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# Common Fixed Point Results on Double-Composed Cone-Metric-Like Spaces for Generalized Rational Contractions with Applications

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**Abstract.** This paper presents a novel concept, known as a double-composed cone-metric-like space, which extends the idea of a double-composed cone-metric space. In this new space, the self-distance may not necessarily be zero; however, if the distance metric is zero, it must be for identical points. Additionally, this text introduces several results pertaining to this innovative concept, including theorems that demonstrate the existence of common fixed points for two mappings that satisfy generalized non-linear rational contractions within our new space. Various examples are provided to illustrate the main results and their relationship with other cone metric spaces. Finally, we demonstrate applications to nonlinear integral equations and boundary value problems (BVPs) to validate our findings.

**2020 Mathematics Subject Classifications**: 47H10, 54E50, 54H25

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## 1. Introduction

Fixed-point theory is a fundamental branch of functional and mathematical analysis that addresses the existence and uniqueness of solutions to integral-differential equations. Building upon the renowned Banach contraction principle [1], numerous scholars have made significant contributions to this field. Various results have emerged concerning mappings that satisfy different contractive conditions across diverse types of metric spaces.

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One such extension is $b$-metric spaces, which were introduced independently by Czerwik [2] and Bakhtin [3]. In recent years, there have been several generalizations of $b$-metric spaces, such as $b_v(s)$-metric spaces proposed by Mitrović et al. [4]. Kamran et al. [5] extended $b$-metric spaces, while in 2018, Mlaiki [6], and Abdeljawad et al. [7] introduced the concept of controlled metric-type spaces and double-controlled metric spaces, respectively. Amini-Harandi [8] (and independence Hitzler et al. [9]) have expanded the concept of partial metric spaces by defining metric-like spaces, also known as (dislocated metric spaces). The most comprehensive generalization, the $b$-metric-like space, was introduced by Alghamdi et al. [10]. Several fixed-point results have been explored in $b$-metric and their predecessors (see [11, 12]). In addition, in 2020, Mlaiki [13] and Ayoob et al. [14] introduced double-controlled metric-like spaces as a further extension of double-controlled metric-type spaces. In 2023, Ayoob et al. [15] proposed an extension of metric spaces known as double-composed metric spaces, which involve two composed functions in the triangular inequality.

Huang et al. [16] introduced the concept of cone metric spaces as an extension of traditional metric spaces. Following this, Hussain et al. [17] introduced cone $b$-metric spaces and Shateri [18] presented fixed-point theorems on double-controlled cone metric spaces. Subsequently, Anas et al. [19] introduced type I and II composed cone metric spaces and [20] extended double-composed metric spaces to double-composed metric-like spaces (see [21–27]). In 2020, Lateef [28] proved Fisher-type fixed point results in controlled metric spaces, with subsequent discussion by authors including Dass and Gupta [29] and Jaggi [30] utilizing a contraction condition of the rational-types. Additionally, Ahmad et al. [31] provided a generalization of rational contractions in double-controlled metric spaces for common fixed point theorems. For further details, see [25, 32–34].

The objective of the current study is to establish common fixed point results for new generalized rational contractions, serving as a generalization of various types of metric spaces mentioned previously. This study introduces a new class known as double-composed cone metric-like spaces (for short, $\mathcal{DCCML}$-space). The goal is to present common fixed point results involving various types of generalized rational contractions, accompanied by examples. Finally, the manuscript introduces applications of nonlinear integral equations and boundary value problems (BVPs) that support our fixed-point theorems within these new spaces.

## 2. Preliminaries

This section revisits some notations basic concepts, definitions, and lemmas from prior research that will be utilized throughout the remainder of this manuscript.

**Definition 1.** *[16] Let $E$ be a real Banach space and $P \subset E$. $P$ is called a cone if it satisfies the following conditions:*

*(P1)* $\{0_E\} \neq P$ *is nonempty and closed,*

*(P2)* $\alpha_1 a + \alpha_2 b \in P$ *for all* $a, b \in P$, *where* $\alpha_1, \alpha_2 \geq 0$,

*(P3)* $P \cap (-P) = \{0_E\}$, *where* $0_E$ *is the zero element of* $E$.

*Consider a cone* $P$, *we can define a partial ordering* $\preceq$ *on* $E$ *with respect to* $P$ *by* $a \preceq b$ *if and only if* $b - a \in P$. *Here,* $a \prec b$ *indicates that* $a \preceq b$ *and* $a \neq b$, *but* $a \ll b$ *stands for* $b - a \in intP$, *such that* $intP$ *denotes the interior of* $P$.

Let $E$ be a Banach space, $P$ be a cone in $E$ such as $intP \neq \phi$ and $\preceq$ is the partial ordering of $P$. The cone $P$ is called normal if there exists a constant number $M > 0$ such that for all $a, b \in E$ and $0_E \preceq a \preceq b \Rightarrow \|a\| \leq M\|b\|$ it holds or equivalently, if

$$inf\{\|a + b\| : a, b \in P, \|a\| = \|b\| = 1\} > 0.$$

For a non-normal cone (see [18]). Moreover, $P$ is called a solid if $intP \neq \phi$.

Now, we present some basic notations of cone $b$-metric spaces and their properties.

**Definition 2.** *[17] Let* $\Gamma$ *be a non-empty set and* $s \geq 1$. *Assume that a mapping* $d_b : \Gamma \times \Gamma \to E$ *satisfies the following conditions: For all* $a, b, c \in \Gamma$,

*(Cb1)* $d_b(a, b) = 0_E$ *if and only if* $a = b$,

*(Cb2)* $d_b(a, b) = d_b(b, a)$,

*(Cb3)* $d_b(a, b) \preceq s\,(d_b(a, c) + d_b(c, a))$.

*The pair* $(\Gamma, d_b)$ *is called a cone* $b$*-metric space. If changing condition (Cb1) in Definition 2 to* $d_b(a, b) = 0_E$, *implies* $a = b$, *then* $(\Gamma, d_b)$ *is called a cone* $b$*-metric-like space.*

Obviously, cone $b$-metric-like spaces are generalized to cone $b$-metric spaces, cone metric-like spaces, cone metric spaces and metric spaces, respectively, but the same is not true vice versa (see [17, 19, 21–25, 27]).

Abdeljawad et al. [7] introduced double-controlled type-metric spaces. Mlaiki et al. [13] generalized double-controlled metric-type spaces (DCMTS) to double-controlled metric-like spaces (DCMLS). Moreover, we expand on the expanded on cone metric space as follows:

**Definition 3.** *[18] Consider a set* $\Gamma \neq \phi$ *and non-comparable functions* $\omega_1, \omega_2 : \Gamma \times \Gamma \to [1, \infty)$. *Assume that a mapping* $\sigma : \Gamma \times \Gamma \to E$ *satisfies the conditions below: For all* $a, b, c \in \Gamma$,

*(*$\mathcal{C}1$*)* $\sigma(a, b) = 0_E$ *implies* $a = b$,

*(*$\mathcal{C}2$*)* $\sigma(a, b) = \sigma(b, a)$,

*(*$\mathcal{C}3$*)* $\sigma(a, b) \preceq \omega_1(a, c)\sigma(a, c) + \omega_2(c, b)\sigma(c, b)$.

*The pair* $(\Gamma, \sigma)$ *is referred to as a double controlled cone-metric-like space (DCCMLS) (see [18, 26, 28, 35]).*

Ayoob et al. [15] introduced generalizations of DCMTS and named it a double-composed metric space (abbreviated as $\mathcal{DCMS}$). In the same vein, Anas et al. [19] extended $\mathcal{DCMS}$ to a type II composed cone-metric space (C2CMS) and [20] further generalized $\mathcal{DCMS}$ to a double composed metric-like space known as $\mathcal{DCML}$-space. In this context, we present the double composed cone-metric-like space, DCCML-space, as outlined below:

**Definition 4.** *[19] Let $\Gamma$ be a non-empty set and $f, g : P \to P$ be nonconstant functions. Consider the mapping $\mathcal{D}_c : \Gamma \times \Gamma \to E$ that adheres to the following conditions: For all $a, b, c \in \Gamma$,*

*($\mathcal{D}1$) $\mathcal{D}_c(a, b) = 0_E$ if and only if $a = b$,*

*($\mathcal{D}2$) $\mathcal{D}_c(a, b) = \mathcal{D}_c(b, a)$,*

*($\mathcal{D}3$) $\mathcal{D}_c(a, b) \preceq f\big(\mathcal{D}_c(a, c)\big) + g\big(\mathcal{D}_c(c, b)\big)$.*

*Then the pair $(\Gamma, \mathcal{D}_c)$ is referred to as a C2CMS.*

**Example 1.** *Let $E = \mathbb{R}^2, P = \{\mathfrak{u} = (\mathfrak{r}, \mathfrak{s}) \in E : \mathfrak{r}, \mathfrak{s} \geq 0\}$, and $\Gamma = \mathbb{R}$. Let $\mathcal{D}_c : \Gamma \times \Gamma \to E$ be defined by $\mathcal{D}_c(a, b) = (\mathfrak{e}_1(a - b)^3, \mathfrak{e}_2(a - b)^2)$, where $\mathfrak{e}_1, \mathfrak{e}_2 \geq 0$.*
*Define $f, g : P \to P$ by*
*$f(\mathfrak{u}) = (e^{4\mathfrak{r}} - 1, e^{2\mathfrak{s}} - 1)$ and $g(\mathfrak{u}) = (4\mathfrak{r}, 2\mathfrak{s}), \mathfrak{u} \in P$.*
*It is not difficult to see that $(a - b)^3 \leq 4a^3 + 4b^3 \leq (e^{4a^3} - 1) + 4b^3$. Also, $(a - b)^2 \leq 2a^2 + 2b^2 \leq (e^{4a^2} - 1) + 2b^2$.*
*Therefore, $(\Gamma,\ \mathcal{D}_c)$ is a C2CMS.*

Now we introduce our generalization of the $\mathcal{DCML}$-spaces.

**Definition 5.** *Let $\Gamma$ be a non-empty set and $f, g : P \to P$ be nonconstant functions. Consider the mapping $\mathcal{L}_c : \Gamma \times \Gamma \to E$ that adheres to the following conditions: For all $a, b, c \in \Gamma$,*

*($\mathcal{L}1$) $\mathcal{L}_c(a, b) = 0_E \Rightarrow a = b$,*

*($\mathcal{L}2$) $\mathcal{L}_c(a, b) = \mathcal{L}_c(b, a)$,*

*($\mathcal{L}3$) $\mathcal{L}_c(a, b) \preceq f\big(\mathcal{L}_c(a, c)\big) + g\big(\mathcal{L}_c(c, b)\big)$.*

*The pair $(\Gamma, \mathcal{L}_c)$ is known as a double-composed cone metric-like space ($\mathcal{DCCML}$-space).*

The following examples demonstrate that every C2CMS is a $\mathcal{DCCML}$-space; however, the converse is not always true.

**Example 2.** *Let $E = \mathbb{R}^2, P = \{\mathfrak{u} = (\mathfrak{r}, \mathfrak{s}) \in E : \mathfrak{r}, \mathfrak{s} \geq 0\}$, and $\Gamma = \mathbb{R}$. Then $\mathcal{L}_c : \Gamma \times \Gamma \to E$ is defined as $\mathcal{L}_c(a, b) = (\mathfrak{e}_1(a + b)^{\mathfrak{p}}, \mathfrak{e}_2(a + b)^{\mathfrak{q}})$, where $\mathfrak{e}_1, \mathfrak{e}_2 \geq 0$ and $\mathfrak{p}, \mathfrak{q} > 1$ and $\mathfrak{p} \neq \mathfrak{q}$.*
*Define $f, g : P \to P$ by*
*$f(\mathfrak{u}) = \left(\frac{(1+2^{\mathfrak{p}-1}\mathfrak{r}^{\mathfrak{p}})^{\mathfrak{n}}-1}{\mathfrak{n}}, \frac{(1+2^{\mathfrak{q}-1}\mathfrak{r}^{\mathfrak{q}})^{\mathfrak{n}}-1}{\mathfrak{n}}\right)$ and $g(\mathfrak{u}) = \left(\frac{(1+2^{\mathfrak{p}-1}\mathfrak{r}^{\mathfrak{p}})^{\mathfrak{m}}-1}{\mathfrak{m}}, \frac{(1+2^{\mathfrak{q}-1}\mathfrak{r}^{\mathfrak{q}})^{\mathfrak{m}}-1}{\mathfrak{m}}\right), \mathfrak{u} \in P, \mathfrak{n} \geq$*

$\mathfrak{m} \geq 1$.
*Obviously, conditions* $(\mathcal{L}1)$ *and* $(\mathcal{L}2)$ *of Definition 5 are satisfied. Note that if* $a, b$ *are two nonnegative real numbers, then* $(a+b)^{\mathfrak{q}} \leq 2^{\mathfrak{q}-1}a^{\mathfrak{q}} + 2^{\mathfrak{q}-1}b^{\mathfrak{q}}, \mathfrak{q} > 1$, *and* $1 + \mathfrak{n}a \leq (1+a)^{\mathfrak{n}}$ *is Bernoulli's inequality. Hence,* $a \leq \frac{(1+a)^{\mathfrak{n}}-1}{\mathfrak{n}}$ *for any* $\mathfrak{n} \geq 1$, *by the same way for* $\mathfrak{p} > 1, \mathfrak{m} > 1$. *We can easily deduce that the condition* $(\mathcal{L}3)$ *is satisfied.*
*Therefore,* $(\Gamma, \mathcal{L}_c)$ *is a* $\mathcal{DCCML}$*-space. Obviously,* $(\Gamma, \mathcal{L}_c)$ *is not a C2CMS because it does not satisfy* $(\mathcal{D}1)$.

**Example 3.** *Consider* $E = \mathbb{R}^2, P = \{\mathfrak{u} = (\mathfrak{r}, \mathfrak{s}) \in E : \mathfrak{r}, \mathfrak{s} \geq 0\}$ *and* $\Gamma = \mathbb{R}$.
*Define* $\mathcal{L}_c : \Gamma \times \Gamma \to E$ *by* $\mathcal{L}_c(a, b) = \big(\sinh(\mathfrak{e}\sigma(a, b)), e^{(a+b)} - 1\big)$, *where* $\mathfrak{e} \geq 0$, *and* $\sigma(a, b)$ *is a DCMLS with two controlled functions* $\omega_1, \omega_2 : \Gamma \times \Gamma \to [1, \infty)$.
*Take* $f, g : P \to P$ *defined by*
$f(\mathfrak{u}) = \big(\sinh(2\omega_1(a, c)\mathfrak{r}), \frac{\mathfrak{s}^2+2\mathfrak{u}}{2}\big)$ *and* $g(\mathfrak{u}) = \big(\sinh(2\omega_2(c, b)\mathfrak{r}), \frac{\mathfrak{s}^2+2\mathfrak{s}}{2}\big)$, *where* $\mathfrak{u} \in P$.

*Evidently,* $(\mathcal{L}1)$ *and* $(\mathcal{L}2)$ *are satisfied. Since* $\sinh(\mathfrak{r})$ *is an increasing function, for all* $a, b \geq 0$,

$$\sinh(a + b) \leq \sinh(2max\{a, b\}) \leq \sinh(2a) + \sinh(2b).$$

*Therefore, for each* $a, b, c \in \Gamma$,

$$\begin{aligned}\sinh\left(\mathfrak{e}\sigma(a, b)\right) &\leq \sinh\left(\mathfrak{e}\omega_1(a, c)\sigma(a, c) + \mathfrak{e}\omega_2(c, b)\sigma(c, b)\right) \\ &\leq \sinh\left(\omega_1(a, c)\sinh(\mathfrak{e}\sigma(a, c)) + \omega_2(c, b)\sinh(\mathfrak{e}\sigma(c, b))\right) \\ &\leq \sinh\left(2\omega_1(a, c)\mathcal{L}_c(a, c)\right) + \sinh\left(2\omega_2(c, b)\mathcal{L}_c(c, b)\right),\end{aligned} \tag{1}$$

*and*

$$\begin{aligned}e^{(a+b)} - 1 &\leq e^{a+2c+b} - 1 = e^{a+c}e^{b+c} - 1 \\ &\leq \frac{e^{2(a+c)} + e^{2(c+b)}}{2} - 1 = \frac{e^{2(a+c)} - 1}{2} + \frac{e^{2(c+b)} - 1}{2}.\end{aligned} \tag{2}$$

*Thus, from* (1), (2), *we get*

$$\mathcal{L}_c(a, b) \preceq f(\mathcal{L}_c(a, c)) + g(\mathcal{L}_c(c, b)).$$

*Then,* $(\Gamma, \mathcal{L}_c)$ *is a* $\mathcal{DCCML}$*-space. Clearly, it is not C2CMS or* $\mathcal{DCML}$*-space.*

Afterwards, we define the topology of the $\mathcal{DCCML}$-space on $\Gamma$.

**Definition 6.** *Let* $(\Gamma, \mathcal{L}_c)$ *be a* $\mathcal{DCCML}$*-space, where* $P$ *is a normal cone with normal constant* $M$, *and* $\{a_n\}$ *be a sequence in* $\Gamma$.

*(i) The sequence* $\{a_n\}$ *is called convergent to* $a_0 \in \Gamma$ *if* $\lim_{n\to\infty} \mathcal{L}_c(a_n, a_0) = \mathcal{L}_c(a_0, a_0)$, *i.e.,* $\lim_{n\to\infty} a_n = a_0$.

*(ii)* $\{a_n\}$ *in* $\Gamma$ *is called* $\mathcal{L}_c$*-Cauchy if* $\lim_{n,m\to\infty} \mathcal{L}_c(a_n, a_m)$ *is a converges in* $\Gamma$, *i.e., for every* $\mathfrak{c} \in E$ *with* $0_E \ll \mathfrak{c}$, *there is a positive integer* $\mathfrak{n}_0 \in \mathbb{N}$ *such that* $\mathcal{L}_c(a_n, a_m) \ll \mathfrak{c}$ *for all* $n, m > \mathfrak{n}_0$.

*(iii) The space* $(\Gamma, \mathcal{L}_c)$ *is said to be* $\mathcal{L}_c$*-complete if every* $\mathcal{L}_c$*-Cauchy sequence in* $\Gamma$ *converges to a point in* $\Gamma$*, i.e.,* $\lim_{n\to\infty} \mathcal{L}_c(a_n, a_0) = \mathcal{L}_c(a_0, a_0) = \lim_{n,m\to\infty} \mathcal{L}_c(a_n, a_m)$.

**Definition 7.** *Let* $(\Gamma, \mathcal{L}_c)$ *be a* $\mathcal{DCCML}$*-space via* $f$ *and* $g$*, where* $P$ *is a normal cone with normal constant* $M$*. Suppose* $a_0 \in \Gamma$ *and* $0_E \prec \mathfrak{c}$*. Then* $\mathcal{L}_c$*-ball with center* $a_0$ *and radius* $\mathfrak{c}$ *is* $\mathfrak{B}(a_0, \mathfrak{c}) = \{b \in \Gamma : |\mathcal{L}_c(a_0, b) - \mathcal{L}_c(a_0, a_0)| \prec \mathfrak{c}\}$*, and put* $\mathcal{B} = \{\mathfrak{B}(a_0, \mathfrak{c}) : a_0 \in \Gamma \, and \, 0_E \ll \mathfrak{c}\}$.

**Lemma 1.** *The collection* $\mathcal{B} = \{\mathfrak{B}(a_0, \mathfrak{c}) : a_0 \in \Gamma \text{ and } 0_E \ll \mathfrak{c}\}$ *of all open balls forms a basis for a topology* $\tau_{\mathcal{L}_c}$ *on* $\Gamma$.

*Proof.* Let $a_0 \in \Gamma$. So, $a_0 \in \mathfrak{B}(a_0, \mathfrak{c})$ for $0_E \prec \mathfrak{c}$, which implies that $a_0 \in \mathfrak{B}(a_0, \mathfrak{c}) \subseteq \bigcup_{\substack{a_0 \in \Gamma \\ 0_E \prec \mathfrak{c}}} \mathfrak{B}(a_0, \mathfrak{c})$. Now, assume that $b \in \mathfrak{B}(a_0, \mathfrak{c}_1) \cap \mathfrak{B}(a_0, \mathfrak{c}_2)$. Then there exists $0_E \prec \mathfrak{c}$ such that $\mathfrak{B}(a_0, \mathfrak{c}) \subseteq \mathfrak{B}(a_0, \mathfrak{c}_1)$ and $\mathfrak{B}(a_0, \mathfrak{c}) \subseteq \mathfrak{B}(a_0, \mathfrak{c}_2)$. Let $d \in \mathfrak{B}(b, \mathfrak{c})$, then $\mathcal{L}^{\mathcal{C}}(b, d) - \mathcal{L}^{\mathcal{C}}(b, b) \ll \mathfrak{c}$. Thus, $\mathfrak{B}(b, \mathfrak{c}) \subseteq \mathfrak{B}(a_0, \mathfrak{c}_1) \cap \mathfrak{B}(a_0, \mathfrak{c}_2)$.

**Definition 8.** *[36] Presume* $P$ *is a solid cone in a Banach space* $E$*. A sequence* $\{a_n\} \subset P$ *is said to converge if for each* $0_E \ll \mathfrak{c}$ *there exists* $\mathcal{N}$ *such that* $a_n \ll \mathfrak{c}$ *for all* $n > \mathcal{N}$.

**Lemma 2.** *[32] If* $E$ *is a real Banach space with a solid cone* $P$ *and* $\{a_n\} \subset P$ *is a sequence with* $\|a_n\| \to 0$*, as* $n \to +\infty$*, then* $\{a_n\}$ *is a convergent sequence.*

**Lemma 3.** *[32] Let* $E$ *be a real Banach space with a solid cone* $P$.

*(i) If* $\mathfrak{r}, \mathfrak{s}, \mathfrak{t} \in E$ *and* $\mathfrak{r} \preceq \mathfrak{s} \ll \mathfrak{t}$*, then* $\mathfrak{r} \ll \mathfrak{t}$.

*(ii) If* $\mathfrak{r} \in P$ *and* $\mathfrak{r} \ll \mathfrak{t}$ *for each* $\mathfrak{t} \gg 0_E$*, then* $\mathfrak{r} = 0_E$.

In general, the limit of a convergent sequence in $\mathcal{DCCML}$-space may not be unique.

**Proposition 1.** *Let* $E$ *be a real Banach space with a solid cone* $P$ *and* $\{a_n\} \subset P$*. Let* $(\Gamma, \mathcal{L}_c)$ *be a* $\mathcal{DCCML}$*-space via* $f$ *and* $g$*. Assume* $\lim_{n\to\infty} \mathcal{L}_c(a_n, a_0) = 0_E$*. Then, every convergent sequence has a unique limit, i.e.,*
$\|\mathcal{L}_c(a_n, a_0)\| \to 0$ *implies that* $\lim_{n\to\infty} a_n = a_0$ *is unique.*

*Proof.* The proof is omitted.

Let $(\Gamma, \mathcal{L}_c)$ be a $\mathcal{DCCML}$-space. Define $\widehat{\mathcal{L}_c} : \Gamma^2 \to E$ by
$\widehat{\mathcal{L}_c}(a, b) = |2\mathcal{L}_c(a, b) - \mathcal{L}_c(a, a) - \mathcal{L}_c(b, b)|, \forall a, b \in \Gamma$.
Obviously, $\widehat{\mathcal{L}_c}(a, a) = 0_E, \forall a \in \Gamma$.

Let $\Psi$ be the family of all onto mappings $\psi : [0, \infty) \to [0, \infty)$ under the following necessities: $\mathfrak{r} \leq \psi(\mathfrak{r})$ for each $\mathfrak{r} \in [0, \infty)$, and $\psi'$ (the derivative of $\psi$) increases [37]. Next, we present the following lemma, utilizing results from the literature.

**Lemma 4.** *[20] Let* $\psi \in \Psi$*, then for all* $x \in [0, 1]$ *and* $0 < \mathfrak{q} \leq 1 \leq \mathfrak{p}$*, we have*

*(i)* $\left(\psi(x^{\mathfrak{p}})\right)^{\frac{1}{\mathfrak{p}}} \leq \psi(x) \leq \left(\psi(x^{\mathfrak{q}})\right)^{\frac{1}{\mathfrak{q}}}$;

*(ii)* $\left(\psi^{-1}(x^{\mathfrak{q}})\right)^{\frac{1}{\mathfrak{q}}} \leq \psi^{-1}(x) \leq \left(\psi^{-1}(x^{\mathfrak{p}})\right)^{\frac{1}{\mathfrak{p}}}$.

## 3. Main results

This section presents some fixed-point results within the framework of $\mathcal{DCCML}$-space. In this work, the first theorem for common fixed points is analogous to the non-linear generalization rational contraction, with the self-mapping of $\mathcal{DCCML}$-space, see [31, 38].

Motivated by Ahmad et al. [31], we denote by $\Delta$ the family of all mappings $\lambda : \Gamma^2 \to [0, 1)$ with any mapping (say) $T : \Gamma \to \Gamma$ satisfying the following conditions:

(i) $\lambda(Ta, b) \leq \lambda(a, b)$ for each $a, b \in \Gamma$.

(ii) $\lambda(a, Tb) \leq \lambda(a, b)$ for each $a, b \in \Gamma$.

Clearly, since $\lambda \in \Delta$ the iterative $\lambda^j(a, b) \to 0$ as $j \to +\infty$. Now, we state and prove the common fixed point results in $\mathcal{DCCML}$-space.

**Theorem 1.** *Assume $(\Gamma, \mathcal{L}_c)$ is an $\mathcal{L}_c$-complete $\mathcal{DCCML}$-space with two non-constant functions $f, g : P \to P$, where $P$ is a normal cone via normal constant $M$. Let $T_1, T_2 : \Gamma \to \Gamma$ be a mappings and there exists $\lambda \in \Delta$ such that*

$$\mathcal{L}_c(T_1a, T_2b) \preceq \lambda(a, b)\widetilde{M}(a, b), \quad \text{for all } a, b \in \Gamma, \tag{3}$$

*where*

$$\widetilde{M}(a, b) = \max\left\{\mathcal{L}_c(a, b), \mathcal{L}_c(a, T_1a), \mathcal{L}_c(b, T_2b), \frac{\mathcal{L}_c(a, T_1a)\mathcal{L}_c(b, T_2b)}{1 + \mathcal{L}_c(a, b)}, \right.$$
$$\left. \frac{\mathcal{L}_c(b, T_2b)\big[1 + \mathcal{L}_c(a, T_1a)\big]}{1 + \mathcal{L}_c(a, b)}, \frac{\big[\mathcal{L}_c(a, T_1a) + \mathcal{L}_c(b, T_2b)\big]\mathcal{L}_c(T_1a, T_2b)}{1 + \mathcal{L}_c(a, b) + \mathcal{L}_c(T_1a, T_2b)}\right\}.$$

*For $a_0 \in \Gamma$, we set a sequence $\{a_n\}$ defined as $a_{2n+1} = T_1a_{2n}$ and $a_{2n+2} = T_2a_{2n+1}$ for every $n \geq 0$. Suppose*

*(i) $f$ and $g$ are bounded and non-decreasing, $g$ is sub-additive and $g(\lambda a) \prec a, \lambda \in (0, 1)$;*

*(ii) $\lim_{n,m\to\infty} \sum_{i=m}^{n-2} \|g^{i-m} f\big(\xi^i \mathcal{L}_c(a_0, a_1)\big)\| + \|g^{n-m-1}\big(\xi^{n-1}\mathcal{L}_c(a_0, a_1)\big)\| = 0$,*

*where $\xi = \lambda(a_0, a_1) < 1$. If for every fixed point $a$, we conclude that $\mathcal{L}_c(a, a) = 0_E$, then $T_1$ and $T_2$ have a unique common fixed point.*

*Proof.* Let $a_0 \in \Gamma$. Then, $\{a_n\}$ is constructed in $\Gamma$ by $a_{2n+1} = T_1a_{2n}$ and $a_{2n+2} = T_2a_{2n+1}$, for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ for which $a_{n_0+1} = a_{n_0}$, then $T_1a_{n_0} = a_{n_0}$. Therefore, there is nothing to prove. Thus, we assume that $a_{n+1} \neq a_n$ for all $n \in \mathbb{N}$. From inequality (3), we obtain

$$\mathcal{L}_c(a_{2n+1}, a_{2n+2}) = \mathcal{L}_c(T_1a_{2n}, T_2a_{2n+1}) \preceq \lambda(a_{2n}, a_{2n+1}))\widetilde{M}(a_{2n}, a_{2n+1}),$$

where

$$\begin{aligned}\widetilde{M}(a_{2n},a_{2n+1}) &= \max\{\mathcal{L}_c(a_{2n},a_{2n+1}),\mathcal{L}_c(a_{2n},T_1a_{2n}),\mathcal{L}_c(a_{2n+1},T_2a_{2n+1}),\\ &\frac{\mathcal{L}_c(a_{2n},T_1a_2n)\mathcal{L}_c(a_{2n+1},T_2a_{2n+1})}{1+\mathcal{L}_c(a_{2n},a_{2n+1})},\frac{\mathcal{L}_c(a_{2n+1},T_2a_{2n+1})[1+\mathcal{L}_c(a_{2n},T_1a_{2n})]}{1+\mathcal{L}_c(a_{2n},a_{2n+1})},\\ &\left.\frac{\left[\mathcal{L}_c(a_{2n},T_1a_{2n})+\mathcal{L}_c(a_{2n+1},T_2a_{2n+1})\right]\mathcal{L}_c(T_1a_{2n},T_2a_{2n+1})}{1+\mathcal{L}_c(a_{2n},a_{2n+1})+\mathcal{L}_c(T_1a_{2n},T_2a_{2n+1})}\right\}\\ &= \max\{\mathcal{L}_c(a_{2n},a_{2n+1}),\mathcal{L}_c(a_{2n},a_{2n+1}),\mathcal{L}_c(a_{2n+1},a_{2n+2}),\\ &\frac{\mathcal{L}_c(a_{2n},a_{2n+1})\mathcal{L}_c(a_{2n+1},a_{2n+2})}{1+\mathcal{L}_c(a_{2n},a_{2n+1})},\frac{\mathcal{L}_c(a_{2n+1},a_{2n+2})[1+\mathcal{L}_c(a_{2n},a_{2n+1})]}{1+\mathcal{L}_c(a_{2n},a_{2n+1})},\\ &\left.\frac{\left[\mathcal{L}_c(a_{2n},a_{2n+1})+\mathcal{L}_c(a_{2n+1},a_{2n+2})\right]\mathcal{L}_c(a_{2n+1},a_{2n+2})}{1+\mathcal{L}_c(a_{2n},a_{2n+1})+\mathcal{L}_c(a_{2n+1},a_{2n+2})}\right\}\\ &\preceq \max\{\mathcal{L}_c(a_{2n},a_{2n+1}),\mathcal{L}_c(a_{2n+1},a_{2n+2})\}.\end{aligned}$$

Hence,

$$\mathcal{L}_c(a_{2n+1},a_{2n+2})\preceq\lambda(a_{2n},a_{2n+1})\widetilde{M}(a_{2n},a_{2n+1}),$$

where

$$\widetilde{M}(a_{2n},a_{2n+1})=\max\{\mathcal{L}_c(a_{2n},a_{2n+1}),\mathcal{L}_c(a_{2n+1},a_{2n+2})\}.$$

By the properties of the function $\lambda$ we deduce that

$$\begin{aligned}\mathcal{L}_c(a_{2n+1},a_{2n+2}) &\preceq \lambda(a_{2n},a_{2n+1})\widetilde{M}(a_{2n},a_{2n+1})=\lambda(T_2T_1a_{2n-2},a_{2n+1})\widetilde{M}(a_{2n},a_{2n+1})\\ &\preceq \lambda(a_{2n-2},a_{2n+1})\widetilde{M}(a_{2n},a_{2n+1})\preceq\cdots\preceq\lambda(a_0,a_{2n+1})\widetilde{M}(a_{2n},a_{2n+1})\\ &= \lambda(a_0,T_1T_2a_{2n-1})\widetilde{M}(a_{2n},a_{2n+1})\preceq\lambda(a_0,a_{2n-1})(\widetilde{M}(a_{2n},a_{2n+1})\\ &\preceq\cdots\preceq\lambda(a_0,a_1)\widetilde{M}(a_{2n},a_{2n+1}).\end{aligned}$$

Thus,

$$\mathcal{L}_c(a_{2n+1},a_{2n+2})\preceq\lambda(a_0,a_1)\widetilde{M}(a_{2n},a_{2n+1}). \tag{4}$$

If $\widetilde{M}(a_{2n},a_{2n+1})=\mathcal{L}_c(a_{2n+1},a_{2n+2})$, then by (4) we get

$$\mathcal{L}_c(a_{2n+1},a_{2n+2})\preceq\lambda(a_0,a_1)\mathcal{L}_c(a_{2n+1},a_{2n+2})\prec\mathcal{L}_c(a_{2n+1},a_{2n+2}),$$

which is a contradiction.
On the other hand, if $\widetilde{M}(a_{2n},a_{2n+1})=\mathcal{L}_c(a_{2n},a_{2n+1})$, then by (4) we have

$$\begin{aligned}\mathcal{L}_c(a_{2n+1},a_{2n+2}) &\preceq \lambda(a_0,a_1)\mathcal{L}_c(a_{2n},a_{2n+1})\\ &\preceq \big(\lambda(a_0,a_1)\big)^2\mathcal{L}_c(a_{2n-1},a_{2n})\preceq\cdots\preceq\big(\lambda(a_0,a_1)\big)^n\mathcal{L}_c(a_0,a_1).\end{aligned}$$

Applying it recursively, we have

$$\mathcal{L}_c(a_n,a_{n+1})\preceq\xi^n\mathcal{L}_c(a_0,a_1),\text{ where }\xi=\lambda(a_0,a_1). \tag{5}$$

For $m < n$, and $n, m \in \mathbb{N}$ and condition (i), we deduce that

$$\begin{aligned}\mathcal{L}_c(a_m, a_n) &\preceq f\big(\mathcal{L}_c(a_m, a_{m+1})\big) + g\big(\mathcal{L}_c(a_{m+1}, a_n)\big)\\ &\preceq f\big(\mathcal{L}_c(a_m, a_{m+1})\big) + gf\big(\mathcal{L}_c(a_{m+1}, a_{m+2})\big) + g^2\big(\mathcal{L}_c(a_{m+2}, a_n)\big)\\ &\vdots\\ &\preceq \sum_{i=m}^{n-2} g^{i-m} f\big(\mathcal{L}_c(a_i, a_{i+1})\big) + g^{n-m-1}\big(\mathcal{L}_c(a_{n-1}, a_n)\big). \qquad (6)\end{aligned}$$

Substituting (5) in (6) implies that

$$\|\mathcal{L}_c(a_m, a_n)\| \preceq M\Big[\|\sum_{i=m}^{n-2} g^{i-m} f\big(\xi^i \mathcal{L}_c(a_0, a_1)\big)\| + \|g^{n-m-1}\big(\xi^{n-1}\mathcal{L}_c(a_0, a_1)\big)\|\Big].$$

Thus, as $n, m \to \infty$, and condition (ii), we have

$$\|\mathcal{L}_c(a_m, a_n)\| = 0.$$

Since the sequence $\{a_n\}$ is $\mathcal{L}_c$-Cauchy in $\Gamma$, which is $\mathcal{L}_c$-complete $\mathcal{DCCML}$-space, there exists an element $a \in \Gamma$ such that $\{a_n\} \to a$. Hence

$$\mathcal{L}_c(a_n, a) = \mathcal{L}_c(a, a) = \mathcal{L}_c(a_n, a_m) = 0_E. \qquad (7)$$

Now, we prove that $T_1 a = T_2 a = a$. Since $\{a_n\} \to a$, as $n \to +\infty$, from condition (i) and $(\mathcal{L}3)$, we deduce that

$$\begin{aligned}\mathcal{L}_c(a, T_1 a) &\preceq f\big(\mathcal{L}_c(a, a_{2n+2})\big) + g\big(\mathcal{L}_c(a_{2n+2}, T_1 a)\big)\\ &= f\big(\mathcal{L}_c(a, a_{2n+2})\big) + g\big(\mathcal{L}_c(T_1 a, T_2 a_{2n+1})\big),\end{aligned}$$

implying that

$$\mathcal{L}_c(T_1 a, T_2 a_{2n+1}) \preceq \lambda(a, a_{2n+1})\widetilde{M}(a, a_{2n+1}),$$

and

$$\begin{aligned}\widetilde{M}(a, a_{2n+1}) =& \max\Big\{\mathcal{L}_c(a, a_{2n+1}), \mathcal{L}_c(a, T_1 a), \mathcal{L}_c(a_{2n+1}, T_2 a_{2n+1}),\\ &\frac{\mathcal{L}_c(a, T_1 a)\mathcal{L}_c(a_{2n+1}, T_2 a_{2n+1})}{1 + \mathcal{L}_c(a, a_{2n+1})}, \frac{\mathcal{L}_c(a_{2n+1}, T_2 a_{2n+1})[1 + \mathcal{L}_c(a, T_1 a)]}{1 + \mathcal{L}_c(a, a_{2n+1})},\\ &\frac{\big[\mathcal{L}_c(a, T_1 a) + \mathcal{L}_c(a_{2n+1}, T_2 a_{2n+1})\big]\mathcal{L}_c(T_1 a, T_2 a_{2n+1})}{1 + \mathcal{L}_c(a, a_{2n+1}) + \mathcal{L}_c(T_1 a, T_2 a_{2n+1})}\Big\}\\ =& \max\Big\{\mathcal{L}_c(a, a_{2n+1}), \mathcal{L}_c(a, T_1 a), \mathcal{L}_c(a_{2n+1}, a_{2n+2}),\\ &\frac{\mathcal{L}_c(a, T_1 a)\mathcal{L}_c(a_{2n+1}, a_{2n+2})}{1 + \mathcal{L}_c(a, a_{2n+1})}, \frac{\mathcal{L}_c(a_{2n+1}, a_{2n+2})[1 + \mathcal{L}_c(a, T_1 a)]}{1 + \mathcal{L}_c(a, a_{2n+1})},\\ &\frac{\big[\mathcal{L}_c(a, T_1 a) + \mathcal{L}_c(a_{2n+1}, a_{2n+2})\big]\mathcal{L}_c(T_1 a, a_{2n+2})}{1 + \mathcal{L}_c(a, a_{2n+1}) + \mathcal{L}_c(T_1 a, a_{2n+2})}\Big\}.\end{aligned}$$

By assuming $f$ and $g$ are bounded and taking the limit in the above relationship and (7), we obtain that

$$0_E \prec \mathcal{L}_c(a, T_1 a) \preceq f(0) + g\big(\lambda(a, a_{2n+1})\mathcal{L}_c(a, T_1 a)\big).$$

We use the fact that $\lambda(a, a_{2n+1}) \in \Delta$ and condition (i), we can take $g(\lambda(a, a_{2n+1})\mathcal{L}_c(a, T_1 a)) \prec \mathcal{L}_c(a, T_1 a)$.
Therefore, $\|\mathcal{L}_c(a, T_1 a)\| = 0$, i.e., $T_1 a = a$. Using the same way as for the proof of $T_1$, we get $T_2 a = a$. Hence, $T_1$ and $T_2$ have a common fixed point $a$.
Let $a$ and $a^*$ be two fixed points of $T_1$ and $T_2$, where $T_1 a = T_2 a = a$ and $T_1 a^* = T_2 a^* = a$. Since $a \neq a^*$, it implies $T_i a \neq T_j a^*$, $i, j = 1, 2$. By (3) we obtain

$$\mathcal{L}_c(a^*, a) = \mathcal{L}_c(T_1 a^*, T_2 a) \preceq \lambda(a^*, a)\widetilde{M}(a^*, a),$$

where

$$\begin{aligned}\widetilde{M}(a^*, a) &= \max\Bigg\{\mathcal{L}_c(a^*, a), \mathcal{L}_c(a^*, T_1 a^*), \mathcal{L}_c(a, T_2 a), \frac{\mathcal{L}_c(a^*, T_1 a^*)\mathcal{L}_c(a, T_2 a)}{1 + \mathcal{L}_c(a^*, a)},\\ &\quad \frac{\mathcal{L}_c(a, T_2 a)[1 + \mathcal{L}_c(a^*, T_1 a^*)]}{1 + \mathcal{L}_c(a^*, a)}, \frac{\big[\mathcal{L}_c(a^*, T_1 a^*) + \mathcal{L}_c(a, T_2 a)\big]\mathcal{L}_c(T_1 a^*, T_2 a)}{1 + \mathcal{L}_c(a^*, a) + \mathcal{L}_c(T_1 a^*, T_2 a)}\Bigg\}\\ &= \max\Bigg\{\mathcal{L}_c(a^*, a), \mathcal{L}_c(a^*, a^*), \mathcal{L}_c(a, a), \frac{\mathcal{L}_c(a^*, a^*)\mathcal{L}_c(a, a)}{1 + \mathcal{L}_c(a^*, a)},\\ &\quad \frac{\mathcal{L}_c(a, a)[1 + \mathcal{L}_c(a^*, a^*)]}{1 + \mathcal{L}_c(a^*, a)}, \frac{\big[\mathcal{L}_c(a^*, a^*) + \mathcal{L}_c(a, a)\big]\mathcal{L}_c(a^*, a)}{1 + \mathcal{L}_c(a^*, a) + \mathcal{L}_c(a^*, a)}\Bigg\}.\end{aligned}$$

By assuming any fixed point $a$ and $a^*$, $\mathcal{L}_c(a^*, a^*) = \mathcal{L}_c(a, a) = 0_E$ and the fact that $\lambda(a^*, a) \in [0, 1)$, the result is $\mathcal{L}_c(a^*, a) \preceq \lambda(a^*, a)\widetilde{M}(a^*, a) \prec \mathcal{L}_c(a^*, a)$. Thus, $\|\mathcal{L}_c(a^*, a)\| = 0$, which is a contradiction. Therefore, $a^* = a$.

**Corollary 1.** *Assume $(\Gamma, \mathcal{D}_c)$ is a complete C2CMS with two non-constant functions $f, g : P \to P$, where $P$ is a normal cone via normal constant $M$. Let $T_1, T_2 : \Gamma \to \Gamma$ be a mappings and there exists $\lambda \in \Delta$ such that*

$$\mathcal{D}_c(T_1 a, T_2 b) \preceq \lambda(a, b)\widetilde{M}(a, b), \text{ for all } a, b \in \Gamma, \tag{8}$$

*where*

$$\begin{aligned}\widetilde{M}(a, b) = \max\Bigg\{&\mathcal{D}_c(a, b), \mathcal{D}_c(a, T_1 a), \mathcal{D}_c(b, T_2 b), \frac{\mathcal{D}_c(a, T_1 a)\mathcal{D}_c(b, T_2 b)}{1 + \mathcal{D}_c(a, b)},\\ &\frac{\mathcal{D}_c(b, T_2 b)\big[1 + \mathcal{D}_c(a, T_1 a)\big]}{1 + \mathcal{D}_c(a, b)}, \frac{\big[\mathcal{D}_c(a, T_1 a) + \mathcal{D}_c(b, T_2 b)\big]\mathcal{D}_c(T_1 a, T_2 b)}{1 + \mathcal{D}_c(a, b) + \mathcal{D}_c(T_1 a, T_2 b)}\Bigg\}.\end{aligned}$$

*For $a_0 \in \Gamma$, we set a sequence $\{a_n\}$ defined as $a_{2n+1} = T_1 a_{2n}$ and $a_{2n+2} = T_2 a_{2n+1}$ for every $n \geq 0$. Suppose* *(i) $f$ and $g$ are bounded and non-decreasing, $g$ is sub-additive and $g(\lambda a) \prec a, \lambda \in (0,1)$;*

*(ii)* $\lim\limits_{n,m\to\infty} \sum_{i=m}^{n-2} \|g^{i-m} f\big(\xi^i \mathcal{D}_c(a_0,a_1)\big)\| + \|g^{n-m-1}\big(\xi^{n-1}\mathcal{D}_c(a_0,a_1)\big)\| = 0,$

*where $\xi = \lambda(a_0,a_1) < 1$. Then $T_1$ and $T_2$ have a unique common fixed point.*

*Proof.* The proof follows from Theorem 1 by taking $(\Gamma, \mathcal{L}_c)$ in C2CMS.

Then, some special cases of Theorem 1 are presented, and since every C2CMS is a $\mathcal{DCCML}$-space, the last cases of Corollary 1 are investigated, while the early cases of Corollary 1 are omitted.

**Corollary 2.** *Let $(\Gamma, \mathcal{L}_c)$ be a $\mathcal{L}_c$-complete $\mathcal{DCCML}$-space with two non-constant functions $f, g : P \to P$, where $P$ is a normal cone via the normal constant $M$. Suppose that $T_1, T_2 : \Gamma \to \Gamma$ is a mappings and $\lambda_j \in \Delta, j = 1, \cdots, 6$, such that*

$$\begin{aligned}\mathcal{L}_c(T_1a,T_2b) \preceq& \lambda_1(a,b)\mathcal{L}_c(a,b) + \lambda_2(a,b)\mathcal{L}_c(a,T_1a) + \lambda_3(a,b)\mathcal{L}_c(b,T_2b)\\ &+ \lambda_4(a,b)\frac{\mathcal{L}_c(a,T_1a)\mathcal{L}_c(b,T_2b)}{1+\mathcal{L}_c(a,b)} + \lambda_5(a,b)\frac{\mathcal{L}_c(b,T_2b)\big[1+\mathcal{L}_c(a,T_1a)\big]}{1+\mathcal{L}_c(a,b)}\\ &+ \lambda_6(a,b)\frac{\big[\mathcal{L}_c(a,T_1a)+\mathcal{L}_c(b,T_2b)\big]\mathcal{L}_c(T_1a,T_2b)}{1+\mathcal{L}_c(a,b)+\mathcal{L}_c(T_1a,T_2b)}, \end{aligned} \tag{9}$$

*for all $a, b \in \Gamma$, where $\sum_{j=1}^{6} \lambda_j(a,b) < 1$.*
*For $a_0 \in \Gamma$, take the sequence $\{a_n\}$ as $a_{2n+1} = T_1 a_{2n}$ and $a_{2n+2} = T_2 a_{2n+1}$ for every $n \geq 0$. Let $\xi = \frac{\lambda_1(a_0,a_1)+\lambda_2(a_0,a_1)}{1-\sum_{j=3}^{6}\lambda_j(a_0,a_1)} < 1$. Suppose*

*(i) $f$ and $g$ are bounded and non-decreasing, $g$ is sub-additive, and $g(\lambda a) \prec a, \lambda \in (0,1)$;*

*(ii)* $\lim\limits_{n,m\to\infty} \sum_{i=m}^{n-2} \|g^{i-m} f\big(\xi^i \mathcal{L}_c(a_0,a_1)\big)\| + \|g^{n-m-1}\big(\xi^{n-1}\mathcal{L}_c(a_0,a_1)\big)\| = 0.$

*If for every fixed point $a$, we conclude that $\mathcal{L}_c(a,a) = 0_E$, then $T_1$ and $T_2$ have a unique common fixed point.*

*Proof.* It is observed that for each $a, b \in \Gamma$, there exist $\lambda_j \in \Delta, j = 1, \cdots, 6$, such that $\lambda(a,b) = \sum_{j=1}^{6} \lambda_j(a,b) < 1$, resulting in

$$\begin{aligned}\mathcal{L}_c(T_1a,T_2b) \preceq& \lambda_1(a,b)\mathcal{L}_c(a,b) + \lambda_2(a,b)\mathcal{L}_c(a,T_1a) + \lambda_3(a,b)\mathcal{L}_c(b,T_2b)\\ &+ \lambda_4(a,b)\frac{\mathcal{L}_c(a,T_1a)\mathcal{L}_c(b,T_2b)}{1+\mathcal{L}_c(a,b)} + \lambda_5(a,b)\frac{\mathcal{L}_c(b,T_2b)\big[1+\mathcal{L}_c(a,T_1a)\big]}{1+\mathcal{L}_c(a,b)}\\ &+ \lambda_6(a,b)\frac{\big[\mathcal{L}_c(a,T_1a)+\mathcal{L}_c(b,T_2b)\big]\mathcal{L}_c(T_1a,T_2b)}{1+\mathcal{L}_c(a,b)+\mathcal{L}_c(T_1a,T_2b)}\\ \preceq& \Big[\sum_{j=1}^{6}\lambda_j(a,b)\Big]\max\Bigg\{\mathcal{L}_c(a,b), \mathcal{L}_c(a,T_1a), \mathcal{L}_c(b,T_2b), \frac{\mathcal{L}_c(a,T_1a)\mathcal{L}_c(b,T_2b)}{1+\mathcal{L}_c(a,b)},\\ &\frac{\mathcal{L}_c(b,T_2b)\big[1+\mathcal{L}_c(a,T_1a)\big]}{1+\mathcal{L}_c(a,b)}, \frac{\big[\mathcal{L}_c(a,T_1a)+\mathcal{L}_c(b,T_2b)\big]\mathcal{L}_c(T_1a,T_2b)}{1+\mathcal{L}_c(a,b)+\mathcal{L}_c(T_1a,T_2b)}\Bigg\}\\ =& \lambda(a,b)\widetilde{M}(a,b). \end{aligned}$$

Therefore, through Theorem 1 we obtain the desired result. Moreover, let $a_0 \in \Gamma$. A sequence $\{a_n\}$ in $\Gamma$ defined as $a_{2n+1} = T_1 a_{2n}$ and $a_{2n+2} = T_2 a_{2n+1}, \forall n \in \mathbb{N}$. Then,

$$\begin{aligned}\mathcal{L}_c(a_{2n+1}, a_{2n+2}) =& \mathcal{L}_c(T_1 a_{2n}, T_2 a_{2n+1}) \\ &\preceq \left[\frac{\lambda_1(a_0, a_1) + \lambda_2(a_0, a_1)}{1 - \sum_{j=3}^{6} \lambda_j(a_0, a_1)}\right] \mathcal{L}_c(a_{2n}, a_{2n+1}).\end{aligned}$$

**Remark 1.** *In Theorem 1 and Corollaries 1 and 2, if $P = \mathbb{R}^+$ is taken, the condition (i) is interchanged to $f$ and $g$ are continuous and non-decreasing, $g$ is sub-additive, and $g(\lambda a) < a, \lambda \in (0,1)$; and condition (ii) to*

$$\sum_{i=m}^{n-2} g^{i-m} f\big(\xi^i \mathcal{L}_c(a_0, a_1)\big) + g^{n-m-1}\big(\xi^{n-1} \mathcal{L}_c(a_0, a_1)\big) \to 0,$$

*as $n, m \to \infty$. Then, the study reveals the same special results in the spaces $\mathcal{DCMS}$ and $\mathcal{DCML}$-spaces, respectively.*

By providing $T_1 = T_2 = T$ in Theorem 1, and Corollaries 1 and 2, respectively, we derive the following corollaries:

**Corollary 3.** *Assume $(\Gamma, \mathcal{L}_c)$ is an $\mathcal{L}_c$-complete $\mathcal{DCCML}$-space, where $P$ is a normal cone via normal constant $M$. Let $T : \Gamma \to \Gamma$ be a mapping and there exists $\lambda \in \Delta$ such that*

$$\mathcal{L}_c(Ta, Tb) \preceq \lambda(a, b) \widetilde{M}(a, b), \quad \text{for all } a, b \in \Gamma, \tag{10}$$

*where*

$$\begin{aligned}\widetilde{M}(a, b) = \max\Bigg\{&\mathcal{L}_c(a, b), \mathcal{L}_c(a, Ta), \mathcal{L}_c(b, Tb), \frac{\mathcal{L}_c(a, Ta)\mathcal{L}_c(b, Tb)}{1 + \mathcal{L}_c(a, b)}, \\ &\frac{\mathcal{L}_c(b, Tb)\big[1 + \mathcal{L}_c(a, Ta)\big]}{1 + \mathcal{L}_c(a, b)}, \frac{\big[\mathcal{L}_c(a, Ta) + \mathcal{L}_c(b, Tb)\big]\mathcal{L}_c(Ta, Tb)}{1 + \mathcal{L}_c(a, b) + \mathcal{L}_c(Ta, Tb)}\Bigg\}.\end{aligned}$$

*For $a_0 \in \Gamma$, we take the sequence $\{a_n\}$ defined as $a_{n+1} = T^n a_0$ for every $n \geq 0$. Suppose*

*(i) $f$ and $g$ are bounded and non-decreasing, $g$ is sub-additive, and $g(\lambda a) \prec a, \lambda \in (0,1)$;*

*(ii) $\lim_{n,m \to \infty} \sum_{i=m}^{n-2} \|g^{i-m} f\big(\xi^i \mathcal{L}_c(a_0, a_1)\big)\| + \|g^{n-m-1}\big(\xi^{n-1} \mathcal{L}_c(a_0, a_1)\big)\| = 0$,*

*where $\xi = \lambda(a_0, a_1) < 1$. If for every fixed point $a$, we conclude that $\mathcal{L}_c(a, a) = 0_E$, then $T$ has a unique fixed point.*

*Proof.* The proof follows from Theorem 1 by taking the self-map $T : \Gamma \to \Gamma$ as $T_1 = T_2 = T$.

**Corollary 4.** *Suppose that* $(\Gamma, \mathcal{L}_c)$ *is a complete C2CMS, where* $P$ *is a normal cone via normal constant* $M$*. Let* $T : \Gamma \to \Gamma$ *be a mapping and there exists* $\lambda \in \Delta$ *such that*

$$\mathcal{D}_c(Ta, Tb) \preceq \lambda(a, b)\widetilde{M}(a, b), \quad \text{for all } a, b \in \Gamma, \tag{11}$$

*where*

$$\widetilde{M}(a,b) = \max\Bigg\{\mathcal{D}_c(a,b), \mathcal{D}_c(a,Ta), \mathcal{D}_c(b,Tb), \frac{\mathcal{D}_c(a,Ta)\mathcal{D}_c(b,Tb)}{1+\mathcal{D}_c(a,b)}, \frac{\mathcal{D}_c(b,Tb)\big[1+\mathcal{D}_c(a,Ta)\big]}{1+\mathcal{D}_c(a,b)}, \frac{\big[\mathcal{D}_c(a,Ta)+\mathcal{D}_c(b,Tb)\big]\mathcal{D}_c(Ta,Tb)}{1+\mathcal{D}_c(a,b)+\mathcal{D}_c(Ta,Tb)}\Bigg\}.$$

*For* $a_0 \in \Gamma$*, we take the sequence* $\{a_n\}$ *defined as* $a_{n+1} = T^n a_0$ *for every* $n \geq 0$*. Assume that*

*(i)* $f$ *and* $g$ *are bounded and non-decreasing,* $g$ *is sub-additive, and* $g(\lambda a) \prec a, \lambda \in (0, 1)$*;*

*(ii)* $\lim\limits_{n,m\to\infty} \sum_{i=m}^{n-2} \|g^{i-m} f\big(\xi^i \mathcal{D}_c(a_0, a_1)\big)\| + \|g^{n-m-1}\big(\xi^{n-1}\mathcal{D}_c(a_0, a_1)\big)\| = 0,$

*where* $\xi = \lambda(a_0, a_1) < 1$*. Then* $T$ *ensures a unique fixed point.*

*Proof.* The proof follows from Corollary 1 by taking the self-map $T : \Gamma \to \Gamma$ as $T_1 = T_2 = T$.

**Corollary 5.** *Suppose that* $(\Gamma, \mathcal{L}_c)$ *is an* $\mathcal{L}_c$*-complete* $\mathcal{DCCML}$*-space, where* $P$ *is a normal cone via normal constant* $M$*. Let* $T : \Gamma \to \Gamma$ *be a mapping and there exists* $\lambda_j \in \Delta, j = 1, \cdots, 5$*, such that*

$$\begin{aligned}\mathcal{L}_c(Ta,Tb) \preceq{}& \lambda_1(a,b)\mathcal{L}_c(a,b) + \lambda_2(a,b)\mathcal{L}_c(a,Ta) + \lambda_3(a,b)\mathcal{L}_c(b,Tb) \\ &+ \lambda_4(a,b)\frac{\mathcal{L}_c(a,Ta)\mathcal{L}_c(b,Tb)}{1+\mathcal{L}_c(a,b)} + \lambda_5(a,b)\frac{\mathcal{L}_c(b,Tb)\big[1+\mathcal{L}_c(a,Ta)\big]}{1+\mathcal{L}_c(a,b)} \\ &+ \lambda_6(a,b)\frac{\big[\mathcal{L}_c(a,Ta)+\mathcal{L}_c(b,Tb)\big]\mathcal{L}_c(Ta,Tb)}{1+\mathcal{L}_c(a,b)+\mathcal{L}_c(Ta,Tb)},\end{aligned} \tag{12}$$

*for all* $a, b \in \Gamma$ *with* $\sum_{j=1}^{6} \lambda_j(a, b) < 1$*.*
*For* $a_0 \in \Gamma$*, we take the sequence* $\{a_n\}$ *as* $a_{n+1} = T^n a_0$ *for every* $n \geq 0$*. Let* $\xi = \frac{\lambda_1(a_0,a_1)+\lambda_2(a_0,a_1)}{1-\sum_{j=3}^{6}\lambda_j(a_0,a_1)} < 1$*. Suppose that*

*(i)* $f$ *and* $g$ *are bounded and non-decreasing,* $g$ *is sub-additive, and* $g(\lambda a) \prec a, \lambda \in (0, 1)$*;*

*(ii)* $\lim\limits_{n,m\to\infty} \sum_{i=m}^{n-2} \|g^{i-m} f\big(\xi^i \mathcal{L}_c(a_0, a_1)\big)\| + \|g^{n-m-1}\big(\xi^{n-1}\mathcal{L}_c(a_0, a_1)\big)\| = 0.$

*If for every fixed point* $a$*, we conclude that* $\mathcal{L}_c(a, a) = 0_E$*, then* $T$ *ensures a unique fixed point.*

*Proof.* The proof follows from Corollary 2 by setting the self-map $T : \Gamma \to \Gamma$ as $T_1 = T_2 = T$.

**Remark 2.** *It is worth noting that the fourth member* $\frac{\mathcal{L}_c(a,Ta)\mathcal{L}_c(b,Tb)}{\mathcal{L}_c(a,b)}$ *in the sources [35, 39] raises some doubts. Indeed, it follows from the proof of the previous references and others results, as well as some examples in these works, from which we obtain the form* $\frac{0}{0}$*, a division by zero. This is incorrect since* $0$ *is the unique fixed point of map* $T$ *or the distance metric approaches zero. The main motivation for our new results complements and provides entirely new observations.*

**Remark 3.** *In the following we show that:*

*(i) Our results represent an improvement and generalization of the findings of Lateef [28]. On one hand, if* $\lambda_2 = \lambda_3 = \lambda_5 = \lambda_6 = 0$*, the new generalized contraction becomes a Fisher contraction or an improvement of Jaggi work [29, 30] regarding common fixed points or merely fixed points. Moreover, the conclusion still holds, i.e.,* $T$ *has a fixed point. On the other hand, we extend the result in* $\mathcal{DCCML}$*-spaces and C2CMS, instead of DCCMLS and DCCMTS. In other words, we broaden the result to* $\mathcal{DCCML}$*-spaces.*

*(ii) Special Cases for Corollaries 2 and 5:*
***Case 1.*** *if* $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_6 = 0$*, then we obtain the result of the Dass and Gupta contraction, where* $\lambda_1(a,b) = k_1$ *and* $\lambda_5(a,b) = k_2, k_1, k_2 \in (0,1)$*, (see, [29]).*
***Case 2.*** *if* $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 0$*, then we obtain the result of extending Banach contraction principle.*
***Case 3.*** *if* $\lambda_1 = \lambda_4 = \lambda_5 = \lambda_6 = 0$*, then we obtain the result of extending Kannan's contraction.*
***Case 4.*** *if* $\lambda_4 = \lambda_5 = \lambda_6 = 0$*, then we obtain the result of extending the Riech-type contraction (see [19]).*

*(iii) Through Remark 1, we know that every C2CMS is a* $\mathcal{DCCML}$*-space, and the self-distance in the latter does not need to be zero. Thus, the new results are still valid in C2CMS.*

*(iv) Towards the six-member in Corollary 2 we can that obtained as the form* $\frac{\mathcal{L}_c(a,T_1a)\mathcal{L}_c(b,T_1a)+\mathcal{L}_c(b,T_2b)\mathcal{L}_c(a,T_2b)}{1+\mathcal{L}_c(b,T_1a)+\mathcal{L}_c(a,T_2b)}$*, which implies that it is less than and equal to* $\mathcal{L}_c(a,T_1a) + \mathcal{L}_c(b,T_2b)$*. So, we go again to the second and third members. Thus, we discuss the result obtained.*

We present some examples below to verify our theorems.

**Example 4.** *Consider* $E = \mathbb{R}^2, P = \{\mathfrak{u} = (\mathfrak{r},\mathfrak{s}) \in E : \mathfrak{r},\mathfrak{s} \geq 0\}$*, and* $\Gamma = [0,1]$*. Now,* $\mathcal{L}_c : \Gamma \times \Gamma \to E$ *is defined by*
$\mathcal{L}_c(a,b) = \left(\sinh\left(\frac{(a+b)^2}{2}\right), 0\right)$*. Then* $\mathcal{L}_c$ *is a* $\mathcal{DCCML}$*-space with two functions,* $f(\mathfrak{u}) = \left(\sinh((a+b+2)\mathfrak{r}), 0\right)$ *and* $g(\mathfrak{u}) = \left(\sinh((a^2+b^2+1)\mathfrak{r}), 0\right)$*, where* $\mathfrak{u} \in P$*, by the same process*

*as for Example 3, where the function* $\frac{(a+b)^2}{2}$ *is a DCMLS. Moreover,* $(a_1,b_1)(a_2,b_2) = (a_1a_2, b_1b_2)$ *is defined in [40].*
*Define* $T_1, T_2 : \Gamma \to \Gamma$ *by* $T_1 a = \frac{a}{2}$ *and* $T_2 a = \frac{a}{3}$*, for* $a \in \mathbb{R}$*. Choose* $\lambda_j : \Gamma \times \Gamma \to [0,1)$*, for* $j = 1, \cdots, 6$ *by*
$\lambda_1(a,b) = \frac{5+a+b}{36}, \lambda_2(a,b) = \frac{3+a+b}{36}, \lambda_3(a,b) = \frac{2+a+b}{36}, \lambda_4(a,b) = \frac{4+a+b}{36},\ \lambda_5(a,b) = \frac{6+a+b}{36}$ *and* $\lambda_6(a,b) = 0$*. Then, evidently,* $\sum_{j=1}^{6} \lambda_j(a,b) = \frac{20+5a+5b}{36} < 1$*. Also,* $\lambda_j(a,b) \in \Delta$ *with two maps* $T_1$ *and* $T_2$ *for all* $j = 1, \cdots, 6$*.*
*Now, consider* $a_0 = 0$ *and* $a, b \in \Gamma$*. Then,*

$$\begin{aligned}
\mathcal{L}_c(T_1 a, T_2 b) &= \big(\sinh\big(\frac{(3a+2b)^2}{72}\big), 0\big) \\
&\preceq \frac{5+a+b}{36}\big(\sinh\big(\frac{(a+b)^2}{2}\big), 0\big) + \frac{3+a+b}{36}\big(\sinh\big(\frac{9a^2}{8}\big), 0\big) \\
&+ \frac{2+a+b}{36}\big(\sinh\big(\frac{16b^2}{18}\big), 0\big) + \frac{4+a+b}{36}\frac{\big(\sinh\big(\frac{9a^2}{8}\big)\sinh\big(\frac{16b^2}{18}\big), 0\big)}{1 + \big(\sinh\big(\frac{(a+b)^2}{2}\big), 0\big)} \\
&+ \frac{6+a+b}{36}\frac{\big(\sinh\big(\frac{16b^2}{18}\big), 0\big) + \big(\sinh\big(\frac{9a^2}{8}\big)\sinh\big(\frac{16b^2}{18}\big), 0\big)}{1 + \big(\sinh\big(\frac{(a+b)^2}{2}\big), 0\big)} \\
&= \lambda_1(a,b)\mathcal{L}_c(a,b) + \lambda_2(a,b)\mathcal{L}_c(a,Ta) + \lambda_3(a,b)\mathcal{L}_c(b,Tb) \\
&+ \lambda_4(a,b)\frac{\mathcal{L}_c(a,Ta)\mathcal{L}_c(b,Tb)}{1+\mathcal{L}_c(a,b)} + \lambda_5(a,b)\frac{\mathcal{L}_c(b,Tb)\big[1+\mathcal{L}_c(a,Ta)\big]}{1+\mathcal{L}_c(a,b)}.
\end{aligned}$$

*Hence, Corollary 2 is fulfilled and* $a = 0 \in \Gamma$ *is a common fixed point such that* $T_1 a = T_2 a = a$*.*

**Example 5.** *Consider* $E = \mathbb{R}$*, and* $P = \mathbb{R}^+$*. Let* $\Gamma = \{1,2,3\}$*. Define a symmetric map* $\mathcal{L}_c : \Gamma \times \Gamma \to E$ *by*
$\mathcal{L}_c(1,1) = \mathcal{L}_c(2,2) = 0, \mathcal{L}_c(3,3) = 2,$ *and*
$\mathcal{L}_c(1,2) = 11, \mathcal{L}_c(1,3) = 6, \mathcal{L}_c(3,2) = 3.$
*Take* $f, g : P \to P$*; it is defined by* $f(\mathfrak{u}) = \sinh\big(\frac{12}{11}\mathfrak{u}\big)$*; and* $g(\mathfrak{u}) = \big(\frac{3}{11}\mathfrak{u}\big)$*, where* $\mathfrak{u} \in P$*.*
*Evidently, valid that* $(\Gamma, \mathcal{L}_c)$ *is a* $\mathcal{L}_c$*-complete* $\mathcal{DCCML}$*-space regarding* $f, g$*, and* $\mathcal{L}_c(3,3) \neq 0$*. Therefore,* $(\Gamma, \mathcal{L}_c)$ *is not a C2CMS, see [20].*
*Further, let us define a map* $T : \Gamma \to \Gamma$ *as*

$$T(a) = \begin{cases} 2 & \text{if } a \in \{2,3\} \\ 3 & \text{if } a = 1. \end{cases}$$

*Then,* $T$ *ensures a unique fixed point.*

*Proof.* Let us take $\lambda(a,b) = \frac{3}{2+a+b} \in \Delta$, for all $a, b \in \Gamma$. Now, consider the following cases to show that Corollary 3 satisfies:
**Case 1.** $a = 1, b = 2,$
$\mathcal{L}_c(T1, T2) = \mathcal{L}_c(3,2) = 3 \preceq \frac{33}{5} = \frac{3}{5} \times 11 = \lambda(1,2)\max\{11, 6, 0, \frac{6\times 0}{1+11}, \frac{0\times[1+6]}{1+11}, \frac{[6+0]\times 3}{1+11+3}\};$

**Case 2.** $a = 1, b = 3$,
$\mathcal{L}_c(T1, T3) = \mathcal{L}_c(3, 2) = 3 \preceq 3 = \frac{3}{6} \times 6 = \lambda(1, 3) \max\{6, 6, 3, \frac{6\times3}{1+6}, \frac{3\times[1+6]}{1+6}, \frac{[6+3]\times3}{1+6+3}\}$;
**Case 3.** $a = 2, b = 3$,
$\mathcal{L}_c(T2, T3) = \mathcal{L}_c(2, 2) = 0 \preceq \frac{9}{7} = \frac{3}{7} \times 3 = \lambda(2, 3) \max\{3, 0, 3, \frac{0\times3}{1+3}, \frac{3\times[1+0]}{1+3}, \frac{[0+3]\times0}{1+3+0}\}$.
Since $\mathcal{L}_c(3, 3) \neq 0$, we further take
**Case 4.** $a = 1, b = 1$,
$\mathcal{L}_c(T1, T1) = \mathcal{L}_c(3, 3) = 2 \preceq \frac{126}{4} = \frac{3}{4} \times 42 = \lambda(1, 2) \max\{0, 6, 6, \frac{6\times6}{1+0}, \frac{6\times[1+6]}{1+0}, \frac{[6+6]\times2}{1+0+2}\}$.
Consider $a_0 = 2 \in \Gamma$. Thus, $a_n = T^n a_0 = 2$ for each $n \geq 1$. For condition (i) in Corollary 3, we reach that $\mathfrak{u} \prec f(\mathfrak{u}) = \sinh\left(\frac{12}{11}\mathfrak{u}\right), \left(\frac{3}{11}\mathfrak{u}\right) = g(\mathfrak{u}) \prec u, 0_E \prec \mathfrak{u}$, and $g(\lambda(a, b)\mathfrak{u}) \prec \mathfrak{u}, \mathfrak{u} \in P$. Moreover, we see that

$$\lim_{\substack{n\to\infty \\ a,b\in\Gamma}} \big(\lambda(a, b)\big)^n = 0.$$

Therefore, all the conditions of Corollary 3 are satisfied and a fixed point is given as $a = 2$.

Afterwards, we present the iterative fixed point $T^k, k > 1$ as follows.

**Theorem 2.** *Suppose $(\Gamma, \mathcal{L}_c)$ is an $\mathcal{L}_c$-complete $\mathcal{DCCML}$-space with two non-constant functions $f, g : P \to P$, where $P$ is a normal cone via normal constant $M$. Let $T : \Gamma \to \Gamma$ be a mapping and there exists $\lambda \in \Delta$ such that*

$$\mathcal{L}_c(T^k a, T^k b) \preceq \lambda(a, b)\widetilde{M}(a, b), \quad \text{for all } a, b \in \Gamma, \tag{13}$$

*where*

$$\widetilde{M}(a, b) = \max\Bigg\{\mathcal{L}_c(a, b), \mathcal{L}_c(a, T^k a), \mathcal{L}_c(b, T^k b), \frac{\mathcal{L}_c(a, T^k a)\mathcal{L}_c(b, T^k b)}{1 + \mathcal{L}_c(a, b)}, \frac{\mathcal{L}_c(b, T^k b)\big[1 + \mathcal{L}_c(a, T^k a)\big]}{1 + \mathcal{L}_c(a, b)}, \frac{\big[\mathcal{L}_c(a, T^k a) + \mathcal{L}_c(b, T^k b)\big]\mathcal{L}_c(T^k a, T^k b)}{1 + \mathcal{L}_c(a, b) + \mathcal{L}_c(T^k a, T^k b)}\Bigg\}.$$

*For $a_0 \in \Gamma$, we set a sequence $\{a_n\}$ defined as $a_{n+1} = T^n a_0$ for every $n \geq 0$. Suppose that*

*(i) $f$ and $g$ are bounded and non-decreasing, $g$ is sub-additive, and $g(\lambda a) \prec a, \lambda \in (0, 1)$;*

*(ii) $\lim_{n,m\to\infty} \sum_{i=m}^{n-2} \|g^{i-m} f\big(\xi^i \mathcal{L}_c(a_0, a_1)\big)\| + \|g^{n-m-1}\big(\xi^{n-1}\mathcal{L}_c(a_0, a_1)\big)\| = 0$,*

*where $\xi = \lambda(a_0, a_1) < 1$. If for every fixed point $a$, we conclude that $\mathcal{L}_c(a, a) = 0_E$, then $T^k$ has a unique fixed point.*

*Proof.* The proof follows from Corollary 3 by taking $T^k a = a$. Subsequently, we observe

$$T^k(Ta) = T(T^k a) = Ta.$$

Hence, $T^k$ has a fixed point $Ta$ and $Ta = a$, which means that $T^k$ has a unique fixed point, where $T$ has a fixed point $a$.

**Corollary 6.** *Suppose* $(\Gamma, \mathcal{L}_c)$ *is a complete C2CMS with two non-constant functions* $f, g : P \to P$*, where* $P$ *is a normal cone via normal constant* $M$*. Let* $T : \Gamma \to \Gamma$ *be a mapping and there exists* $\lambda \in \Delta$ *such that*

$$\mathcal{D}_c(T^k a, T^k b) \preceq \lambda(a,b)\widetilde{M}(a,b), \quad \textit{for all } a, b \in \Gamma, \tag{14}$$

*where*

$$\widetilde{M}(a,b) = \max\Bigg\{\mathcal{D}_c(a,b), \mathcal{D}_c(a,T^k a), \mathcal{D}_c(b,T^k b), \frac{\mathcal{D}_c(a,T^k a)\mathcal{D}_c(b,T^k b)}{1+\mathcal{D}_c(a,b)}, \frac{\mathcal{D}_c(b,T^k b)\big[1+\mathcal{D}_c(a,T^k a)\big]}{1+\mathcal{D}_c(a,b)}, \frac{\big[\mathcal{D}_c(a,T^k a)+\mathcal{D}_c(b,T^k b)\big]\mathcal{D}_c(T^k a,T^k b)}{1+\mathcal{D}_c(a,b)+\mathcal{D}_c(T^k a,T^k b)}\Bigg\}.$$

*For* $a_0 \in \Gamma$*, take the sequence* $\{a_n\}$ *defined as* $a_{n+1} = T^n a_0$ *for every* $n \geq 0$*. Suppose*

*(i)* $f$ *and* $g$ *are bounded and non-decreasing,* $g$ *is sub-additive, and* $g(\lambda a) \prec a, \lambda \in (0,1)$*;*

*(ii)* $\lim_{n,m\to\infty} \sum_{i=m}^{n-2} \|g^{i-m} f\big(\xi^i \mathcal{D}_c(a_0,a_1)\big)\| + \|g^{n-m-1}\big(\xi^{n-1}\mathcal{D}_c(a_0,a_1)\big)\| = 0,$

*where* $\xi = \lambda(a_0,a_1) < 1$*. Then* $T^k$ *ensures a unique fixed point.*

**Corollary 7.** *Assume* $(\Gamma, \mathcal{L}_c)$ *is an* $\mathcal{L}_c$*-complete* $\mathcal{DCCML}$*-space with two non-constant functions* $f, g : P \to P$*, where* $P$ *is a normal cone via normal constant* $M$*. Let* $T : \Gamma \to \Gamma$ *be a mapping and there exists* $\lambda_j \in \Delta, j = 1, \cdots, 6$*, such that*

$$\begin{aligned}\mathcal{L}_c(T^k a, T^k b) \preceq & \lambda_1(a,b)\mathcal{L}_c(a,b) + \lambda_2(a,b)\mathcal{L}_c(a,T^k a) + \lambda_3(a,b)\mathcal{L}_c(b,T^k b) \\ & + \lambda_4(a,b)\frac{\mathcal{L}_c(a,T^k a)\mathcal{L}_c(b,T^k b)}{1+\mathcal{L}_c(a,b)} + \lambda_5(a,b)\frac{\mathcal{L}_c(b,T^k b)\big[1+\mathcal{L}_c(a,T^k a)\big]}{1+\mathcal{L}_c(a,b)} \\ & + \lambda_6(a,b)\frac{\big[\mathcal{L}_c(a,T^k a)+\mathcal{L}_c(b,T^k b)\big]\mathcal{L}_c(T^k a,T^k b)}{1+\mathcal{L}_c(a,b)+\mathcal{L}_c(T^k a,T^k b)},\end{aligned} \tag{15}$$

*for all* $a, b \in \Gamma$*, where* $\sum_{j=1}^{6} \lambda_j(a,b) < 1$*.*
*For* $a_0 \in \Gamma$*, take the sequence* $\{a_n\}$ *as* $a_{n+1} = T^n a_0$ *for every* $n \geq 0$*. Let* $\xi = \frac{\lambda_1(a_0,a_1)+\lambda_2(a_0,a_1)}{1-\sum_{j=3}^{6}\lambda_j(a_0,a_1)} < 1$*. Suppose that*

*(i)* $f$ *and* $g$ *are bounded and non-decreasing,* $g$ *is sub-additive, and* $g(\lambda a) \prec a, \lambda \in (0,1)$*;*

*(ii)* $\lim_{n,m\to\infty} \sum_{i=m}^{n-2} \|g^{i-m} f\big(\xi^i \mathcal{L}_c(a_0,a_1)\big)\| + \|g^{n-m-1}\big(\xi^{n-1}\mathcal{L}_c(a_0,a_1)\big)\| = 0.$

*If for every fixed point* $a$*, we conclude that* $\mathcal{L}_c(a,a) = 0_E$*, then* $T^k$ *possesses a unique fixed point.*

## 4. Applications

The fixed-point results play a vital role in the existence of theory of various classes of equations, particularly, for solving differential equations, integral equations, and fractional differential equations. This has led to significant improvements in the applications of fixed-point techniques.

### 4.1. Non-linear Integral Equations

Consider $\Gamma = C[0,1]$, the class of all continuous functions on $[0,1]$. Let $E = C[0,1]$ so that $P = \{h(t) \in E : h(t) \geq 0, t \in [0,1]\}$ is equipped via the norm $\|\eta\| = \|\eta\|_\infty + \|\eta^{'}\|_\infty$. We endow $\Gamma$ with $\mathcal{DCCML}$-space as follows,

$$\mathcal{L}_c(\eta_1,\eta_2)(t) = \left(\sup_{t\in[0,1]} \sinh\left(|\eta_1(t)| + |\eta_2(t)|\right)^{\mathfrak{p}}\right)^{\frac{1}{\mathfrak{p}}} e^t, \text{ for each } \eta_1,\eta_2 \in \Gamma, \text{ and } \mathfrak{p} \geq 1. \quad (16)$$

Evidently, $(\Gamma, \mathcal{L}_c)$ is an $\mathcal{L}_c$-complete $\mathcal{DCCML}$-space, where $f(\mathfrak{u}) = [\sinh((1+\eta_1+\eta_2)(2\mathfrak{u})^{\mathfrak{p}})]^{\frac{1}{\mathfrak{p}}}$, and $g(\mathfrak{u}) = [\sinh((2+\eta_1^2+\eta_2^2)(2\mathfrak{u})^{\mathfrak{p}})]^{\frac{1}{\mathfrak{p}}}, \mathfrak{u} \in P$.

**Theorem 3.** *Assume that for each $\eta_1, \eta_1 \in \Gamma = C[0,1]$,*

*(i) There exist a function $\lambda \in \Delta$, and $0 < \beta < 1$, such that,*

$$|\Xi(t,\nu,\eta_1(\nu))| + |\Xi(t,\nu,\eta_2(\nu))| < \frac{\beta}{2}\lambda(\eta_1(\nu),\eta_2(\nu))(|\eta_1(\nu)| + |\eta_2(\nu)|); \quad (17)$$

*(ii) $\Xi\left(t,\nu,\int_0^1 \Xi(t,\nu,\eta(\nu))d\nu\right) < \Xi(t,\nu,\eta(\nu))$ for some $t,\nu \in [0,1]$.*

*Then, this integral equation*

$$\eta(\nu) = \int_0^1 \Xi(t,\nu,\eta(\nu))d\nu,$$

*admits a unique solution in $C[0,1]$.*

*Proof.* Let $T : \Gamma \to \Gamma$ be continuous defined by $T\eta(\nu) = \int_0^1 \Xi(t,\nu,\eta(\nu))d\nu$. Then

$$\mathcal{L}_c(T\eta_1,T\eta_2)(t) = \big(\sup_{t\in[0,1]} \sinh\Big(|T\eta_1(t)| + |T\eta_2(t)|\Big)^{\mathfrak{p}}\Big)^{\frac{1}{\mathfrak{p}}} e^t,$$

from Lemma 4, we have

$$\begin{aligned}
\big(\sinh\Big(|T\eta_1(t)| + |T\eta_2(t)|\Big)^{\mathfrak{p}}\Big)^{\frac{1}{\mathfrak{p}}} &\preceq \sinh|T\eta_1(t)| + |T\eta_2(t)| \preceq 2|T\eta_1(t)| + |T\eta_2(t)| \\
&= 2\left|\int_0^1 \Xi(t,\nu,\eta_1(\nu))d\nu\right| + \left|\int_0^1 \Xi(t,\nu,\eta_2(\nu))d\nu\right| \\
&\preceq 2\int_0^1 |\Xi(t,\nu,\eta_1(\nu))|\,d\nu + \int_0^1 |\Xi(t,\nu,\eta_2(\nu))|\,d\nu \\
&= 2\int_0^1 |\Xi(t,\nu,\eta_1(\nu))| + |\Xi(t,\nu,\eta_2(\nu))|\,d\nu \\
&\preceq 2\int_0^1 \frac{\beta}{2}\lambda(\eta_1(\nu),\eta_2(\nu))(|\eta_1(\nu)| + |\eta_2(\nu)|)d\nu
\end{aligned}$$

$$
\begin{aligned}
&= \beta \int_0^1 \lambda(\eta_1(\nu), \eta_2(\nu)) \left[(|\eta_1(\nu)| + |\eta_2(\nu)|)^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}} d\nu \\
&\preceq \beta \int_0^1 \lambda(\eta_1(\nu), \eta_2(\nu)) \left[\sinh(|\eta_1(\nu)| + |\eta_2(\nu)|)^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}} d\nu \\
&\preceq \beta \mathcal{L}_c(\eta_1, \eta_2)(t) \int_0^1 \lambda(\eta_1(\nu), \eta_2(\nu)) d\nu \\
&\preceq \beta \lambda(\eta_1(\nu), \eta_2(\nu)) \mathcal{L}_c(\eta_1, \eta_2)(t).
\end{aligned}
$$

We observe that $\mathcal{L}_c(T\eta_1, T\eta_2)(t) \preceq \beta\lambda(\eta_1(\nu), \eta_2(\nu))\mathcal{L}_c(\eta_1, \eta_2)(t)$, where $0 < \beta < 1$ and $\lambda \in \Delta$. Hence, all of the requirements for Corollary 5 have been met. We obtain the desired results.

### 4.2. Boundary Value Problems

The current study results will be applied to solve the first-order periodic BVPs:

$$
\begin{aligned}
\varpi'(\mathfrak{t}) &= h(\mathfrak{t}, \varpi(\mathfrak{t})),\ \mathfrak{t} \in [0, 1] \\
\varpi(0) &= \varpi(1),
\end{aligned} \tag{18}
$$

where $h : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function on $[0, 1]$. The problem above can be formulated as:

$$
\begin{aligned}
\varpi'(\mathfrak{t}) + \delta\varpi(\mathfrak{t}) &= h(\mathfrak{t}, \varpi(\mathfrak{t})) + \delta\varpi(\mathfrak{t}),\ \mathfrak{t} \in [0, 1] \\
\varpi(0) &= \varpi(1).
\end{aligned} \tag{19}
$$

The problem (19) is equivalent to the following integral equation:

$$
\varpi(\mathfrak{t}) = \int_0^1 \mathbb{G}(\mathfrak{t}, \mathfrak{u})\big(h(\mathfrak{u}, \varpi(\mathfrak{u})) + \delta\varpi(\mathfrak{u})\big) d\mathfrak{u}, \tag{20}
$$

where $\mathbb{G}$ is a Green function defined by

$$
\mathbb{G}(\mathfrak{t}, \mathfrak{u}) = \begin{cases} \frac{e^{\delta(\mathfrak{u}-\mathfrak{t}+1)}}{e^{\delta}-1} & 0 \leq \mathfrak{u} \leq \mathfrak{t} \\ \frac{e^{\delta(\mathfrak{u}-\mathfrak{t})}}{e^{\delta}-1} & \mathfrak{t} \leq \mathfrak{u} \leq 1. \end{cases}
$$

Thus, it is noticed that $\int_0^1 \mathbb{G}(\mathfrak{t}, \mathfrak{u}) d\mathfrak{u} = \frac{1}{\delta}$.
Let $\Gamma = C[0, 1]$. Define $\mathcal{L}_c : \Gamma \times \Gamma \to E$, where $E = C[0, 1], P = \{\varphi(\mathfrak{t}) \in E : \varphi(\mathfrak{t}) \geq 0, \mathfrak{t} \in [0, 1]\}$ is a $\mathcal{DCCML}$-space, by

$$
\mathcal{L}_c(\varpi_1, \varpi_2)(t) = \left(e^{|\varpi_1(\mathfrak{t})| + |\varpi_2(\mathfrak{t})|} - 1\right) \varphi(\mathfrak{t}), \tag{21}
$$

where $\varphi(\mathfrak{t}) = e^{\mathfrak{t}} > 0$, and $(\Gamma, \mathcal{L}_c)$ is an $\mathcal{L}_c$-complete $\mathcal{DCCML}$-space via $f(\mathfrak{u}) = g(\mathfrak{u}) = \left(\frac{\mathfrak{u}^2+2\mathfrak{u}}{2}\right)e^{\mathfrak{t}}$. Moreover, let $T : \Gamma \to \Gamma$ be a mapping defined by

$$
T\varpi(\mathfrak{t}) = \int_0^1 \mathbb{G}(\mathfrak{t}, \mathfrak{u})\big(h(\mathfrak{u}, \varpi(\mathfrak{u})) + \delta\varpi(\mathfrak{u})\big) d\mathfrak{u}, \tag{22}
$$

Corollary 3 is utilized to show that $T$ has a unique fixed point, which is the solution for the BVP (18).

**Theorem 4.** *Assume that there exists $\delta > 0$ such that, for each $\varpi_1, \varpi_2 \in \Gamma$,*

$$|h(\mathrm{t}, \varpi_1(\mathrm{t})) + \delta\varpi_1(\mathrm{t})| + |h(\mathrm{t}, \varpi_2(\mathrm{t})) + \delta\varpi_2(\mathrm{t})| \leq \frac{\delta}{3}(|\varpi_1(\mathrm{t})| + |\varpi_2(\mathrm{t})|).$$

*Then, BVP (18) possesses a unique solution in $\Gamma$.*

*Proof.* Let $\mathcal{L}_c$ be a mapping given in (21), $T$ be the operator function in (22). Then

$$\begin{aligned}
\mathcal{L}_c(T\varpi_1, T\varpi_2)(t) &= \left(e^{|T\varpi_1(\mathrm{t})|+|T\varpi_2(\mathrm{t})|} - 1\right) e^{\mathrm{t}} \\
&= \left(e^{\left|\int_0^1 \mathbb{G}(\mathrm{t},\mathrm{u})(h(\mathrm{u},\varpi_1(\mathrm{u}))+\delta\varpi_1(\mathrm{u}))du\right| + \left|\int_0^1 \mathbb{G}(\mathrm{t},\mathrm{u})(h(\mathrm{u},\varpi_2(\mathrm{u}))+\delta\varpi_2(\mathrm{u}))du\right|} - 1\right) e^{\mathrm{t}} \\
&\preceq \left(e^{\int_0^1 \mathbb{G}(\mathrm{t},\mathrm{u})|h(\mathrm{u},\varpi_1(\mathrm{u})+\delta\varpi_1(\mathrm{u}))| + |h(\mathrm{u},\varpi_2(\mathrm{u}))+\delta\varpi_2(\mathrm{u})|du} - 1\right) e^{\mathrm{t}} \\
&\preceq \left(e^{\int_0^1 \mathbb{G}(\mathrm{t},\mathrm{u})\frac{\delta}{3}(|\varpi_1(\mathrm{u})|+|\varpi_2(\mathrm{u})|)du} - 1\right) e^{\mathrm{t}} \\
&\preceq \left(e^{\frac{\delta}{3}(|\varpi_1(\mathrm{t})|+|\varpi_2(\mathrm{t})|)\int_0^1 \mathbb{G}(\mathrm{t},\mathrm{u})du} - 1\right) e^{\mathrm{t}} \\
&\preceq \left(e^{\delta(|\varpi_1(\mathrm{t})|+|\varpi_2(\mathrm{t})|)\frac{1}{\delta}} - 1\right) \frac{e^{\mathrm{t}}}{3}, \quad (\text{since } e^{rt} - 1 \leq r(e^t - 1), r = \frac{1}{3} \in (0,1)). \\
&\preceq \mathcal{L}_c(\varpi_1, \varpi_2) \sup_{\mathrm{t}\in[0,1]} \frac{e^{\mathrm{t}}}{3}. \qquad (\text{Since } 1 \leq e^t, t \in [0,1]).
\end{aligned}$$

We deduce that $\mathcal{L}_c(T\varpi_1, T\varpi_2) \preceq \lambda(\mathrm{t})\mathcal{L}_c(\varpi_1, \varpi_2) \preceq \lambda(\mathrm{t})\widetilde{M}(\varpi_1, \varpi_2)$, where $\widetilde{M}(\varpi_1, \varpi_2)$ in (10), and $\lambda(\mathrm{t}) = \sup_{\mathrm{t}\in[0,1]} \frac{e^{\mathrm{t}}}{3} \in (0,1)$. Therefore, $T$ is a generalized rational contraction, and all the conditions in Corollary 3 hold. Thus, we obtain the desired result.

## 5. Conclusions

This research introduces a novel concept in the realm of generalized metric spaces, called double-composed cone-metric-like spaces, which is illustrated through a series of examples. We derived generalization rational-type contraction theorems for a variety of mappings, termed common fixed points in double-composed cone-metric-like spaces and provided a number of related results to support our theorems. Furthermore, we presented numerous examples to substantiate the main results of our study. The study demonstrates applications of nonlinear integral equations and BVPs, proving the existence of solutions. This particular new generalization provides valuable tools for studying fixed point theorems.

The following points outline potential open problems and avenues for future research:

- Explore new generalizations of double-composed cone-metric-like spaces, such as fuzzy double-composed metric-like spaces, fuzzy double-composed cone-metric-like spaces, and neutrosophic double-composed cone-metric-like spaces.

- Establish new fixed point results in various types of contractions, including new nonlinear rational contractions, weak contractions, almost-contraction, and $(\phi, F)$-contraction, among others.

- Develop deep and non-trivial applications of our main results to further expand the scope of our research.

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## Author Contributions

A. A., L. Kh., S. A. and N. M. wrote the main manuscript text. All authors reviewed the manuscript.

## Conflicts of Interest

The authors declare no conflicts of interest.

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