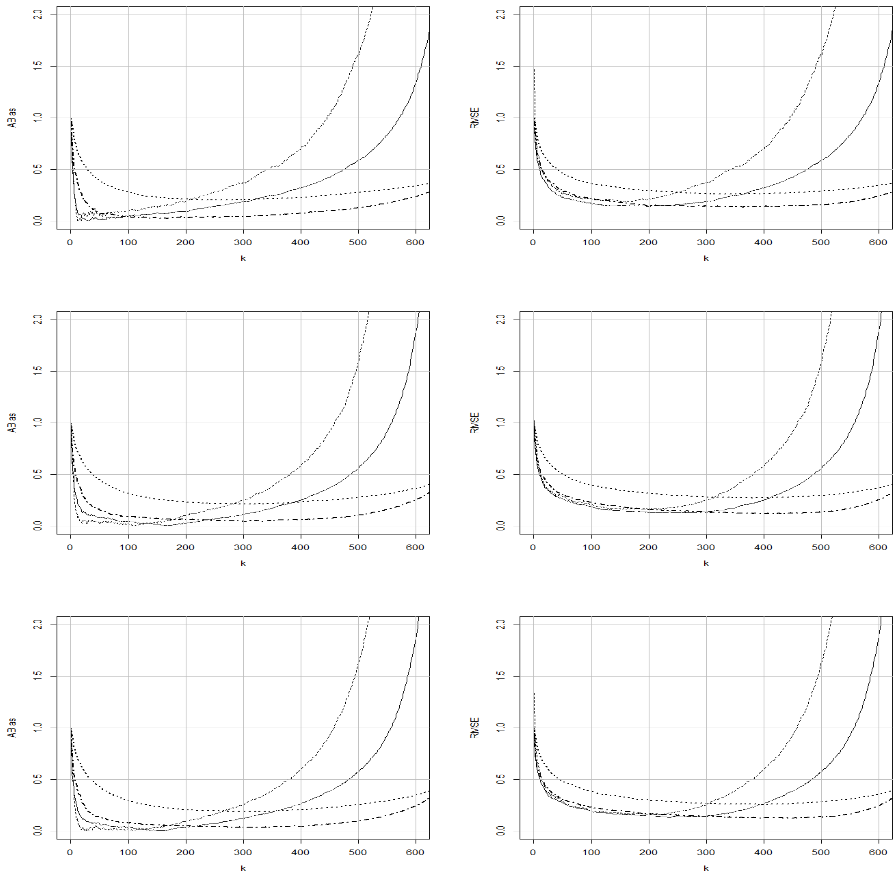

Figure 1: Absolute Bias of the median (left column) and Root median squared error (right column) of $\widehat{\pi}^{(K)}_{\beta,k,n}$ (full line), $\widehat{\pi}^{(K_{2,\overline{\rho}})}_{\beta,k,n}$ (dashed line), $\widehat{\pi}^{(K_{L,\kappa})}_{\beta,k,n}$ (dotted line) and $\widetilde{\pi}^{(K_{\widehat{\Delta}^*_{opt}})}_{\beta,k,n,\widehat{\rho}}$ (dotdash line) as a function of k based on $N = 1000$ samples of size 1000 of the models in Table 1: independence model (top), AR(1) model (middle) and MA(1) model (down) for the distortion risk premiums with aversion index $\beta = 1$.

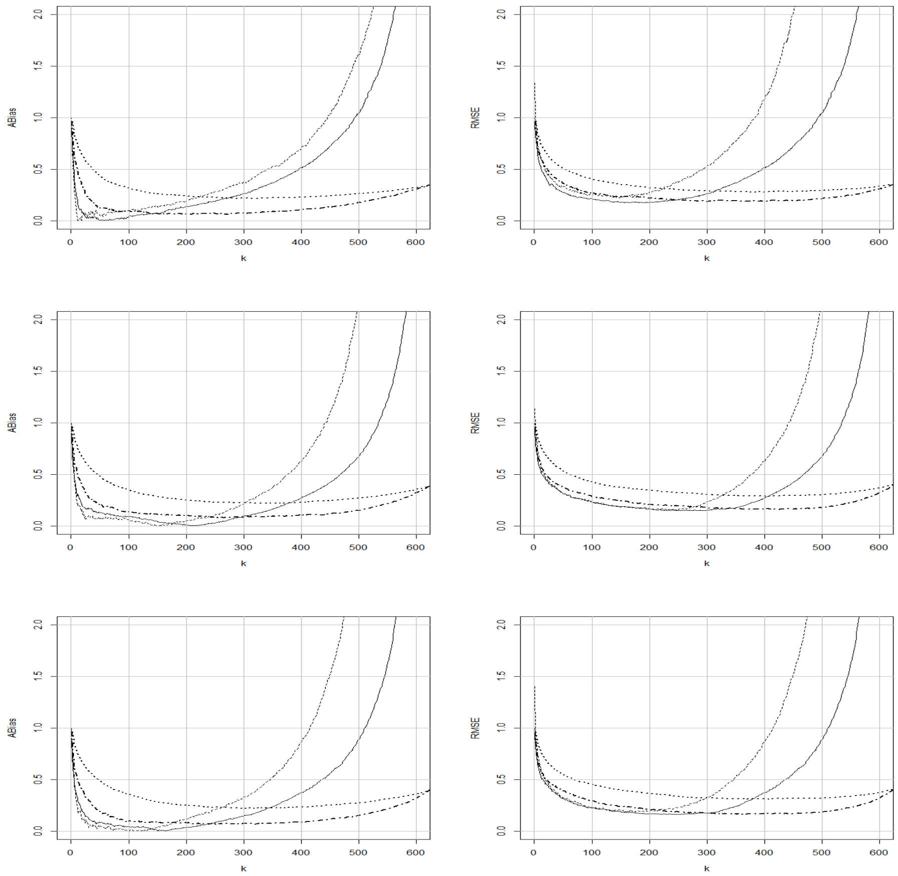

Figure 2: Absolute Bias of the median (left column) and Root median squared error (right column) of $\widehat{\pi}^{(K)}_{\beta,k,n}$ (full line), $\widehat{\pi}^{(K_{2,\overline{\rho}})}_{\beta,k,n}$ (dashed line), $\widehat{\pi}^{(K_{L,\kappa})}_{\beta,k,n}$ (dotted line) and $\widetilde{\pi}^{(K_{\widehat{\Delta}^{*}_{opt}})}_{\beta,k,n,\widehat{\rho}}$ (dotdash line) as a function of k based on $N = 1000$ samples of size 1000 of the models in Table 1: independence model (top), AR(1) model (middle) and MA(1) model (down) for the distortion risk premiums with aversion index $\beta = 1.1$.

Independance Model (defined in Table 1)															
$\beta = 1, \quad \pi_{\beta,n} = 0.830$								$\beta = 1.1, \quad \pi_{\beta,n} = 1.126$							
$\widehat{\pi}^{(\underline{K})}_{\beta,k,n}$	1.010	$\widehat{\pi}^{(K_{2,\overline{\rho}})}_{\beta,k,n}$	1.202	$\widehat{\pi}^{(K_{L,\kappa})}_{\beta,k,n}$	0.657	$\widetilde{\pi}^{(K_{\widehat{\Delta}^*_{opt}})}_{\beta,k,n,\widehat{\rho}}$	0.791	$\widehat{\pi}^{(\underline{K})}_{\beta,k,n}$	1.403	$\widehat{\pi}^{(K_{2,\overline{\rho}})}_{\beta,k,n}$	1.678	$\widehat{\pi}^{(K_{L,\kappa})}_{\beta,k,n}$	0.879	$\widetilde{\pi}^{(K_{\widehat{\Delta}^*_{opt}})}_{\beta,k,n,\widehat{\rho}}$	1.054
ABais	0.216	ABais	0.447	ABais	0.208	ABais	0.047	ABais	0.245	ABais	0.489	ABais	0.218	ABais	0.064
RMSE	0.216	RMSE	0.447	RMSE	0.259	RMSE	0.137	RMSE	0.247	RMSE	0.489	RMSE	0.280	RMSE	0.176
B-inf	0.067	B-inf	0.090	B-inf	0.102	B-inf	0.179	B-inf	0.046	B-inf	0.068	B-inf	0.154	B-inf	0.291
B-sup	6.308	B-sup	6.588	B-sup	0.969	B-sup	0.812	B-sup	8.180	B-sup	7.081	B-sup	1.350	B-sup	1.126
Cover	6.241	Cover	6.498	Cover	0.867	Cover	0.633	Cover	8.134	Cover	7.013	Cover	1.196	Cover	0.835
AR(1) model (defined in Table 1)															
$\beta = 1, \quad \pi_{\beta,n} = 0.912$								$\beta = 1.1, \quad \pi_{\beta,n} = 1.256$							
$\widehat{\pi}^{(\underline{K})}_{\beta,k,n}$	1.039	$\widehat{\pi}^{(K_{2,\overline{\rho}})}_{\beta,k,n}$	1.1.207	$\widehat{\pi}^{(K_{L,\kappa})}_{\beta,k,n}$	0.799	$\widetilde{\pi}^{(K_{\widehat{\Delta}^*_{opt}})}_{\beta,k,n,\widehat{\rho}}$	0.864	$\widehat{\pi}^{(\underline{K})}_{\beta,k,n}$	1.395	$\widehat{\pi}^{(K_{2,\overline{\rho}})}_{\beta,k,n}$	1.588	$\widehat{\pi}^{(K_{L,\kappa})}_{\beta,k,n}$	0.988	$\widetilde{\pi}^{(K_{\widehat{\Delta}^*_{opt}})}_{\beta,k,n,\widehat{\rho}}$	1.172
ABais	0.139	ABais	0.324	ABais	0.218	ABais	0.051	ABais	0.110	ABais	0.263	ABais	0.213	ABais	0.067
RMSE	0.153	RMSE	0.324	RMSE	0.270	RMSE	0.1294	RMSE	0.157	RMSE	0.268	RMSE	0.314	RMSE	0.178
B-inf	0.067	B-inf	0.013	B-inf	0.099	B-inf	0.286	B-inf	0.026	B-inf	0.098	B-inf	0.169	B-inf	0.427
B-sup	5.119	B-sup	6.268	B-sup	1.702	B-sup	1.062	B-sup	6.710	B-sup	7.047	B-sup	2.957	B-sup	1.279
Cover	5.052	Cover	6.255	Cover	1.603	Cover	0.776	Cover	6.684	Cover	6.949	Cover	2.788	Cover	0.852
MA(1) model (defined in Table 1)															
$\beta = 1, \quad \pi_{\beta,n} = 0.873$								$\beta = 1.1, \quad \pi_{\beta,n} = 1.275$							
$\widehat{\pi}^{(\underline{K})}_{\beta,k,n}$	0.989	$\widehat{\pi}^{(K_{2,\overline{\rho}})}_{\beta,k,n}$	1.119	$\widehat{\pi}^{(K_{L,\kappa})}_{\beta,k,n}$	0.701	$\widetilde{\pi}^{(K_{\widehat{\Delta}^*_{opt}})}_{\beta,k,n,\widehat{\rho}}$	0.843	$\widehat{\pi}^{(\underline{K})}_{\beta,k,n}$	1.534	$\widehat{\pi}^{(K_{2,\overline{\rho}})}_{\beta,k,n}$	1.826	$\widehat{\pi}^{(K_{L,\kappa})}_{\beta,k,n}$	0.988	$\widetilde{\pi}^{(K_{\widehat{\Delta}^*_{opt}})}_{\beta,k,n,\widehat{\rho}}$	1.198
ABais	0.132	ABais	0.281	ABais	0.197	ABais	0.034	ABais	0.202	ABais	0.431	ABais	0.224	ABais	0.060
RMSE	0.149	RMSE	0.281	RMSE	0.271	RMSE	0.134	RMSE	0.209	RMSE	0431	RMSE	0.309	RMSE	0.170
B-inf	0.178	B-inf	0.241	B-inf	0.312	B-inf	0.180	B-inf	0.253	B-inf	0.328	B-inf	0.150	B-inf	0.391
B-sup	5.180	B-sup	6.571	B-sup	2.574	B-sup	0.998	B-sup	7.1632	B-sup	7.2436	B-sup	2.087	B-sup	1.280
Cover	5.002	Cover	6.330	Cover	2.262	Cover	0.818	Cover	6.910	Cover	6.915	Cover	1.937	Cover.	0.889

Table 2: Estimation results of $\widehat{\pi}^{(\underline{K})}_{\beta,k,n}$, $\widehat{\pi}^{(K_{2,\overline{\rho}})}_{\beta,k,n}$,$\widehat{\pi}^{(K_{L,\kappa})}_{\beta,k,n}$ and $\widetilde{\pi}^{(K_{\widehat{\Delta}^*_{opt}})}_{\beta,k,n,\widehat{\rho}}$ estimators of the true distortion premium $\pi_{\beta,n} := \pi_\beta(Q(1-k^*/n))$ for $\beta = 1$; 1.1 and with their 95% confidence intervals, computed with their associated optimal numbers of top statistics k^*, based on $N = 1000$ samples of size $n = 1000$, from the different three (03) models listed in Table 1.

5. Conclusion

In this paper, we introduced a large class of asymptotically normal estimators of the distortion risk measures at the optimal retention level for stationary insured risks with heavy-tailed marginals. From that class, we derived a bias reduction procedure and we proposed an unbiased estimator of the distortion risk measures. Comparing the bias reduction procedure to the alternative estimators, our unbiased estimator provides, in addition to lower absolute bias and mean squared error in general, more stability over the number of top statistics k, especially when bias of the alternative estimators are strong. The comparison are also made at their optimal point of top statistics and with their 95% confidence intervals, constructed from a Bootstrap methodology. The results show that, the reduced bias estimator is more efficient than alternative estimators regardless to the absolute bias, the medan squared errors and the coverage. An important feature expected in this type of of bias reduction approach to be used in practice. In reinsurance application, the unbiased estimator can be proposed to any dependence heavy-tailed loses for which distortion risk premiums need to be calculated.

6. Proofs of the results

Proof of Theorem 1. Recall that

$$\pi_{\beta,n} := \pi_{g_\beta}(Q(1-k/n)) = \int_{Q(1-k/n)}^{\infty} g_\beta(\overline{F}(x))dx$$

and

$$\widehat{\pi}_{\beta,k,n}^{(K)} = \frac{\widehat{\gamma}_k^{(K)}}{\frac{1}{\beta} - \widehat{\gamma}_k^{(K)}}\, g_\beta(k/n) X_{n-k,n}.$$

Next, we have

$$\widehat{\pi}_{\beta,k,n}^{(K)} - \pi_{\beta,n} = \frac{\widehat{\gamma}_k^{(K)}}{\frac{1}{\beta} - \widehat{\gamma}_k^{(K)}}\, g_\beta(k/n) X_{n-k,n} - \int_{Q(1-k/n)}^{\infty} g_\beta(\overline{F}(x))dx$$

Let consider

$$\begin{aligned}
A_1 &= \left(\frac{\widehat{\gamma}_k^{(K)}}{\frac{1}{\beta} - \widehat{\gamma}_k^{(K)}} - \frac{\gamma}{\frac{1}{\beta} - \gamma} \right) g_\beta(k/n) X_{n-k,n} \\
A_2 &= \frac{\gamma}{\frac{1}{\beta} - \gamma} \left(\frac{X_{n-k,n}}{Q(1-k/n)} - 1 \right) g_\beta(k/n) Q(1-k/n) \\
A_3 &= \frac{\gamma}{\frac{1}{\beta} - \gamma} g_\beta(k/n) Q(1-k/n) - \int_{Q(1-k/n)}^{\infty} g_\beta(\overline{F}(x))dx.
\end{aligned}$$

It easy to verify that:

$$\widehat{\pi}_{\beta,n}^{(K)} - \pi_{\beta,n} = A_1 + A_2 + A_3.$$

Under the assumptions of Theorem 1, if $k^{1/2}A(n/k) \to \lambda \in \mathbb{R}$, as $n \to \infty$, we have from (16),

$$k^{1/2}\left(\widehat{\gamma}_{n,k}^{(K)} - \gamma\right) \stackrel{d}{=} \lambda \int_0^1 t^{-\rho} K(t)dt + \gamma \int_0^1 \left(t^{-1}W(t) - W(1)\right) d(tK(t)) + o_{\mathbb{P}}(1),$$

as $n \to \infty$, where $(W(t))_{t\in[0,1]}$ is a Gaussian process with covariance function $r(.,.)$ given in (C_R). In particular, this lead to $\widehat{\gamma}_k^{(K)} \xrightarrow{\mathbb{P}} \gamma$, as $n \to \infty$.

Therefore, using the Delta-method procedure, we get for all n large enough:

$$A_1 \stackrel{d}{=} (1 + o_{\mathbb{P}}(1)) \frac{\beta}{(1-\beta\gamma)^2} k^{-1/2} g_\beta(k/n) X_{n-k,n} \left\{ \lambda \int_0^1 t^{-\rho} K(t)dt + \gamma \int_0^1 \Big(t^{-1}W(t) - W(1)\Big) d(tK(t)) \right\}.$$

Next, from [19], Proposition 1, we have for all n large enough:

$$\sqrt{k}\left(\frac{X_{n-k,n}}{Q(1-k/n)} - 1\right) \stackrel{d}{=} \gamma W(1) + o_{\mathbb{P}}(1). \tag{30}$$

This implies that $X_{n-k,n}/Q(1-k/n) = 1 + o_{\mathbb{P}}(1)$. And then, for all n large enough, we get

$$\frac{\sqrt{k}\, A_1}{g_\beta(k/n)Q(1-k/n)} \stackrel{d}{=} (1+o_{\mathbb{P}}(1)) \frac{\beta}{(1-\beta\gamma)^2} \left\{ \lambda \int_0^1 t^{-\rho} K(t)dt + \gamma \int_0^1 \Big(t^{-1}W(t) - W(1)\Big) d(tK(t)) \right\}. \tag{31}$$

Similarly, using again (30), we get for all n large enough:

$$\frac{\sqrt{k}\, A_2}{g_\beta(k/n)Q(1-k/n)} \stackrel{d}{=} (1 + o_{\mathbb{P}}(1)) \frac{\beta\gamma^2}{1-\beta\gamma} W(1). \tag{32}$$

For the term A_3, we have:

$$\frac{\sqrt{k}\, A_3}{g_\beta(k/n)Q(1-k/n)} = \sqrt{k}\left(\frac{\gamma\beta}{1-\beta\gamma} - \frac{\pi_{\beta,n}}{Q(1-k/n)g_\beta(k/n)}\right).$$

Since $\pi_{\beta,n} = \int_{U(n/k)}^{\infty} g_\beta(\overline{F}(x))dx$ and $U(t) = Q(1-1/t)$, $t \geq 1$, a change of variables with $x = U(nt/k)$ yields to:

$$\pi_{\beta,n} = \int_1^\infty g_\beta(k/nt)\, dU(nt/k).$$

Since $g_\beta(x) = x^{1/\beta}$ and froù (8), $U(\cdot)$ is a regularly varying function with index $\gamma > 0$, then $g_\beta(k/nt)U(k/nt) \to 0$, as $t \to \infty$. Thus, an integration by parts yields to:

$$\pi_{\beta,n} = g_\beta(k/n)\frac{1}{\beta}\int_1^\infty t^{-1/\beta-1}(U(nt/k) - U(n/k))dt.$$

Therefore, by using again $U(n/k) = Q(1-k/n)$, we get:

$$\begin{aligned} \frac{\sqrt{k}A_3}{g_\beta(k/n)Q(1-k/n)} &= \sqrt{k}\left[\frac{\beta\gamma}{1-\beta\gamma} - \frac{1}{\beta}\int_1^\infty t^{-1/\beta-1}\left(\frac{U(nt/k)}{U(n/k)} - 1\right)dt\right] \\ &= -\frac{1}{\beta}\sqrt{k}\int_1^\infty t^{-1-1/\beta}\left(\frac{U(nt/k)}{U(n/k)} - t^\gamma\right)dt. \end{aligned} \tag{33}$$

Assume that the second order condition (C_{SO}) holds. From Theorem B.2.18 in [11], we have we have for a possibly different function $\widetilde{A}$, with $\widetilde{A}(z) \sim A(z)$, $z \to \infty$, and for all $\varepsilon,\ \delta > 0$, there exists some positive number $z_0 = z_0(\varepsilon, \delta)$ such that for $tz \geq z_0$:

$$\left| \frac{\frac{U(tz)}{U(z)} - t^\gamma}{\tilde{A}(z)} - t^\gamma \frac{t^\rho - 1}{\rho} \right| \leq \varepsilon t^{\rho+\gamma} \max(t^\delta, t^{-\delta}). \tag{34}$$

Since $k^{1/2}A(n/k) \to \lambda \in \mathbb{R}$, then from (33) and (34), we have for $0 < \gamma < 1$, $\gamma < 1/\beta$ and for all values of n large enough:

$$\begin{aligned}\frac{\sqrt{k}A_3}{g_\beta(k/n)Q(1-k/n)} &= -\frac{\lambda}{\beta}\int_1^\infty t^{\gamma-1/\beta-1}\frac{t^\rho-1}{\rho}dt(1+o(1)) \\ &= -\frac{\lambda\beta}{(1-\beta\gamma-\beta\rho)(1-\beta\gamma)}(1+o(1)).\end{aligned} \tag{35}$$

Finlay, form (31), (32) and (35), we get as $n \to \infty$:

$$\begin{aligned}\frac{\sqrt{k}\left(\widehat{\pi}^{(K)}_{\beta,k,n}-\pi_{\beta,n}\right)}{g_\beta(k/n)Q(1-k/n)} \overset{d}{\to} \lambda &\left\{\frac{\beta}{(1-\beta\gamma)^2}\int_0^1 t^{-\rho}K(t)dt - \frac{\beta}{(1-\beta\gamma-\beta\rho)(1-\beta\gamma)}\right\} \\ &+ \frac{\gamma\beta}{(1-\beta\,\gamma)^2}\int_0^1\left(t^{-1}W(t)-W(1)\right)d(tK(t)) + \frac{\beta\gamma^2}{1-\beta\,\gamma}W(1).\end{aligned} \tag{36}$$

Proof of Corollary 1. Computing the variance of the Gaussian terms appeared in the right side of (36) with respect to the covariance structure $r(.,.)$, the proof of Corollary 1 holds.

Proof of Theorem 2. Recall that

$$\widetilde{\pi}^{(K_{\hat{\Delta}^*_{opt}})}_{\beta,k,n\widehat{\rho}} : = \left\{\frac{\widehat{\gamma}^{(K_{\hat{\Delta}^*_{\rm opt}})}_{n,k}}{\frac{1}{\beta}-\widehat{\gamma}^{(K_{\hat{\Delta}^*_{\rm opt}})}_{n,k}} + \frac{\widehat{A}_{n,k}(\widehat{\rho})}{\left(\frac{1}{\beta}-\widehat{\gamma}^{(K_{\hat{\Delta}^*_{\rm opt}})}_{n,k}\right)\left(1-\beta\widehat{\gamma}^{(K_{\hat{\Delta}^*_{\rm opt}})}_{n,k}-\beta\widehat{\rho}\right)}\right\}g_\beta(k/n)\ X_{n-k,n}.$$

where

$$\widehat{A}_{n,k}(\widehat{\rho}) := -\frac{(1-\widehat{\rho})(1-2\widehat{\rho})}{\widehat{\rho}^{\,2}}\left\{\widehat{\gamma}^{(\underline{K})}_{n,k} - \widehat{\gamma}^{(K_{2,\widehat{\rho}})}_{n,k}\right\}$$

is the estimator of $A(n/k)$.
Next, as in the proof of Theorem 1, we have

$$\widehat{\pi}^{(K_{\hat{\Delta}^*_{opt}})}_{\beta,R} - \pi_{\beta,n} = H_1 + H_2 + H_3$$

with

$$\begin{aligned}H_1 &= \left(\frac{\widehat{\pi}^{(K_{\hat{\Delta}^*_{opt}})}}{\frac{1}{\beta}-\widehat{\pi}^{(K_{\hat{\Delta}^*_{opt}})}} - \frac{\gamma}{\frac{1}{\beta}-\gamma}\right)g_\beta(k/n)X_{n-k,n} \\ H_2 &= \frac{\gamma}{\frac{1}{\beta}-\gamma}\left(\frac{X_{n-k,n}}{Q(1-k/n)}-1\right)g_\beta(k/n)Q(1-k/n) \\ H_3 &= \frac{\gamma}{\frac{1}{\beta}-\gamma}g_\beta(k/n)Q(1-k/n) - \int_{Q(1-k/n)}^\infty g_\beta(\overline{F}(x))dx \\ H_4 &= \frac{\widehat{A}_{n,k}(\widehat{\rho})}{\left(\frac{1}{\beta}-\widehat{\gamma}^{(K_{\hat{\Delta}^*_{\rm opt}})}_{n,k}\right)\left(1-\beta\widehat{\gamma}^{(K_{\hat{\Delta}^*_{\rm opt}})}_{n,k}-\beta\widehat{\rho}\right)}g_\beta(k/n)\ X_{n-k,n}.\end{aligned}$$

Under assumptions and for all n large enough, we have from (23),

$$\sqrt{k}\left(\widehat{\gamma}^{(K_{\hat{\Delta}^*_{opt}})}_k - \gamma\right) \overset{d}{=} \gamma\int_0^1\left(t^{-1}W(t)-W(1)\right)d(tK_{\Delta^*_{opt}}(t))\left\{1+o_{\mathbb{P}}(1)\right\}. \tag{37}$$

Therefore, using the Delta-method procedure, we get for all n large enough:

$$H_1 \overset{d}{=} (1+o_{\mathbb{P}}(1))\frac{\beta}{(1-\beta\,\gamma)^2}k^{-1/2}g_\beta(k/n)X_{n-k,n}\left\{\gamma\int_0^1\Big(t^{-1}W(t)-W(1)\Big)d(tK_{\hat{\Delta}^*_{opt}}(t)))\right\},$$

Next, using (30), we have implies $X_{n-k,n}/Q(1-k/n)=1+o_{\mathbb{P}}(1)$, as $n\to\infty$ and

$$\frac{\sqrt{k}\,H_1}{g_\beta(k/n)Q(1-k/n)} \overset{d}{=} (1+o_{\mathbb{P}}(1))\frac{\gamma\beta}{(1-\beta\,\gamma)^2}\int_0^1\Big(t^{-1}W(t)-W(1)\Big)d(tK_{\hat{\Delta}^*_{opt}}(t)), \tag{38}$$

as $n\to\infty$. Similarly, we have for all n large enough:

$$\frac{\sqrt{k}\,H_2}{g_\beta(k/n)Q(1-k/n)} \overset{d}{=} (1+o_{\mathbb{P}}(1))\frac{\gamma\beta}{1-\beta\,\gamma}\sqrt{k}\left(\frac{X_{n-k,n}}{Q(1-k/n)}-1\right).$$

Using again (30), we get:

$$\frac{\sqrt{k}\,H_2}{g_\beta(k/n)Q(1-k/n)} \overset{d}{=} (1+o_{\mathbb{P}}(1))\frac{\beta\gamma^2}{1-\beta\,\gamma}W(1). \tag{39}$$

For the term H_3, we note that it is exactly equal to the term A_3 in the proof of the Theorem 1. Therefore, using the statement in (35), we have for all values of n large enough,

$$\frac{\sqrt{k}H_3}{g_\beta(k/n)Q(1-k/n)} = -\sqrt{k}A(n/k)\,\frac{\beta}{(1-\beta\gamma)(1-\beta\gamma-\beta\rho)}(1+o(1)). \tag{40}$$

Now, for the term H_4, using the fact that $X_{n-k,n}/Q(1-k/n)=1+o_{\mathbb{P}}(1)$, we have

$$\frac{\sqrt{k}H_4}{g_\beta(k/n)Q(1-k/n)} = \sqrt{k}\,\widehat{A}_{n,k}(\widehat{\rho})\frac{\beta}{\Big(1-\beta\,\widehat{\gamma}_{n,k}^{(K_{\hat{\Delta}^*_{\rm opt}})}\Big)\Big(1-\beta\,\widehat{\gamma}_{n,k}^{(K_{\hat{\Delta}^*_{\rm opt}})}-\beta\,\widehat{\rho}\Big)}(1+o_{\mathbb{P}}(1)).$$

Since $\widehat{\gamma}_k^{(K_{\hat{\Delta}^*_{opt}})}$ and $\widehat{\rho}:=\hat{\rho}_{k_\rho}$ are respectively consistent to γ and ρ, we obtain:

$$\frac{\sqrt{k}H_4}{g_\beta(k/n)Q(1-k/n)} \overset{d}{=} \sqrt{k}\,\widehat{A}_{n,k}(\widehat{\rho})\,\frac{\beta}{(1-\beta\,\gamma)(1-\beta\,\gamma-\beta\,\rho)}(1+o_{\mathbb{P}}(1)). \tag{41}$$

Hence, from (40) and (41), we get for all large n,

$$\frac{\sqrt{k}(H_3+H_4)}{g_\beta(k/n)Q(1-k/n)} = \sqrt{k}\,\left(\widehat{A}_{n,k}(\widehat{\rho})-A(n/k)\right)\frac{\beta}{(1-\beta\gamma)(1-\beta\,\gamma-\beta\,\rho)}(1+o_{\mathbb{P}}(1)).$$

Recall from that

$$\widehat{A}_{n,k}(\widehat{\rho}\,) := -\frac{(1-\widehat{\rho})(1-2\widehat{\rho})}{\widehat{\rho}^{\,2}}\left\{\widehat{\gamma}_{n,k}^{(\underline{K})}-\widehat{\gamma}_{n,k}^{(K_{2,\widehat{\rho}})}\right\}.$$

Using again the consistency of $\widehat{\rho}:=\hat{\rho}_{k_\rho}$ to ρ and the expansion in (16), we get for all large values of n:

$$\begin{aligned}\sqrt{k}\left(\widehat{\gamma}_{n,k}^{(\underline{K})}-\widehat{\gamma}_{n,k}^{(K_{2,\widehat{\rho}})}\right) &\overset{d}{=} \sqrt{k}\left(\widehat{\gamma}_{n,k}^{(\underline{K})}-\gamma\right)-\sqrt{k}\left(\widehat{\gamma}_{n,k}^{(K_{2,\widehat{\rho}})}-\gamma\right)+o_{\mathbb{P}}(1)\\ &\overset{d}{=} -\sqrt{k}A(n/k)\frac{\rho^2}{(1-\rho)(1-2\rho)}+\gamma\int_0^1\Big(t^{-1}W(t)-W(1)\Big)d\Big\{t\,(1-K_{2,\rho}(t))\Big\}+o_{\mathbb{P}}(1).\end{aligned}$$

With the consistency of $\widehat{\rho} := \hat{\rho}_{k_\rho}$ to ρ, this leads to

$$\widehat{A}_{n,k}(\widehat{\rho}) \overset{d}{=} A(n/k) - \frac{\gamma(1-\rho)(1-2\rho)}{\rho^2 k^{1/2}} \int_0^1 \Big(t^{-1}W(t) - W(1)\Big) d\Big\{t\,(\underline{K} - K_{2,\rho}(t))\Big\} + o_{\mathbb{P}}(k^{-1/2}).$$

This implies that for all large values of n:

$$\frac{\sqrt{k}(H_3+H_4)}{g_\beta(k/n)Q(1-k/n)} \overset{d}{=} -\frac{\beta\gamma(1-\rho)(1-2\rho)}{\rho^2(1-\beta\gamma)(1-\beta\,\gamma-\beta\,\rho)} \int_0^1 \Big(t^{-1}W(t) - W(1)\Big) d\Big\{t\,(1 - K_{2,\rho}(t))\Big\} + o_{\mathbb{P}}(1). \tag{42}$$

Finlay, form (38), (39) and (42), we get as $n \to \infty$:

$$\frac{\sqrt{k}\big(\widetilde{\pi}^{(K_{\widehat{\Delta}^*_{opt}})}_{\beta,k,n\widehat{\rho}} - \pi_{\beta,n}\big)}{g_\beta(k/n)Q(1-k/n)} \overset{d}{\to} \frac{\gamma\beta}{(1-\beta\,\gamma)^2} \int_0^1 \Big(t^{-1}W(t) - W(1)\Big) d(tK(t)) + \frac{\beta\gamma^2}{1-\beta\,\gamma} W(1).$$

Proof of Corollary 1. Computing the variance of the Gaussian terms appeared in the right side of (36) with respect to the covariance structure $r(.,.)$, the proof of Corollary 1 holds.

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