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The Composition of Modified Reflection Operators and Their Fixed Point Sets

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Abstract. The modified reflection operator plays a crucial role in optimization, particularly in algorithms designed to solve constrained optimization problems. By effectively transforming feasible solutions while maintaining their viability within defined constraints, this operator enables smoother navigation through the solution space. It enhances convergence rates and stability in iterative methods, such as projected gradient descent and proximal algorithms. In this paper, we investigate the fixed point sets of the compositions of three modified reflection operators onto linear closed subspaces. We also derive formulas for the compositions under different parameters.

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1. Introduction

The modified reflection operator is a vital component in optimization, particularly in algorithms that address constrained optimization problems. This operator facilitates the transformation of feasible solutions while ensuring they remain within the defined constraints, allowing for a more efficient exploration of the solution space. By enhancing the convergence rates and stability of iterative methods, such as projected gradient descent and proximal algorithms, the modified reflection operator significantly improves the performance of optimization algorithms (see [1], [2], [3], [4], [5], [6], and [7] for more information). Moreover, it is particularly effective in navigating non-convex landscapes, as it aids in the exploration of local minima, thereby reducing the risk of stagnation in suboptimal solutions. Its adaptability to various constraints further underscores its versatility, making it an essential tool in a wide range of applications, from machine learning

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to engineering design. Overall, the modified reflection operator contributes to more robust and efficient optimization processes, ensuring that algorithms can effectively tackle real-world problems (see [8] for more information). Throughout, we assume that

$$\mathcal{H} \text{ is } \textit{a real Hilbert space} \text{ with inner product } \langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{R}, \tag{1}$$

and induced norm $\|\cdot\| \colon \mathcal{H} \to \mathbb{R} \colon x \mapsto \sqrt{\langle x, x \rangle}$. Let $A : \mathcal{H} \rightrightarrows \mathcal{H}$ be an arbitrary set valued operator, i.e., $Ax \subseteq \mathcal{H} \ \ (\forall x \in \mathcal{H})$. The *graph* of A, donoted by Then gra A, is defined as

$$\operatorname{gra} A = \Big\{ (x, y) \in \mathcal{H} \times \mathcal{H} \mid y \in Ax \Big\}.$$

A set-valued operator $A : \mathcal{H} \rightrightarrows \mathcal{H}$ is a *monotone* if

$$\Big(\forall (x, u) \in \operatorname{gra} A\Big) \ \Big(\forall (y, v) \in \operatorname{gra} A\Big) \quad \langle x - y, u - v \rangle \geq 0.$$

A monotone operator A is a *maximally monotone* if there exists no monotone operator B such that gra $A \subset$ gra B. That is, for every $(x, u) \in \mathcal{H} \times \mathcal{H}$,

$$\Big((x, u) \in \operatorname{gra} A\Big) \ \Leftrightarrow \ \Big(\forall\ (y, v) \in \operatorname{gra} A\Big) \quad \langle x - y, u - v \rangle \geq 0.$$

The Id is the *identity operator* defines as $\operatorname{Id} : \mathcal{H} \to \mathcal{H} : x \to x$ and satisfies (8).

Definition 1. [9, Definition 3.28] Let A be a monotone operator from $\mathcal{H} \rightrightarrows \mathcal{H}$ and denote the associated *resolvent* by

$$J_A = (\operatorname{Id} + A)^{-1}. \tag{2}$$

The *reflected resolvent* of A is denoted by R_A and defined by

$$R_A = 2J_A - \operatorname{Id}. \tag{3}$$

Example 1. Let $A = \operatorname{Id}$. Then $J_A = \frac{1}{2}\operatorname{Id}$ and $R_A = 0$. To show that let $y \in \mathcal{H}$ and set $x = J_A y$. Then $y \in \big(\operatorname{Id} + A\big)x$. This implies that $y = x + x \Leftrightarrow y = 2x \Leftrightarrow x = \frac{1}{2}y$. Therefore, $J_A = \frac{1}{2}\operatorname{Id}$. Using (3) gives $R_A = 2(1/2)\operatorname{Id} - \operatorname{Id} = 0$.

Definition 2. [9, Definition 4.1] Let U be a nonempty subset of $\mathcal{H}$. A mapping $T : U \to \mathcal{H}$ is *nonexpansive* or *Lipschitz continuous* with constant 1, i.e.,

$$\big(\forall x \in U\big) \ \big(\forall y \in U\big) \quad \|Tx - Ty\| \leq \|x - y\|. \tag{4}$$

Moreover, $T : U \to \mathcal{H}$ is *firmly nonexpansive* if

$$\big(\forall x \in U\big)\big(\forall y \in U\big) \quad \|Tx - Ty\|^2 + \|(\operatorname{Id} - T)x - (\operatorname{Id} - T)y\|^2 \leq \|x - y\|^2. \tag{5}$$

The Fix T is *the set of fixed points* of T defined as

$$\operatorname{Fix} T := \{x \in \mathcal{H} \mid x = Tx\}. \tag{6}$$

Definition 3. Let U be a nonempty closed and convex subset of $\mathcal{H}$ and let $z \in \mathcal{H}$. The *projection operator* (this is also known as the closet point mapping) of z onto U is the unique point in U denoted by $\mathrm{P}_U z$ that satisfies

$$\|z - x\| = \inf\|u - z\|, \quad \text{where } x = \mathrm{P}_U z.$$

Let U be closed linear subspace. Then,

$$R_U := 2\,\mathrm{P}_U - \mathrm{Id}\,. \tag{7}$$

and

$$\mathrm{Id} := \mathrm{P}_U + \mathrm{P}_{U^\perp}\,. \tag{8}$$

Example 2. Let U be a nonempty closed convex subset of $\mathcal{H}$ and let $R_U = 2\,\mathrm{P}_U - \mathrm{Id}$ be a nonexpansive operator on U. Then Fix $R_U = C$. To show that let $x \in \mathcal{H}$. Then $x = R_U x$. Using (3) gives $x = 2\,\mathrm{P}_U\, x - x$. This implies that $\mathrm{P}_U\, x = x \Leftrightarrow x \in U$.

Example 3. Let $A = a + \mathrm{P}_U$, where U is a closed linear subspace of $\mathcal{H}$ and $a \in \mathcal{H}$. Then $J_A = \big(\mathrm{Id} - \frac{1}{2}\,\mathrm{P}_U\big) + \big(\frac{1}{2}\,\mathrm{P}_U - \mathrm{Id}\big)a$ and $R_A = \big(\mathrm{Id} - \mathrm{P}_U\big) + \big(\mathrm{P}_U - 2\,\mathrm{Id}\big)a$.

Proof. See [9, Lemma 4.3 (i) and (ii)]. ■

For more details about the composition of reflectors, see [10], [9], [11], [2], [12], and [13]. A comprehensive analysis of nonexpansive mappings under the condition of isometry of finite order of R was provided in [9, Lemma] and [14, Section 3]. We refer the reader to [15, Exercise 12.16], [2, Example 20.29], and [16]

In this paper, we study the composition of three modified reflection operators and their fixed point sets.. Our results can be summarized as follows:

- Lemma 1 and Lemma 2 provide key properties concerning the fixed point set of the composition of two modified reflection operators.

- Theorem 1, Theorem 2, and Theorem 3 show that the sequence in which three modified reflection operators are applied affects the fixed point set of their composition. These theorems offer valuable insights into the fixed point set resulting from the composition of three modified reflection operators.

- Under different parameters $\gamma, \beta, \alpha \in (0, 1]$, we derive formulas for the composition of two modified reflection operators (see Lemma 1 and Lemma 2). Additionally, formulas for the composition of three modified reflection operators are given in Theorem 1, Theorem 2, and Theorem 3.

The notation employed in this paper is standard and closely aligns with that in [9, 17], and [2].

2. Results

In this section, we will present significant new results regarding the composition of two and three modified reflection operators, as well as their corresponding fixed point sets. We will explore the properties and interactions of these operators, highlighting how their compositions influence the structure of the fixed point sets.

Lemma 1. Let U be a closed linear subspace of $\mathcal{H}$, and let $U^\perp$ denote the orthogonal complement of U. Let $\beta \in]0,1]$. Recall from (7) that

$$R_U := 2\,\mathrm{P}_U - \mathrm{Id},$$

where P_U is defined in Definition 3. The following results hold:

(i) $R_{U^\perp} R_U = -\mathrm{Id} = R_U R_{U^\perp}$.
(ii) $\big(2\beta\,\mathrm{P}_U - \mathrm{Id}\big) = \beta R_U + (1-\beta)(-\mathrm{Id})$.
(iii) $\big(2\beta\,\mathrm{P}_{U^\perp} - \mathrm{Id}\big) = \beta R_{U^\perp} + (1-\beta)(-\mathrm{Id})$.
(iv) $-\big(2\beta\,\mathrm{P}_U - \mathrm{Id}\big) = \beta R_{U^\perp} + (1-\beta)\,\mathrm{Id}$.
(v) $\big(2\beta\,\mathrm{P}_U - \mathrm{Id}\big) \circ (-\mathrm{Id}) = -\big(2\beta\,\mathrm{P}_U - \mathrm{Id}\big)$.
(vi) $\mathrm{Fix}\,\big(\big(2\beta\,\mathrm{P}_U - \mathrm{Id}\big) \circ (-\mathrm{Id})\big) = \mathrm{Fix}\,\big(-\big(2\beta\,\mathrm{P}_U - \mathrm{Id}\big)\big) = \mathrm{Fix}\,\big(\beta R_U + (1-\beta)(-\mathrm{Id})\big) = U^\perp$.

Proof.
(i): See [9, Lemma 6.2 (i)].
(ii): Using (7) gives

$$\begin{aligned}
\big(2\beta\,\mathrm{P}_U - \mathrm{Id}\big) &= 2\beta\,\mathrm{P}_U - \mathrm{Id} + \beta\,\mathrm{Id} - \beta\,\mathrm{Id} \\
&= \big(2\beta\,\mathrm{P}_U - \beta\,\mathrm{Id}\big) + (1-\beta)(-\mathrm{Id}) \\
&= \beta R_U + (1-\beta)(-\mathrm{Id}).
\end{aligned}$$

(iii): Applying (7) yields

$$\begin{aligned}
\big(2\beta\,\mathrm{P}_{U^\perp} - \mathrm{Id}\big) &= 2\beta\,\mathrm{P}_{U^\perp} - \mathrm{Id} + \beta\,\mathrm{Id} - \beta\,\mathrm{Id} \\
&= \big(2\beta\,\mathrm{P}_{U^\perp} - \beta\,\mathrm{Id}\big) + (1-\beta)(-\mathrm{Id}) \\
&= \beta R_{U^\perp} + (1-\beta)(-\mathrm{Id}).
\end{aligned}$$

(iv): Using (7), (8), and (ii), we obtain

$$\begin{aligned}
-\big(2\beta\,\mathrm{P}_U - \mathrm{Id}\big) &= -\big(\beta R_U + (1-\beta)(-\mathrm{Id})\big) \\
&= -\beta R_U + (1-\beta)\,\mathrm{Id} \\
&= -\beta\big(2\,\mathrm{P}_U - \mathrm{Id}\big) + (1-\beta)\,\mathrm{Id} \\
&= -\beta\big(\,\mathrm{P}_U - \big(\mathrm{Id} - \mathrm{P}_U\big)\big) + (1-\beta)\,\mathrm{Id} \\
&= -\beta\big(\,\mathrm{P}_U - \mathrm{P}_{U^\perp}\big) + (1-\beta)\,\mathrm{Id} \\
&= -\beta\big(\,\mathrm{P}_U - \mathrm{P}_{U^\perp} + \mathrm{P}_{U^\perp} - \mathrm{P}_{U^\perp}\big) + (1-\beta)\,\mathrm{Id}
\end{aligned}$$

$$
\begin{aligned}
&= -\beta\big(\,\mathrm{P}_U + \mathrm{P}_{U^\perp} - 2\,\mathrm{P}_{U^\perp}\,\big) + (1-\beta)\,\mathrm{Id} \\
&= -\beta\big(\,\mathrm{Id} - 2\,\mathrm{P}_{U^\perp}\,\big) + (1-\beta)\,\mathrm{Id} \\
&= \beta R_{U^\perp} + (1-\beta)\,\mathrm{Id},
\end{aligned}
$$

as required.

(v): Let $x \in \mathcal{H}$. By using (ii) and (iv), we have

$$
\begin{aligned}
\big(2\beta\,\mathrm{P}_U - \mathrm{Id}\,\big)\circ(-\,\mathrm{Id})(x) &= \big(2\beta\,\mathrm{P}_U - \mathrm{Id}\,\big)(-x) \\
&= 2\beta\,\mathrm{P}_U(-x) - (-x) \\
&= -2\beta\,\mathrm{P}_U(x) + x \\
&= -\big(2\beta\,\mathrm{P}_U\,x - x\big) \\
&= \beta R_{U^\perp} x + (1-\beta)x
\end{aligned}
$$

(vi): It follows from (ii), (iv), and (v) that

$$
\mathrm{Fix}\,\big(\big(2\beta\,\mathrm{P}_U - \mathrm{Id}\,\big)\circ(-\,\mathrm{Id})\big) = \mathrm{Fix}\,\big(-\,\big(2\beta\,\mathrm{P}_U - \mathrm{Id}\,\big)\big) = \mathrm{Fix}\,\big(\beta R_U + (1-\beta)(-\,\mathrm{Id})\big).
$$

Let $x \in \mathcal{H}$. Then

$$
\begin{aligned}
x &= \beta R_{U^\perp}(x) + (1-\beta)x \\
&= \beta R_{U^\perp}(x) + x - \beta x,
\end{aligned}
$$

therefore,

$$
\begin{aligned}
x - x &= \beta R_{U^\perp}(x) - \beta x \\
0 &= \beta\big(2\,\mathrm{P}_{U^\perp}\,x - x\big) - \beta x \\
0 &= 2\beta\,\mathrm{P}_{U^\perp}\,x - 2\beta x,
\end{aligned}
$$

and

$$
2\beta x = 2\beta\,\mathrm{P}_{U^\perp}\,x.
$$

Hence,

$$
x = \mathrm{P}_{U^\perp}\,x \quad \Leftrightarrow \quad \mathrm{Fix}\,\big(\big(2\beta\,\mathrm{P}_U - \mathrm{Id}\,\big)\circ(-\,\mathrm{Id})\big) = U^\perp.
$$

■

Lemma 2. Let U be a closed linear subspace of $\mathcal{H}$, and let $\gamma, \beta \in]0,1]$. Then the following holds:

$$
\big(2\beta\,\mathrm{P}_{U^\perp} - \mathrm{Id}\,\big)\big(2\gamma\,\mathrm{P}_U - \mathrm{Id}\,\big) = \begin{cases} -\,\mathrm{Id}, & \text{for } \beta = \gamma = 1 \\ \mathbf{0}, & \text{for } \beta = \gamma = 1/2 \\ \mathrm{Id} - 2\big(\beta\,\mathrm{P}_{U^\perp} + \gamma\,\mathrm{P}_U\,\big), & \text{for } \beta, \gamma \neq 1 \end{cases}
$$

$$
= \big(2\gamma\,\mathrm{P}_U - \mathrm{Id}\,\big)\big(2\beta\,\mathrm{P}_{U^\perp} - \mathrm{Id}\,\big).
$$

Proof. Using Lemma 1, (i), (ii), and (iii) yields

$$\begin{aligned}(2\beta \mathrm{P}_{U^\perp} - \mathrm{Id})(2\gamma \mathrm{P}_U - \mathrm{Id}) &= (\beta R_{U^\perp} + (1-\beta)(-\mathrm{Id}))(\gamma R_U + (1-\gamma)(-\mathrm{Id}))\\ &= -\beta\gamma \mathrm{Id} + \beta(1-\gamma)R_U - (1-\beta)\gamma R_U + (1-\beta)(1-\gamma)\mathrm{Id}\\ &= \beta R_U - \gamma R_U - (\beta+\gamma)\mathrm{Id} + \mathrm{Id}\\ &= \mathrm{Id} - 2(\beta \mathrm{P}_{U^\perp} + \gamma \mathrm{P}_U).\end{aligned}$$

There are 3 cases:
Case 1: If $\beta = \gamma = 1$, then

$$\begin{aligned}(2\beta \mathrm{P}_{U^\perp} - \mathrm{Id})(2\gamma \mathrm{P}_U - \mathrm{Id}) &= \mathrm{Id} - 2(\mathrm{P}_{U^\perp} + \mathrm{P}_U)\\ &= \mathrm{Id} - 2\,\mathrm{Id} = -\mathrm{Id},\end{aligned}$$

by (8).

Case 2: If $\beta = \gamma = 1/2$, then

$$\begin{aligned}(\mathrm{P}_{U^\perp} - \mathrm{Id})(\mathrm{P}_U - \mathrm{Id}) &= \mathrm{Id} - (\mathrm{P}_{U^\perp} + \mathrm{P}_U)\\ &= \mathrm{Id} - \mathrm{Id} = \mathbf{0},\end{aligned}$$

Case 3: If $\beta, \gamma \neq 1$ and $\beta, \gamma \neq 1/2$,then

$$(2\beta \mathrm{P}_{U^\perp} - \mathrm{Id})(2\gamma \mathrm{P}_U - \mathrm{Id}) = \mathrm{Id} - 2(\beta \mathrm{P}_{U^\perp} + \gamma \mathrm{P}_U).$$

Applying Lemma 1, (i), (ii), and (iii), the same strategy can be appied to show that

$$(2\gamma \mathrm{P}_U - \mathrm{Id})(2\beta \mathrm{P}_{U^\perp} - \mathrm{Id}) = \begin{cases} -\mathrm{Id}, & \text{for } \gamma = \beta = 1\\ \mathbf{0}, & \text{for } \gamma = \beta = 1/2\\ \mathrm{Id} - 2(\gamma \mathrm{P}_U + \beta \mathrm{P}_{U^\perp}), & \text{for } \gamma, \beta \neq 1 \end{cases}$$

Therefore,

$$(2\beta \mathrm{P}_{U^\perp} - \mathrm{Id})(2\gamma \mathrm{P}_U - \mathrm{Id}) = (2\gamma \mathrm{P}_U - \mathrm{Id})(2\beta \mathrm{P}_{U^\perp} - \mathrm{Id}).$$

■

Example 4. *Let $X = \mathbb{R}^2$ and suppose that $U = \mathbb{R} \times \{0\}$ and $x = (2,2)$. Then $U^\perp = \{0\} \times \mathbb{R}$, and by Lemma 2, we obtain the following:*

Case 1. *If $\beta = \gamma = 1$, then*

$$\begin{aligned}\Big(2\,\mathrm{P}_{\{0\}\times\mathbb{R}} - \mathrm{Id}\Big)\Big(2\,\mathrm{P}_{\mathbb{R}\times\{0\}}(2,2) - (2,2)\Big) &= \Big(2\,\mathrm{P}_{\mathbb{R}\times\{0\}} - \mathrm{Id}\Big)\Big(2\,\mathrm{P}_{\{0\}\times\mathbb{R}}(2,2) - (2,2)\Big)\\ &= (-2,-2).\end{aligned}$$

Case 2. *If* $\beta = \gamma = 1/2$, *then*

$$\Big(\mathrm{P}_{\{0\}\times\mathbb{R}} - \mathrm{Id}\Big)\Big(\mathrm{P}_{\mathbb{R}\times\{0\}}(2,2) - (2,2)\Big) = \Big(\mathrm{P}_{\mathbb{R}\times\{0\}} - \mathrm{Id}\Big)\Big(\mathrm{P}_{\{0\}\times\mathbb{R}}(2,2) - (2,2)\Big)$$
$$= \big(0,0\big).$$

Case 3. *If* $\beta,\gamma \neq 1/2$ *and* $\beta,\gamma \neq 1$, *then*

$$\big(2\beta\,\mathrm{P}_{\{0\}\times\mathbb{R}} - \mathrm{Id}\,\big)\big(2\gamma\,\mathrm{P}_{\mathbb{R}\times\{0\}}(2,2) - (2,2)\big) = \big(2\gamma\,\mathrm{P}_{\mathbb{R}\times\{0\}} - \mathrm{Id}\,\big)\big(2\beta\,\mathrm{P}_{\{0\}\times\mathbb{R}}(2,2) - (2,2)\big)$$
$$= (2,2) - 2\big(\beta(0,2) + \gamma(2,0)\big)$$

Note that when $\beta,\gamma \to 0$, *then*

$$\big(2\beta\,\mathrm{P}_{\{0\}\times\mathbb{R}} - \mathrm{Id}\,\big)\big(2\gamma\,\mathrm{P}_{\mathbb{R}\times\{0\}}(2,2) - (2,2)\big) \to (2,2).$$

Additionally, when $\beta,\gamma \to 1$, *then*

$$\big(2\beta\,\mathrm{P}_{\{0\}\times\mathbb{R}} - \mathrm{Id}\,\big)\big(2\gamma\,\mathrm{P}_{\mathbb{R}\times\{0\}}(2,2) - (2,2)\big) \to (-2,-2),$$

which satisfies the first case.

From now and on, deffine the modified reflector operators;

$$R_{U,\gamma} := 2\gamma\,\mathrm{P}_U - \mathrm{Id}, \tag{9}$$

$$R_{U^\perp,\beta} := 2\beta\,\mathrm{P}_{U^\perp} - \mathrm{Id}, \tag{10}$$

$$R_{V,\alpha} := 2\alpha\,\mathrm{P}_V - \mathrm{Id}\,. \tag{11}$$

Theorem 1. Let U and V be claosed linear subspaces of $\mathcal{H}$. Suppose that $\beta,\gamma,\alpha \in]0,1]$ and recall from (9), (10), and (11) the modified reflector operators. If $\beta,\gamma = 1$, then the following are holds true:

(i) $R_{V,\alpha}R_{U^\perp,\beta}R_{U,\gamma} = \alpha R_{V^\perp} + (1-\alpha)\,\mathrm{Id}$

(ii) $R_{V,\alpha}R_{U,\gamma}R_{U^\perp,\beta} = \alpha R_{V^\perp} + (1-\alpha)\,\mathrm{Id}$

(iii) $R_{U^\perp,\beta}R_{U,\gamma}R_{V,\alpha} = \alpha R_{V^\perp} + (1-\alpha)\,\mathrm{Id}$

(iv) $R_{U,\gamma}R_{U^\perp,\beta}R_{V,\alpha} = \alpha R_{V^\perp} + (1-\alpha)\,\mathrm{Id}$

(v) $\mathrm{Fix}\,\big(R_{V,\alpha}R_{U^\perp,\beta}R_{U,\gamma}\big) = V^\perp$

(vi) $\mathrm{Fix}\,\big(R_{V,\alpha}R_{U,\gamma}R_{U^\perp,\beta}\big) = V^\perp$

(vii) $\mathrm{Fix}\,\big(R_{U^\perp,\beta}R_{U,\gamma}R_{V,\alpha}\big) = V^\perp$

(viii) $\mathrm{Fix}\,\big(R_{U,\gamma}R_{U^\perp,\beta}R_{V,\alpha}\big) = V^\perp$

Additionally, if $\beta,\gamma \neq 1$ and $\alpha = 1$, then the following hold true:

(ix) $R_{V,\alpha}R_{U^\perp,\beta}R_{U,\gamma} = 2\,\mathrm{P}_V + 2\gamma\,\mathrm{P}_U + 2\beta\,\mathrm{P}_{U^\perp} - 4\gamma\,\mathrm{P}_V\,\mathrm{P}_U - 4\beta\,\mathrm{P}_V\,\mathrm{P}_{U^\perp} - \mathrm{Id}$.
(x) $R_{V,\alpha}R_{U,\gamma}R_{U^\perp,\beta} = R_{V,\alpha}R_{U^\perp,\beta}R_{U,\gamma}$
(xi) $R_{U^\perp,\beta}R_{U,\gamma}R_{V,\alpha} = 2\,\mathrm{P}_V + 2\gamma\,\mathrm{P}_U + 2\beta\,\mathrm{P}_{U^\perp} - 4\gamma\,\mathrm{P}_U\,\mathrm{P}_V - 4\beta\,\mathrm{P}_{U^\perp}\,\mathrm{P}_V - \mathrm{Id}$
(xii) $R_{U,\gamma}R_{U^\perp,\beta}R_{V,\alpha} = R_{U^\perp,\beta}R_{U,\gamma}R_{V,\alpha}$.

Moreover, if $\beta, \gamma, \alpha \neq 1$, then the following hold true:
(xiii) $R_{V,\alpha}R_{U^\perp,\beta}R_{U,\gamma} = 2\alpha\,\mathrm{P}_V + 2\gamma\,\mathrm{P}_U + 2\beta\,\mathrm{P}_{U^\perp} - 4\alpha\gamma\,\mathrm{P}_V\,\mathrm{P}_U - 4\alpha\beta\,\mathrm{P}_V\,\mathrm{P}_{U^\perp} - \mathrm{Id}$.
(xiv) $R_{V,\alpha}R_{U,\gamma}R_{U^\perp,\beta} = R_{V,\alpha}R_{U^\perp,\beta}R_{U,\gamma}$
(xv) $R_{U^\perp,\beta}R_{U,\gamma}R_{V,\alpha} = 2\alpha\,\mathrm{P}_V + 2\gamma\,\mathrm{P}_U + 2\beta\,\mathrm{P}_{U^\perp} - 4\gamma\alpha\,\mathrm{P}_U\,\mathrm{P}_V - 4\beta\alpha\,\mathrm{P}_{U^\perp}\,\mathrm{P}_V - \mathrm{Id}$
(xvi) $R_{U,\gamma}R_{U^\perp,\beta}R_{V,\alpha} = R_{U^\perp,\beta}R_{U,\gamma}R_{V,\alpha}$.

Proof. (i): Using Lemma 1, (i), (iv), and (v) gives

$$R_{V,\alpha}R_{U^\perp,1}R_{U,1} = R_{V,\alpha}\big(-\mathrm{Id}\big) = -R_{V,\alpha} = \alpha R_{V^\perp} + (1-\alpha)\,\mathrm{Id}$$

(ii), (iii), (iv): The proof follows a similar approach as in (i).
(v): It follows from (i) that $R_{V,\alpha}R_{U^\perp,\beta}R_{U,\gamma} = \alpha R_{V^\perp} + (1-\alpha)\,\mathrm{Id}$, and using (6) and (i) gives $\mathrm{Fix}\,\big(R_{V,\alpha}R_{U^\perp,\beta}R_{U,\gamma}\big) = \mathrm{Fix}\,\big(\alpha R_{V^\perp} + (1-\alpha)\,\mathrm{Id}\big)$. Next, applying Lemma 1 (vi) yields, $x \in \mathrm{Fix}\,\big(R_{V,\alpha}R_{U^\perp,\beta}R_{U,\gamma}\big) \Leftrightarrow x \in V^\perp$. (vi): The proof follows a similar approach as in statement (v), combining (6), (ii), and Lemma 1 with (vi). (vii): The proof adopts a similar method to that used in statement (v), combining (6), (iii) and Lemma 1 (vi).
(viii): The proof follows a similar approach as in statement (v), combining (6), (iv) and Lemma 1 with (vi).
(ix): Using Lemma 2 and (3) gives

$$\begin{aligned} R_{V,\alpha}R_{U^\perp,\beta}R_{U,\gamma} &= R_{V,1}R_{U^\perp,\beta}R_{U,\gamma} \\ &= R_{V,1} - R_{V,1}2\gamma\,\mathrm{P}_U + R_{V,1}2\beta\,\mathrm{P}_{U^\perp} \\ &= \big(2\,\mathrm{P}_V - \mathrm{Id}\big) - \big(2\,\mathrm{P}_V - \mathrm{Id}\big)2\gamma\,\mathrm{P}_U + \big(2\,\mathrm{P}_V - \mathrm{Id}\big)2\beta\,\mathrm{P}_{U^\perp} \\ &= 2\,\mathrm{P}_V + 2\gamma\,\mathrm{P}_U + 2\beta\,\mathrm{P}_{U^\perp} - 4\gamma\,\mathrm{P}_V\,\mathrm{P}_U - 4\beta\,\mathrm{P}_V\,\mathrm{P}_{U^\perp} - \mathrm{Id}\,. \end{aligned}$$

(x): Following the same approach as in statement (ix) gives

$$\begin{aligned} R_{V,\alpha}R_{U,\gamma}R_{U^\perp,\beta} &= R_{V,1}R_{U,\gamma}R_{U^\perp} \\ &= 2\,\mathrm{P}_V + 2\gamma\,\mathrm{P}_U + 2\beta\,\mathrm{P}_{U^\perp} - 4\gamma\,\mathrm{P}_V\,\mathrm{P}_U - 4\beta\,\mathrm{P}_V\,\mathrm{P}_{U^\perp} - \mathrm{Id}\,. \end{aligned}$$

Combining this result with (x) illustrates that when $\alpha = 1$

$$R_{V,\alpha}R_{U,\gamma}R_{U^\perp,\beta} = R_{V,\alpha}R_{U,\gamma}R_{U^\perp,\beta}.$$

(xi): Using Lemma 2 and (3) yields

$$\begin{aligned} R_{U^\perp,\beta}R_{U,\gamma}R_{V,\alpha} &= R_{U^\perp,\beta}R_{U,\gamma}R_{V,1} \\ &= R_{V,1} - (2\gamma\,\mathrm{P}_U)R_{V,1} + (2\beta\,\mathrm{P}_{U^\perp})R_{V,1} \end{aligned}$$

$$= \big(2\,\mathrm{P}_V - \mathrm{Id}\,\big) - 2\gamma\,\mathrm{P}_U\,\big(2\,\mathrm{P}_V - \mathrm{Id}\,\big) + 2\beta\,\mathrm{P}_{U^\perp}\,\big(2\,\mathrm{P}_V - \mathrm{Id}\,\big)$$
$$= 2\,\mathrm{P}_V + 2\gamma\,\mathrm{P}_U + 2\beta\,\mathrm{P}_{U^\perp} - 4\gamma\,\mathrm{P}_U\,\mathrm{P}_V - 4\beta\,\mathrm{P}_{U^\perp}\,\mathrm{P}_V - \mathrm{Id}\,.$$

(xii): Applying the same method as in statement (xi) gives

$$\begin{aligned} R_{U,\gamma}R_{U^\perp,\beta}R_{V,\alpha} &= R_{U,\gamma}R_{U^\perp}R_{V,1} \\ &= 2\,\mathrm{P}_V + 2\gamma\,\mathrm{P}_U + 2\beta\,\mathrm{P}_{U^\perp} - 4\gamma\,\mathrm{P}_U\,\mathrm{P}_V - 4\beta\,\mathrm{P}_{U^\perp}\,\mathrm{P}_V - \mathrm{Id}\,. \end{aligned}$$

Combining this result with (xi) illustrates that when $\alpha = 1$

$$R_{U,\gamma}R_{U^\perp,\beta}R_{V,\alpha} = R_{U,\gamma}R_{U^\perp,\beta}R_{V,\alpha}.$$

(xiii): Using Lemma 2 and (11) gives

$$\begin{aligned} R_{V,\alpha}R_{U^\perp,\beta}R_{U,\gamma} &= R_{V,\alpha} - R_{V,\alpha}2\gamma\,\mathrm{P}_U + R_{V,\alpha}2\beta\,\mathrm{P}_{U^\perp} \\ &= \big(2\alpha\,\mathrm{P}_V - \mathrm{Id}\,\big) - \big(2\alpha\,\mathrm{P}_V - \mathrm{Id}\,\big)2\gamma\,\mathrm{P}_U + \big(2\alpha\,\mathrm{P}_V - \mathrm{Id}\,\big)2\beta\,\mathrm{P}_{U^\perp} \\ &= 2\alpha\,\mathrm{P}_V + 2\gamma\,\mathrm{P}_U + 2\beta\,\mathrm{P}_{U^\perp} - 4\alpha\gamma\,\mathrm{P}_V\,\mathrm{P}_U - 4\alpha\beta\,\mathrm{P}_V\,\mathrm{P}_{U^\perp} - \mathrm{Id}\,. \end{aligned}$$

(xiv): Following the same approach as in statement (xiii) gives

$$R_{V,\alpha}R_{U,\gamma}R_{U^\perp,\beta} = 2\alpha\,\mathrm{P}_V + 2\gamma\,\mathrm{P}_U + 2\beta\,\mathrm{P}_{U^\perp} - 4\alpha\gamma\,\mathrm{P}_V\,\mathrm{P}_U - 4\alpha\beta\,\mathrm{P}_V\,\mathrm{P}_{U^\perp} - \mathrm{Id}\,.$$

Combining this result with (xiii) illustrates that when $\alpha \neq 1$

$$R_{V,\alpha}R_{U,\gamma}R_{U^\perp,\beta} = R_{V,\alpha}R_{U,\gamma}R_{U^\perp,\beta}.$$

(xv): Using Lemma 2 and (11) yields

$$\begin{aligned} R_{U^\perp,\beta}R_{U,\gamma}R_{V,\alpha} &= R_{V,\alpha} - (2\gamma\,\mathrm{P}_U)R_{V,\alpha} + (2\beta\,\mathrm{P}_{U^\perp})R_{V,\alpha} \\ &= \big(2\alpha\,\mathrm{P}_V - \mathrm{Id}\,\big) - 2\gamma\,\mathrm{P}_U\,\big(2\alpha\,\mathrm{P}_V - \mathrm{Id}\,\big) + 2\beta\,\mathrm{P}_{U^\perp}\,\big(2\alpha\,\mathrm{P}_V - \mathrm{Id}\,\big) \\ &= 2\alpha\,\mathrm{P}_V + 2\gamma\,\mathrm{P}_U + 2\beta\,\mathrm{P}_{U^\perp} - 4\gamma\alpha\,\mathrm{P}_U\,\mathrm{P}_V - 4\beta\alpha\,\mathrm{P}_{U^\perp}\,\mathrm{P}_V - \mathrm{Id}\,. \end{aligned}$$

(xvi): Applying the same method as in statement (xv) gives

$$R_{U,\gamma}R_{U^\perp,\beta}R_{V,\alpha} = 2\alpha\,\mathrm{P}_V + 2\gamma\,\mathrm{P}_U + 2\beta\,\mathrm{P}_{U^\perp} - 4\gamma\alpha\,\mathrm{P}_U\,\mathrm{P}_V - 4\beta\alpha\,\mathrm{P}_{U^\perp}\,\mathrm{P}_V - \mathrm{Id}\,.$$

Combining this result with (xv) illustrates that when $\alpha \neq 1$

$$R_{U,\gamma}R_{U^\perp,\beta}R_{V,\alpha} = R_{U,\gamma}R_{U^\perp,\beta}R_{V,\alpha}.$$

■

Remark 1. Regarding Theorem 1, if $\beta = \gamma = \alpha = 1$, then the following holds:

(i) $R_{V,\alpha}R_{U^\perp,\beta}R_{U,\gamma} = R_{V^\perp}$

(ii) $R_{V,\alpha}R_{U,\gamma}R_{U^\perp,\beta} = R_{V^\perp}$

(iii) $R_{U^\perp,\beta}R_{U,\gamma}R_{V,\alpha} = R_{V^\perp}$

(iv) $R_{U,\gamma}R_{U^\perp,\beta}R_{V,\alpha} = R_{V^\perp}$

(v) $\operatorname{Fix}\left(R_{V,\alpha}R_{U^\perp,\beta}R_{U,\gamma}\right) = V^\perp$

(vi) $\operatorname{Fix}\left(R_{V,\alpha}R_{U,\gamma}R_{U^\perp,\beta}\right) = V^\perp$

(vii) $\operatorname{Fix}\left(R_{U^\perp,\beta}R_{U,\gamma}R_{V,\alpha}\right) = V^\perp$

(viii) $\operatorname{Fix}\left(R_{U,\gamma}R_{U^\perp,\beta}R_{V,\alpha}\right) = V^\perp$

Proof. (i), (ii), (iii), (iv), (v), (vi), (vii), and (viii): See [9, Proposition 6.3 (i), (ii), (iii), (iv), (v), (vi), (vii), and (viii)] for the proof. ■

Theorem 2. Let U and V be claosed linear subspaces of $\mathcal{H}$. Suppose that $\beta, \gamma, \alpha \in]0,1]$ and recall from (9), (10), and (11) the modified reflector operators. If $\beta = \gamma = \alpha = 1$, then the following holds:

(i) $R_{U,\gamma}R_{V,\alpha}R_{U^\perp,\beta} = \mathrm{Id} + 2\,\mathrm{P}_V + 8\,\mathrm{P}_U\,\mathrm{P}_V\,\mathrm{P}_{U^\perp} - 4\,\mathrm{P}_U\,\mathrm{P}_V - 4\,\mathrm{P}_V\,\mathrm{P}_{U^\perp}$

(ii) $R_{U^\perp,\beta}R_{V,\alpha}R_{U,\gamma} = \mathrm{Id} + 2\,\mathrm{P}_V + 8\,\mathrm{P}_{U^\perp}\,\mathrm{P}_V\,\mathrm{P}_U - 4\,\mathrm{P}_{U^\perp}\,\mathrm{P}_V - 4\,\mathrm{P}_V\,\mathrm{P}_U$

(iii) $\operatorname{Fix}\left(R_{U,\gamma}R_{V,\alpha}R_{U^\perp,\beta}\right) = R_U\left(V^\perp\right)$

(iv) $\operatorname{Fix}\left(R_{U^\perp,\beta}R_{V,\alpha}R_{U,\gamma}\right) = R_U\left(V^\perp\right)$.

Additionally, if $\beta, \gamma = 1$, then the following holds:

(v) $R_{U,\gamma}R_{V,\alpha}R_{U^\perp,\beta} = \mathrm{Id} + 2\alpha\,\mathrm{P}_V + 8\alpha\,\mathrm{P}_U\,\mathrm{P}_V\,\mathrm{P}_{U^\perp} - 4\alpha\,\mathrm{P}_U\,\mathrm{P}_V - 4\alpha\,\mathrm{P}_V\,\mathrm{P}_{U^\perp}$

(vi) $R_{U^\perp,\beta}R_{V,\alpha}R_{U,\gamma} = \mathrm{Id} + 2\alpha\,\mathrm{P}_V + 8\alpha\,\mathrm{P}_{U^\perp}\,\mathrm{P}_V\,\mathrm{P}_U - 4\alpha\,\mathrm{P}_{U^\perp}\,\mathrm{P}_V - 4\alpha\,\mathrm{P}_V\,\mathrm{P}_U$

(vii) $\operatorname{Fix}\left(R_{U,\gamma}R_{V,\alpha}R_{U^\perp,\beta}\right) = R_{U,1}\left(V^\perp\right)$

(viii) $\operatorname{Fix}\left(R_{U^\perp,\beta}R_{V,\alpha}R_{U,\gamma}\right) = R_{U,1}\left(V^\perp\right)$

Moreover, if $\beta, \gamma \neq 1$ and $\alpha = 1$, then the following holds:

(ix) We obtain

$$R_{U,\gamma}R_{V,\alpha}R_{U^\perp,\beta} = 8\gamma\beta\,\mathrm{P}_U\,\mathrm{P}_V\,\mathrm{P}_{U^\perp} - 4\gamma\,\mathrm{P}_U\,\mathrm{P}_V - 4\beta\,\mathrm{P}_V\,\mathrm{P}_{U^\perp} + 2\left(\gamma\,\mathrm{P}_U + \beta\,\mathrm{P}_{U^\perp} + \mathrm{P}_V\right) - \mathrm{Id}$$

(x) Also,

$$R_{U^\perp,\beta}R_{V,\alpha}R_{U,\gamma} = 8\beta\gamma\,\mathrm{P}_{U^\perp}\,\mathrm{P}_V\,\mathrm{P}_U - 4\beta\,\mathrm{P}_{U^\perp}\,\mathrm{P}_V - 4\gamma\,\mathrm{P}_V\,\mathrm{P}_U + 2\left(\beta\,\mathrm{P}_{U^\perp} + \gamma\,\mathrm{P}_U + \mathrm{P}_V\right) - \mathrm{Id}$$

Proof. (i): From (3), we have

$$\begin{aligned}
R_{U,\gamma}R_{V,\alpha}R_{U^\perp,\beta} &= R_{U,1}R_{V,1}R_{U^\perp,1} \\
&= \left(2\,\mathrm{P}_U - \mathrm{Id}\right)\left(4\,\mathrm{P}_V\,\mathrm{P}_{U^\perp} - 2\,\mathrm{P}_V - 2\,\mathrm{P}_{U^\perp} + \mathrm{Id}\right) \\
&= 8\,\mathrm{P}_U\,\mathrm{P}_V\,\mathrm{P}_{U^\perp} - 4\,\mathrm{P}_U\,\mathrm{P}_V + 2\,\mathrm{P}_U - 4\,\mathrm{P}_V\,\mathrm{P}_{U^\perp} + 2\,\mathrm{P}_V + 2\,\mathrm{P}_{U^\perp} - \mathrm{Id} \\
&= 8\,\mathrm{P}_U\,\mathrm{P}_V\,\mathrm{P}_{U^\perp} - 4\,\mathrm{P}_U\,\mathrm{P}_V + 2\left(\mathrm{P}_U + \mathrm{P}_{U^\perp}\right) - \mathrm{Id} - 4\,\mathrm{P}_V\,\mathrm{P}_{U^\perp} + 2\,\mathrm{P}_V \\
&= 8\,\mathrm{P}_U\,\mathrm{P}_V\,\mathrm{P}_{U^\perp} - 4\,\mathrm{P}_U\,\mathrm{P}_V + \mathrm{Id} - 4\,\mathrm{P}_V\,\mathrm{P}_{U^\perp} + 2\,\mathrm{P}_V \\
&= \mathrm{Id} + 2\,\mathrm{P}_V + 8\,\mathrm{P}_U\,\mathrm{P}_V\,\mathrm{P}_{U^\perp} - 4\,\mathrm{P}_U\,\mathrm{P}_V - 4\,\mathrm{P}_V\,\mathrm{P}_{U^\perp}
\end{aligned}$$

(ii): The proof follows a similar approach as in (i). (iii), (iv): See [9, Proposition 6.5 (i) and (ii)]. (v): Using (3) yields

$$R_{U,\gamma}R_{V,\alpha}R_{U^\perp,\beta} = R_{U,1}R_{V,\alpha}R_{U^\perp,1}$$

$$
\begin{aligned}
&= \left(2\,\mathrm{P}_U - \mathrm{Id}\right)\left(4\alpha\,\mathrm{P}_V\,\mathrm{P}_{U^\perp} - 2\alpha\,\mathrm{P}_V - 2\,\mathrm{P}_{U^\perp} + \mathrm{Id}\right)\\
&= 8\alpha\,\mathrm{P}_U\,\mathrm{P}_V\,\mathrm{P}_{U^\perp} - 4\alpha\,\mathrm{P}_U\,\mathrm{P}_V + 2\,\mathrm{P}_U - 4\alpha\,\mathrm{P}_V\,\mathrm{P}_{U^\perp} + 2\alpha\,\mathrm{P}_V + 2\,\mathrm{P}_{U^\perp} - \mathrm{Id}\\
&= 8\alpha\,\mathrm{P}_U\,\mathrm{P}_V\,\mathrm{P}_{U^\perp} - 4\alpha\,\mathrm{P}_U\,\mathrm{P}_V + 2\left(\mathrm{P}_U + \mathrm{P}_{U^\perp}\right) - \mathrm{Id} - 4\alpha\,\mathrm{P}_V\,\mathrm{P}_{U^\perp} + 2\alpha\,\mathrm{P}_V\\
&= 8\alpha\,\mathrm{P}_U\,\mathrm{P}_V\,\mathrm{P}_{U^\perp} - 4\alpha\,\mathrm{P}_U\,\mathrm{P}_V + \mathrm{Id} - 4\alpha\,\mathrm{P}_V\,\mathrm{P}_{U^\perp} + 2\alpha\,\mathrm{P}_V\\
&= \mathrm{Id} + 2\alpha\,\mathrm{P}_V + 8\alpha\,\mathrm{P}_U\,\mathrm{P}_V\,\mathrm{P}_{U^\perp} - 4\alpha\,\mathrm{P}_U\,\mathrm{P}_V - 4\alpha\,\mathrm{P}_V\,\mathrm{P}_{U^\perp}
\end{aligned}
$$

(vi): The proof adopts a similar method to that used in statement (v).
(vii): Using [9, Lemma 6.4] and Lemma 1 (vi) gives

$$
\begin{aligned}
\operatorname{Fix}\left(R_{U,\gamma}R_{V,\alpha}R_{U^\perp,\beta}\right) &= \operatorname{Fix}\left(R_{U,1}R_{V,\alpha}R_{U^\perp,1}\right)\\
&= R_{U,1}\left(\operatorname{Fix} R_{V,\alpha}R_{U^\perp,1}R_{U,1}\right)\\
&= R_{U,1}\left(\operatorname{Fix}\left(R_{V,\alpha}(-\mathrm{Id})\right)\right)\\
&= R_{U,1}\left(\operatorname{Fix} R_{V^\perp,\alpha}\right)\\
&= R_{U,1}\left(V^\perp\right),
\end{aligned}
$$

as required.
(viii): The proof follows a similar approach as in (vii).
(ix): Using (9), (10), and (11) gives

$$
\begin{aligned}
R_{U,\gamma}R_{V,\alpha}R_{U^\perp,\beta} &= R_{U,\gamma}R_{V,1}R_{U^\perp,\beta}\\
&= R_{U,\gamma}\left(4\beta\,\mathrm{P}_V\,\mathrm{P}_{U^\perp} - 2\,\mathrm{P}_V - 2\beta\,\mathrm{P}_{U^\perp} + \mathrm{Id}\right)\\
&= \left(2\gamma\,\mathrm{P}_U - \mathrm{Id}\right)\left(4\beta\,\mathrm{P}_V\,\mathrm{P}_{U^\perp} - 2\,\mathrm{P}_V - 2\beta\,\mathrm{P}_{U^\perp} + \mathrm{Id}\right)\\
&= 8\gamma\beta\,\mathrm{P}_U\,\mathrm{P}_V\,\mathrm{P}_{U^\perp} - 4\gamma\,\mathrm{P}_U\,\mathrm{P}_V - 4\beta\,\mathrm{P}_V\,\mathrm{P}_{U^\perp} + 2\left(\gamma\,\mathrm{P}_U + \beta\,\mathrm{P}_{U^\perp} + \mathrm{P}_V\right) - \mathrm{Id},
\end{aligned}
$$

as required. (x): The proof follows a similar approach as in (ix). ■

Theorem 3. *let X, Y, and Z be closed subspaces of $\mathcal{H}$. Suppose that $\beta, \gamma, \alpha \in]0,1]$ and recall from* (9) *the modified reflector operator. Then*

$$
\operatorname{Fix}(R_{X,\alpha}R_{Y,\beta}R_{Z,\gamma}) = \operatorname{Fix}\Big(\alpha\,\mathrm{P}_X + \beta\,\mathrm{P}_Y + \gamma\,\mathrm{P}_Z + 4\alpha\beta\gamma\,\mathrm{P}_X\,\mathrm{P}_Y\,\mathrm{P}_Z - 2\alpha\beta\,\mathrm{P}_X\,\mathrm{P}_Y - 2\alpha\gamma\,\mathrm{P}_X\,\mathrm{P}_Z - 2\beta\gamma\,\mathrm{P}_Y\,\mathrm{P}_Z\Big)
$$

and

$$
\operatorname{Fix}(R_{X,\alpha}R_{Y,\beta}R_{Z,\gamma}) \subseteq X + Y + Z \tag{12}
$$

Proof. Using (9) implies that

$$
R_{Y,\beta}R_{Z,\gamma} = \left(2\beta\,\mathrm{P}_Y - \mathrm{Id}\right)\left(2\gamma\,\mathrm{P}_Z - \mathrm{Id}\right) = 4\beta\gamma\,\mathrm{P}_Y\,\mathrm{P}_Z - 2\beta\,\mathrm{P}_Y - 2\gamma\,\mathrm{P}_Z + \mathrm{Id}\,.
$$

Next,

$$
R_{X,\alpha}\left(4\beta\gamma\,\mathrm{P}_Y\,\mathrm{P}_Z - 2\beta\,\mathrm{P}_Y - 2\gamma\,\mathrm{P}_Z + \mathrm{Id}\right) = \left(2\alpha\,\mathrm{P}_X - \mathrm{Id}\right)\left(4\beta\gamma\,\mathrm{P}_Y\,\mathrm{P}_Z - 2\beta\,\mathrm{P}_Y - 2\gamma\,\mathrm{P}_Z + \mathrm{Id}\right)
$$

$$= 2\alpha\,\mathrm{P}_X + 2\beta\,\mathrm{P}_Y + 2\gamma\,\mathrm{P}_Z + 8\alpha\beta\gamma\,\mathrm{P}_X\,\mathrm{P}_Y\,\mathrm{P}_Z - 4\alpha\beta\,\mathrm{P}_X\,\mathrm{P}_Y - 4\alpha\gamma\,\mathrm{P}_X\,\mathrm{P}_Z - 4\beta\gamma\,\mathrm{P}_Y\,\mathrm{P}_Z - \mathrm{Id}\,.$$

Therefore,

$$\begin{aligned} R_{X,\alpha}R_{Y,\beta}R_{Z,\gamma} &= 2\alpha\,\mathrm{P}_X + 2\beta\,\mathrm{P}_Y + 2\gamma\,\mathrm{P}_Z + 8\alpha\beta\gamma\,\mathrm{P}_X\,\mathrm{P}_Y\,\mathrm{P}_Z - 4\alpha\beta\,\mathrm{P}_X\,\mathrm{P}_Y - 4\alpha\gamma\,\mathrm{P}_X\,\mathrm{P}_Z \\ &\quad - 4\beta\gamma\,\mathrm{P}_Y\,\mathrm{P}_Z - \mathrm{Id}\,. \end{aligned}$$

Let $x \in \mathrm{Fix}\left(R_{X,\alpha}R_{Y,\beta}R_{Z,\gamma}\right)$. Then we obtain

$$\begin{aligned} & x = R_{X,\alpha}R_{Y,\beta}R_{Z,\gamma}(x) \\ & \Leftrightarrow 2x = 2\alpha\,\mathrm{P}_X(x) + 2\beta\,\mathrm{P}_Y(x) + 2\gamma\,\mathrm{P}_Z(x) + 8\alpha\beta\gamma\,\mathrm{P}_X\,\mathrm{P}_Y\,\mathrm{P}_Z(x) - 4\alpha\beta\,\mathrm{P}_X\,\mathrm{P}_Y(x) \\ & \qquad - 4\alpha\gamma\,\mathrm{P}_X\,\mathrm{P}_Z(x) - 4\beta\gamma\,\mathrm{P}_Y\,\mathrm{P}_Z(x) \\ & \Leftrightarrow x = \alpha\,\mathrm{P}_X(x) + \beta\,\mathrm{P}_Y(x) + \gamma\,\mathrm{P}_Z(x) + 4\alpha\beta\gamma\,\mathrm{P}_X\,\mathrm{P}_Y\,\mathrm{P}_Z(x) - 2\alpha\beta\,\mathrm{P}_X\,\mathrm{P}_Y(x) \\ & \qquad - 2\alpha\gamma\,\mathrm{P}_X\,\mathrm{P}_Z(x) - 2\beta\gamma\,\mathrm{P}_Y\,\mathrm{P}_Z(x) \end{aligned}$$

As a result,

$$\begin{aligned} \mathrm{Fix}(R_{X,\alpha}R_{Y,\beta}R_{Z,\gamma}) = \mathrm{Fix}\Big(&\alpha\,\mathrm{P}_X + \beta\,\mathrm{P}_Y + \gamma\,\mathrm{P}_Z + 4\alpha\beta\gamma\,\mathrm{P}_X\,\mathrm{P}_Y\,\mathrm{P}_Z - 2\alpha\beta\,\mathrm{P}_X\,\mathrm{P}_Y \\ & - 2\alpha\gamma\,\mathrm{P}_X\,\mathrm{P}_Z - 2\beta\gamma\,\mathrm{P}_Y\,\mathrm{P}_Z\Big) \end{aligned}$$

Consequently,

$$\mathrm{Fix}(R_{X,\alpha}R_{Y,\beta}R_{Z,\gamma}) \subseteq X + Y + Z.$$

■

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