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The Conformable Double Laplace-Sawi Transform

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Abstract. In this study, we introduce the conformable double Laplace-Sawi transform, a method for solving fractional partial differential equations that appear in various physical and engineering models. These models use derivatives and integrals based on the newly defined conformable derivative. The study first explores key properties of the conformable double Laplace-Sawi transform. Then, as an application, the method is applied to solving the conformable telegraph equation, the conformable heat equation, and the conformable Klein-Gordon equation, which are widely used in scientific and engineering fields.

2020 Mathematics Subject Classifications: 44A05, 44A10

Key Words and Phrases: Laplace transform, Sawi transform, double Laplace-Sawi transform, the conformable double Laplace-Sawi transform

1. Introduction

Fractional partial differential equations are important in modeling real-world problems in physics, electrical circuits, fluid dynamics, optics, and mathematical biology. One useful concept introduced in [1] is the conformable fractional derivative, which keeps many familiar properties of standard derivatives.

Recently, researchers have developed various methods to solve conformable fractional partial differential equations, including the conformable double Laplace transform see [2], [3] and the conformable double Sumudu transform see [4].

More recently, researchers have developed a new technique called Double Laplace-Sawi transform [5], which has been successfully applied to different types of partial differential equations. for more details about integral transform see [6–12].

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In this study, we introduce the conformable double Laplace-Sawi transform (CLSW) as a new approach to analyzing conformable partial differential equations. We first explore its fundamental properties, including the conditions for its existence and its behavior with differentiation. We explore new methods for solving conformable partial differential equations, providing a new viewpoint that may lead to more advances in math and real-world uses.

2. Conformable Fractional Derivative

In this section, we present fundamental definitions and theorems related to conformable fractional derivatives.

Definition 1. *[1] "Let $0 < \alpha \leq 1$ and $\omega : (0, \infty) \to \mathbb{R}$. The conformable fractional derivative of order α is defined as:*

$$\frac{d^{\alpha}}{d\gamma^{\alpha}}\omega(\gamma) = \lim_{\tau \to 0} \frac{\omega(\gamma + \tau\gamma^{1-\alpha}) - \omega(\gamma)}{\tau}$$

where $\gamma > 0$, and $\frac{\partial^{\alpha}}{\partial\gamma^{\alpha}}$ is referred to as the fractional derivative of order α."

Definition 2. *[13] "Let $0 < \alpha_1, \alpha_2 \leq 1$ and $\omega(\gamma, \eta) : (0, \infty) \times (0, \infty) \to \mathbb{R}$. The conformable partial derivatives of orders α_1 and α_2 of the function $\omega(\gamma, \eta)$ are defined as:*

$$\frac{\partial^{\alpha_1}}{\partial\gamma^{\alpha_1}}\omega(\gamma, \eta) = \lim_{\tau \to 0} \frac{\omega(\gamma + \tau\gamma^{1-\alpha_1}, \eta) - \omega(\gamma, \eta)}{\tau}$$

$$\frac{\partial^{\alpha_2}}{\partial\eta^{\alpha_2}}\omega(\gamma, \eta) = \lim_{\tau \to 0} \frac{\omega(\gamma, \eta + \tau\eta^{1-\alpha_2}) - \omega(\gamma, \eta)}{\tau}$$

where $\gamma, \eta > 0$, $\frac{\partial^{\alpha_1}}{\partial\gamma^{\alpha_1}}$ and $\frac{\partial^{\alpha_2}}{\partial\eta^{\alpha_2}}$ are referred to as fractional derivatives of orders α_1 and α_2, respectively."

Theorem 1. *[14]Suppose that $\omega(\gamma, \eta)$ is differentiable at a point $\gamma, \eta > 0$, $0 < \alpha_1, \alpha_2 \leq 1$, then:*

$$\frac{\partial^{\alpha_1}\omega}{\partial\gamma^{\alpha_1}} = \gamma^{1-\alpha_1}\frac{\partial\omega}{\partial\gamma},$$

$$\frac{\partial^{\alpha_2}\omega}{\partial\eta^{\alpha_2}} = \eta^{1-\alpha_2}\frac{\partial\omega}{\partial\eta}.$$

3. The Conformable Double Laplace-Sawi transform

This section serves to introduce the CLSW. We commence by delineating its fundamental properties, encompassing aspects like linearity. Subsequently, we reveal a novel result associated with partial derivatives. Ultimately, we illustrate how these insights enable us to compute the CLSW for various essential functions.

Definition 3. *Let $\omega(\gamma,\eta)$ be a continuous function on $(0,\infty)\times(0,\infty)$. Then*
1- The Conformable Laplace transformation (CL) of $\omega(\gamma,\eta)$, denoted by $L_\gamma^\alpha[\omega(\gamma,\eta)]$, is defined as:

$$H(\mu)=L_\gamma^\alpha(\omega(\gamma,\eta))=\int_0^\infty e^{-\mu\frac{\gamma^\alpha}{\alpha}}\omega(\gamma,\eta)\gamma^{\alpha-1}d\gamma,\ \mu\in\mathbb{C}$$

2- The Conformable Sawi transformation (CSW) of $\omega(\gamma,\eta)$, denoted by $L_\eta^\alpha[\omega(\gamma,\eta)]$, is defined as:

$$S(\tau)=W_\eta^\alpha(\omega(\gamma,\eta))=\frac{1}{\tau^2}\int_0^\infty e^{-\frac{\eta^\alpha}{\tau\alpha}}\omega(\gamma,\eta)\eta^{\alpha-1}d\eta,\ \tau\in\mathbb{C}$$

3- The Conformable Laplace Sawi transformation (CLSW) of $\omega(\gamma,\eta)$, denoted by $L_\gamma^{\alpha_1}W_\eta^{\alpha_2}[\omega(\gamma,\eta)]$, is defined as:

$$\Omega(\mu,\tau)=L_\gamma^{\alpha_1}W_\eta^{\alpha_2}[\omega(\gamma,\eta)]=\frac{1}{\tau^2}\int_0^\infty\int_0^\infty e^{-\left(\mu\frac{\gamma^{\alpha_1}}{\alpha_1}+\frac{\eta^{\alpha_2}}{\tau\alpha_2}\right)}\omega(\gamma,\eta)\gamma^{\alpha_1-1}\eta^{\alpha_2-1}d\gamma d\eta.$$

Theorem 2. *Assume that $\omega:(0,\infty)\times(0,\infty)\to\mathbb{R}$ such that $\Omega(\mu,\tau)=L_\gamma^{\alpha_1}W_\eta^{\alpha_2}[\omega(\frac{\gamma^{\alpha_1}}{\alpha_1},\frac{\eta^{\alpha_2}}{\alpha_2})]$ exist, then*

$$L_\gamma^{\alpha_1}W_\eta^{\alpha_2}[\omega(\frac{\gamma^{\alpha_1}}{\alpha_1},\frac{\eta^{\alpha_2}}{\alpha_2})]=L_\gamma W_\eta[\omega(\gamma,\eta)],$$

where

$$L_\gamma W_\eta[\omega(\gamma,\eta)]=\frac{1}{\tau^2}\int_0^\infty\int_0^\infty e^{-\left(\mu\gamma+\frac{\eta}{\tau}\right)}\omega(\gamma,\eta)\,d\gamma\,d\eta.$$

Lemma 1. *$L_\gamma^{\alpha_1}W_\eta^{\alpha_2}(\omega(\gamma,\eta))$ is a linear transformation.*

Proof. for nonzero constants λ and ν, we have

$$\begin{aligned}
&L_\gamma^{\alpha_1}W_\eta^{\alpha_2}(\lambda\omega_1(\gamma,\eta)+\nu\omega_2(\gamma,\eta))\\
&=\frac{1}{\tau^2}\int_0^\infty\int_0^\infty e^{-\left(\mu\frac{\gamma^{\alpha_1}}{\alpha_1}+\frac{\eta^{\alpha_2}}{\tau\alpha_2}\right)}(\lambda\omega_1(\gamma,\eta)+\nu\omega_2(\gamma,\eta))\gamma^{\alpha_1-1}\eta^{\alpha_2-1}d\gamma d\eta,\\
&=\lambda\frac{1}{\tau^2}\int_0^\infty\int_0^\infty e^{-\left(\mu\frac{\gamma^{\alpha_1}}{\alpha_1}+\frac{\eta^{\alpha_2}}{\tau\alpha_2}\right)}\omega_1(\gamma,\eta)\gamma^{\alpha_1-1}\eta^{\alpha_2-1}d\gamma d\eta+\nu\frac{1}{\tau^2}\int_0^\infty\int_0^\infty e^{-\left(\mu\frac{\gamma^{\alpha_1}}{\alpha_1}+\frac{\eta^{\alpha_2}}{\tau\alpha_2}\right)}\omega_2(\gamma,\eta)\gamma^{\alpha_1-1}\eta^{\alpha_2-1}d\gamma d\eta\\
&=\lambda L_\gamma^{\alpha_1}W_\eta^{\alpha_2}(\omega_1(\gamma,\eta))+\nu L_\gamma^{\alpha_1}W_\eta^{\alpha_2}(\omega_2(\gamma,\eta)).
\end{aligned}$$

If $\omega(\gamma,\eta)$ can be written as $\omega(\gamma,\eta)=p(\gamma)q(\eta)$ for some continuous functions p and q, then $L_\gamma^{\alpha_1}W_\eta^{\alpha_2}(\omega(\gamma,\eta))=L_\gamma^{\alpha_1}(p(\gamma))W_\eta^{\alpha_2}(q(\eta))$. In fact

$$L_\gamma^{\alpha_1}W_\eta^{\alpha_2}(\omega(\gamma,\eta))\ =\ L_\gamma^{\alpha_1}W_\eta^{\alpha_2}(p(\gamma)q(\eta))$$

$$
\begin{aligned}
&= \frac{1}{\tau^2}\int_0^\infty\int_0^\infty e^{-\left(\mu\frac{\gamma^{\alpha_1}}{\alpha_1}+\frac{\eta^{\alpha_2}}{\tau\alpha_2}\right)} p(\gamma)q(\eta)\gamma^{\alpha_1-1}\eta^{\alpha_2-1}d\gamma d\eta \\
&= \left(\int_0^\infty e^{-\mu\frac{\gamma^{\alpha_1}}{\alpha_1}} p(\gamma)\gamma^{\alpha_1-1}d\gamma\right)\left(\frac{1}{\tau^2}\int_0^\infty e^{-\frac{\eta^{\alpha_2}}{\tau\alpha_2}} q(\eta)\eta^{\alpha_2-1}d\eta\right) \\
&= L_\gamma^{\alpha_1}(p(\gamma))W_\eta^{\alpha_2}(q(\eta)).
\end{aligned}
$$

3.1. The Conformable Double Laplace-Sawi transform for some basic functions

(i)

$$L_\gamma^{\alpha_1}W_\eta^{\alpha_2}[c] = L_\gamma W_\eta[c] = \frac{c}{\mu\tau}, \quad c\in\mathbb{R},$$

(ii)

$$L_\gamma^{\alpha_1}W_\eta^{\alpha_2}\left[\left(\frac{\gamma^{\alpha_1}}{\alpha_1}\right)^\lambda\left(\frac{\eta^{\alpha_2}}{\alpha_2}\right)^\nu\right]$$

$$
\begin{aligned}
&= L_\gamma W_\eta[\gamma^\lambda\eta^\nu] \\
&= \frac{\tau^{\nu-1}}{\mu^{\lambda+1}}\Gamma(\lambda+1)\Gamma(\nu+1),\ \mathrm{Re}(\mu)>0 \text{ and } \mathrm{Re}(\lambda)>-1,
\end{aligned}
$$

(iii)

$$L_\gamma^{\alpha_1}W_\eta^{\alpha_2}\left[e^{\lambda\frac{\gamma^{\alpha_1}}{\alpha_1}+\nu\frac{\eta^{\alpha_2}}{\alpha_2}}\right]$$

$$= L_\gamma W_\eta[e^{\lambda\gamma+\nu\eta}] = \frac{1}{\tau\left(\mu-\lambda\right)\left(1-\nu\tau\right)}, \mathrm{Re}(\mu)>\mathrm{Re}(\lambda).$$

3.2. Existence condition for the Conformable Double Laplace-Sawi transform

Definition 4. *Let $0<\alpha_1,\alpha_2\le 1$. Then a function $\omega(\gamma,\eta)$ is said to be of conformable exponential orders λ and ν on $0<\gamma<\infty$ and $0<\eta<\infty$. If there exist $K,X,Y>0$ such that $|\omega(\gamma,\eta)|\le Ke^{\lambda\frac{\gamma^{\alpha_1}}{\alpha_1}+\nu\frac{\eta^{\alpha_2}}{\alpha_2}}$, for all $\frac{\gamma^{\alpha_1}}{\alpha_1}>X$, $\frac{\eta^{\alpha_2}}{\alpha_2}>Y$.*

Theorem 3. *Let $0<\alpha_1,\alpha_2\le 1$ and $\omega(\gamma,\eta)$ be a continuous function on the region $(0,\infty)\times(0,\infty)$ of conformable exponential orders λ and ν. Then $\Omega(\mu,\tau)=L_\gamma^{\alpha_1}W_\eta^{\alpha_2}[\omega(\gamma,\eta)]$ exists for μ,τ whenever $\mathrm{Re}\left(\mu\right)>\lambda$ and $\mathrm{Re}\left(\frac{1}{\tau}\right)>\nu$.*

Proof. We have

$$
\begin{aligned}
|\Omega(\mu,\tau)| &= \left| \frac{1}{\tau^2}\int_0^\infty\int_0^\infty e^{-\left(\mu\frac{\gamma^{\alpha_1}}{\alpha_1}+\frac{\eta^{\alpha_2}}{\tau\alpha_2}\right)}\omega(\gamma,\eta)\gamma^{\alpha_1-1}\eta^{\alpha_2-1}\ d\gamma d\eta \right| \\
&\leq \frac{1}{\tau^2}\int_0^\infty\int_0^\infty e^{-\left(\mu\frac{\gamma^{\alpha_1}}{\alpha_1}+\frac{\eta^{\alpha_2}}{\tau\alpha_2}\right)} |\omega(\gamma,\eta)|\, \gamma^{\alpha_1-1}\eta^{\alpha_2-1} d\gamma d\eta \\
&\leq K\frac{1}{\tau^2}\int_0^\infty\int_0^\infty e^{-\left(\mu\frac{\gamma^{\alpha_1}}{\alpha_1}+\frac{\eta^{\alpha_2}}{\tau\alpha_2}\right)} e^{\lambda\frac{\gamma^{\alpha_1}}{\alpha_1}+\nu\frac{\eta^{\alpha_2}}{\alpha_2}}\gamma^{\alpha_1-1}\eta^{\alpha_2-1} d\gamma d\eta \\
&= K\left(\int_0^\infty e^{-(\mu-\lambda)\frac{\gamma^{\alpha_1}}{\alpha_1}}\gamma^{\alpha_1-1}d\gamma\right)\left(\frac{1}{\tau^2}\int_0^\infty e^{-(\frac{1}{\tau}-\nu)\frac{\eta^{\alpha_2}}{\alpha_2}}\eta^{\alpha_2-1}d\eta\right) \\
&= \frac{K}{\tau\,(\mu-\lambda)\,(1-\nu\tau)},
\end{aligned}
$$

where $\mathrm{Re}\,(\mu) > \lambda$ and $\mathrm{Re}\left(\frac{1}{\tau}\right) > \nu$.

3.3. Derivatives properties

Now, we present some basic properties of the CLSW

Let $\Omega(\mu,\tau) = L_\gamma^{\alpha_1}W_\eta^{\alpha_2}\left(\omega(\gamma,\eta)\right)$ where $\omega(\gamma,\eta)$ is a continuous function on $(0,\infty)\times(0,\infty)$. Then

(i)

$$L_\gamma^{\alpha_1}W_\eta^{\alpha_2}\left(\frac{\partial^{\alpha_1}\omega(\gamma,\eta)}{\partial\gamma^{\alpha_1}}\right) = \mu\Omega(\mu,\tau) - W_\eta^{\alpha_2}(\omega(0,\eta)), \tag{1}$$

(ii)

$$L_\gamma^{\alpha_1}W_\eta^{\alpha_2}\left(\frac{\partial^{2\alpha_1}\omega(\gamma,\eta)}{\partial\gamma^{2\alpha_1}}\right) = \mu^2\Omega(\mu,\tau) - \mu W_\eta^{\alpha_2}(\omega(0,\eta)) - W_\eta^{\alpha_2}(\frac{\partial^{\alpha_1}\omega(0,\eta)}{\partial\gamma^{\alpha_1}}), \tag{2}$$

(iii)

$$L_\gamma^{\alpha_1}W_\eta^{\alpha_2}\left(\frac{\partial^{\alpha_2}\omega(\gamma,\eta)}{\partial\eta^{\alpha_2}}\right) = \frac{1}{\tau}\Omega(\mu,\tau) - \frac{1}{\tau^2}L_\gamma^{\alpha_1}(\omega(\gamma,0)), \tag{3}$$

(iv)

$$L_\gamma^{\alpha_1}W_\eta^{\alpha_2}\left(\frac{\partial^{2\alpha_2}\omega(\gamma,\eta)}{\partial\eta^{2\alpha_2}}\right) = \frac{1}{\tau^2}\Omega(\mu,\tau) - \frac{1}{\tau^3}L_\gamma^{\alpha_1}(\omega(\gamma,0)) - \frac{1}{\tau^2}L_\gamma^{\alpha_1}(\frac{\partial^{\alpha_2}\omega(\gamma,0)}{\partial\eta^{\alpha_2}}). \tag{4}$$

Proof. Proof of Equation 1 $L_{\gamma}^{\alpha_1}W_{\eta}^{\alpha_2}\left(\frac{\partial^{\alpha_1}\omega(\gamma,\eta)}{\partial\gamma^{\alpha_1}}\right)=\frac{1}{\tau^2}\int_0^\infty\int_0^\infty e^{-\left(\mu\frac{\gamma^{\alpha_1}}{\alpha_1}+\frac{\eta^{\alpha_2}}{\tau\alpha_2}\right)}\frac{\partial^{\alpha_1}\omega(\gamma,\eta)}{\partial\gamma^{\alpha_1}}\gamma^{\alpha_1-1}\eta^{\alpha_2-1}d\gamma d\eta.$
By Theorem 1, we have $\frac{\partial^{\alpha_1}\omega(\gamma,\eta)}{\partial\gamma^{\alpha_1}}=\gamma^{1-\alpha_1}\frac{\partial\omega(\gamma,\eta)}{\partial\gamma}$. So,
$L_{\gamma}^{\alpha_1}W_{\eta}^{\alpha_2}\left(\frac{\partial^{\alpha_1}\omega(\gamma,\eta)}{\partial\gamma^{\alpha_1}}\right)=\frac{1}{\tau^2}\int_0^\infty e^{-\frac{\eta^{\alpha_2}}{\tau\alpha_2}}\eta^{\alpha_2-1}\int_0^\infty e^{-\mu\frac{\gamma^{\alpha_1}}{\alpha_1}}\frac{\partial\omega(\gamma,\eta)}{\partial\gamma}d\gamma d\eta.$
By integrating by parts, we get

$L_{\gamma}^{\alpha_1}W_{\eta}^{\alpha_2}\left(\frac{\partial^{\alpha_1}\omega(\gamma,\eta)}{\partial\gamma^{\alpha_1}}\right)=\frac{1}{\tau^2}\int_0^\infty e^{-\frac{\eta^{\alpha_2}}{\tau\alpha_2}}\eta^{\alpha_2-1}\left(-\omega(0,\eta)+\mu\int_0^\infty e^{-\mu\frac{\gamma^{\alpha_1}}{\alpha_1}}\omega(\gamma,\eta)\gamma^{\alpha_1-1}\ d\gamma\right)d\eta$
$=-\frac{1}{\tau^2}\int_0^\infty e^{-\frac{\eta^{\alpha_2}}{\tau\alpha_2}}\omega(0,\eta)\eta^{\alpha_2-1}d\eta+\frac{\mu}{\tau^2}\int_0^\infty\int_0^\infty e^{-\left(\mu\frac{\gamma^{\alpha_1}}{\alpha_1}+\frac{\eta^{\alpha_2}}{\tau\alpha_2}\right)}\omega(\gamma,\eta)\gamma^{\alpha_1-1}\eta^{\alpha_2-1}d\gamma d\eta$
$=\mu\Omega(\mu,\tau)-W_{\eta}^{\alpha_2}(\omega(0,\eta)).$
The proof of Equations 2, 3 and 4 can be obtained in the same manner.

In Table 1, we have the CLSW of some basic functions.

Table 1: Table of CLSW

$\omega(\gamma,\eta)$	$L_{\gamma}^{\alpha_1}W_{\eta}^{\alpha_2}(\omega(\gamma,\eta))$
c	$\frac{c}{\mu\tau}$, $\text{Re}(\mu)>0$
$\left(\frac{\gamma^{\alpha_1}}{\alpha_1}\right)^{\lambda}\left(\frac{\eta^{\alpha_2}}{\alpha_2}\right)^{\nu}$	$\frac{\tau^{\nu-1}}{\mu^{\lambda+1}}\Gamma(\lambda+1)\Gamma(\nu+1)$, $\text{Re}(\mu)>0$ and $\text{Re}(\lambda)>-1$
$e^{\lambda\frac{\gamma^{\alpha_1}}{\alpha_1}+\nu\frac{\eta^{\alpha_2}}{\alpha_2}}$	$\frac{1}{\tau(\mu-\lambda)(1-\nu\tau)}$, $\text{Re}(\mu)>\text{Re}(\lambda)$
$e^{i\left(\lambda\frac{\gamma^{\alpha_1}}{\alpha_1}+\nu\frac{\eta^{\alpha_2}}{\alpha_2}\right)}$	$\frac{i}{\tau(\mu-i\lambda)(i+\nu\tau)}$, $\text{Im}(\lambda)+\text{Re}(\mu)>0$
$\sin\left(\lambda\frac{\gamma^{\alpha_1}}{\alpha_1}+\nu\frac{\eta^{\alpha_2}}{\alpha_2}\right)$	$\frac{\lambda+\mu\tau\nu}{\tau(\mu^2+\lambda^2)(1+\nu^2\tau^2)}$, $\lvert\text{Im}(\lambda)\rvert<\text{Re}(\mu)$
$\cos\left(\lambda\frac{\gamma^{\alpha_1}}{\alpha_1}+\nu\frac{\eta^{\alpha_2}}{\alpha_2}\right)$	$\frac{\mu-\tau\lambda\nu}{\tau(\mu^2+\lambda^2)(1+\nu^2\tau^2)}$, $\lvert\text{Im}(\lambda)\rvert<\text{Re}(\mu)$
$\sinh\left(\lambda\frac{\gamma^{\alpha_1}}{\alpha_1}+\nu\frac{\eta^{\alpha_2}}{\alpha_2}\right)$	$\frac{\lambda+\mu\tau\nu}{\tau(\mu^2-\lambda^2)(1-\nu^2\tau^2)}$, $\text{Re}(\mu)>\text{Re}(\lambda)$ and $\text{Re}(\mu)+\text{Re}(\lambda)>0$
$\cosh\left(\lambda\frac{\gamma^{\alpha_1}}{\alpha_1}+\nu\frac{\eta^{\alpha_2}}{\alpha_2}\right)$	$\frac{\mu+\tau\lambda\nu}{\tau(\mu^2-\lambda^2)(1-\nu^2\tau^2)}$, $\text{Re}(\mu)>\text{Re}(\lambda)$ and $\text{Re}(\mu)+\text{Re}(\lambda)>0$
$p(\gamma)q(\eta)$	$L_{\gamma}^{\alpha_1}(p(\gamma))W_{\eta}^{\alpha_2}(q(\eta))$

4. Applications

In this section, we use the CLSW for solving conformable partial differential equations

Example 1. *Consider the conformable telegraph equation*

$$\frac{\partial^{2\alpha_1}\omega(\gamma,\eta)}{\partial\gamma^{2\alpha_1}}-2\frac{\partial^{2\alpha_2}\omega(\gamma,\eta)}{\partial\eta^{2\alpha_2}}+\frac{\partial^{\alpha_1}\omega(\gamma,\eta)}{\partial\gamma^{\alpha_1}}=4\omega(\gamma,\eta),\ \textit{where}\ \gamma,\eta>0 \tag{5}$$

With initial conditions (ICs)
$\omega(\gamma,0)=e^{2\frac{\gamma^{\alpha_1}}{\alpha_1}}$, $\frac{\partial^{\alpha_2}\omega(\gamma,0)}{\partial\eta^{\alpha_2}}=-e^{2\frac{\gamma^{\alpha_1}}{\alpha_1}}$,

and boundary conditions (BCs)
$\omega\left(0,\eta\right)=e^{-\frac{\eta^{\alpha_1}}{\alpha_1}}$, $\frac{\partial^{\alpha_1}\omega(0,\eta)}{\partial\gamma^{\alpha_1}}=2e^{-\frac{\eta^{\alpha_1}}{\alpha_1}}$.

Solution 1. *By applying the CL to the ICs and the CSW to the BCs, we get*
$L_{\gamma}^{\alpha_1}\left(e^{2\frac{\gamma^{\alpha_1}}{\alpha_1}}\right)=\frac{1}{\mu-2}$, $L_{\gamma}^{\alpha_1}\left(-e^{2\frac{\gamma^{\alpha_1}}{\alpha_1}}\right)=\frac{-1}{\mu-2}$, $W_{\eta}^{\alpha_2}\left(e^{-\frac{\eta^{\alpha_1}}{\alpha_1}}\right)=\frac{1}{\tau(1+\tau)}$, $W_{\eta}^{\alpha_2}\left(2e^{-\frac{\eta^{\alpha_1}}{\alpha_1}}\right)=\frac{2}{\tau(1+\tau)}$
Apply the CLSW to Equation 5, we get

$$\mu^2\Omega-\frac{\mu}{\tau\left(1+\tau\right)}-\frac{2}{\tau\left(1+\tau\right)}-\frac{2}{\tau^2}\Omega+\frac{2}{\tau^3\left(\mu-2\right)}-\frac{2}{\tau^2\left(\mu-2\right)}+\mu\Omega-\frac{1}{\tau\left(1+\tau\right)}=4\Omega$$

So,

$$\begin{aligned}\Omega(\mu,\tau) &= \frac{\frac{\mu+3}{\tau(1+\tau)}-\frac{2}{\tau^3(\mu-2)}+\frac{2}{\tau^2(\mu-2)}}{\mu^2-\frac{2}{\tau^2}+\mu-4}\\ &= \frac{\frac{\tau^2(\mu-2)(\mu+3)-2(1+\tau)+2\tau(1+\tau)}{\tau^3(\mu-2)(1+\tau)}}{\frac{\mu^2\tau^2+\mu\tau^2-4\tau^2-2}{\tau^2}}\end{aligned}$$

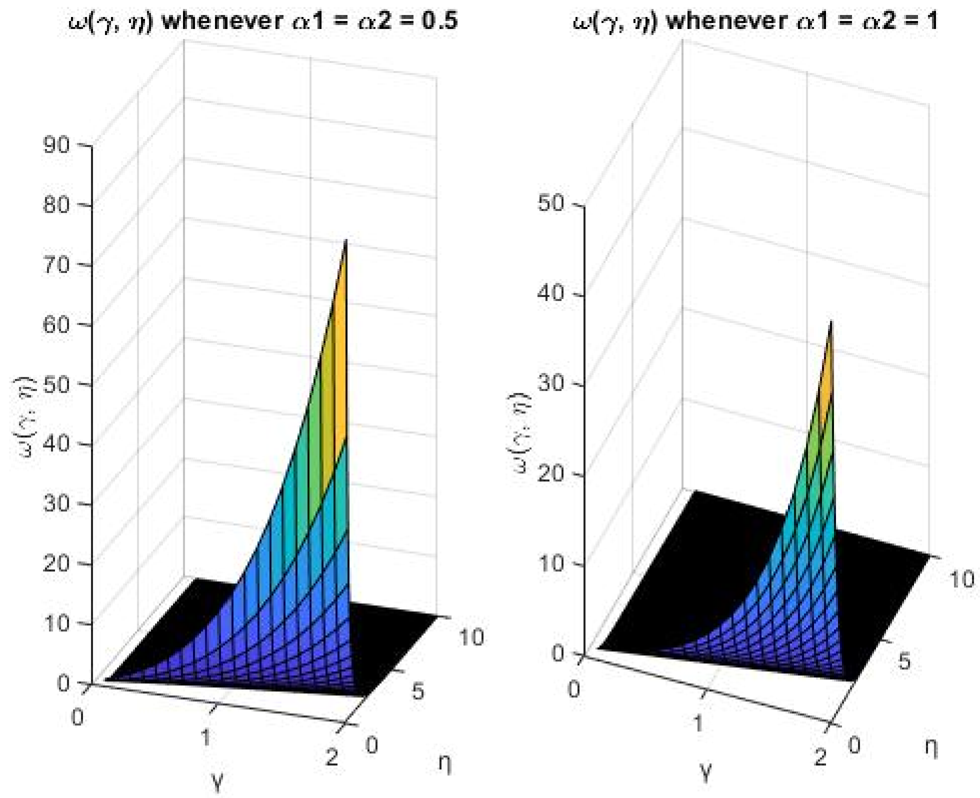
By simplify,

$$\Omega(\mu,\tau)=\frac{1}{\tau\left(\mu-2\right)\left(1+\tau\right)}.$$

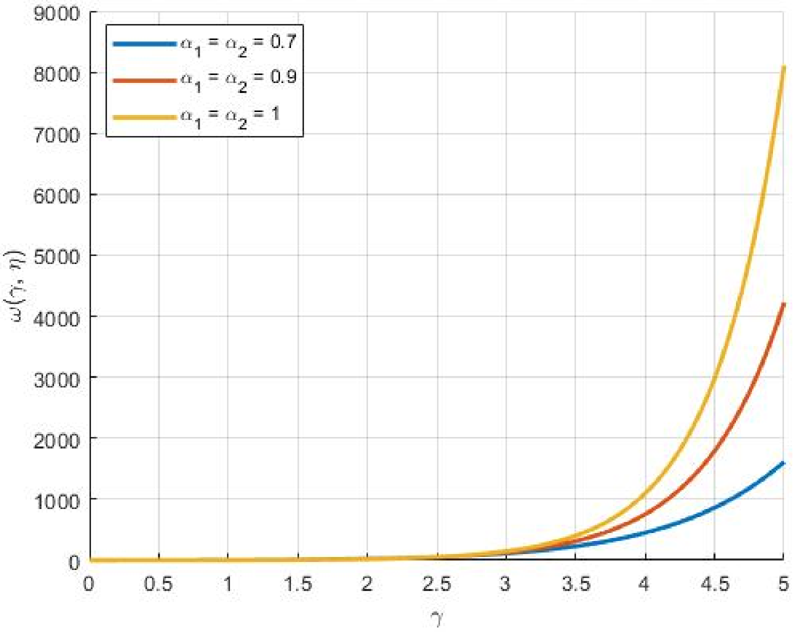
So,

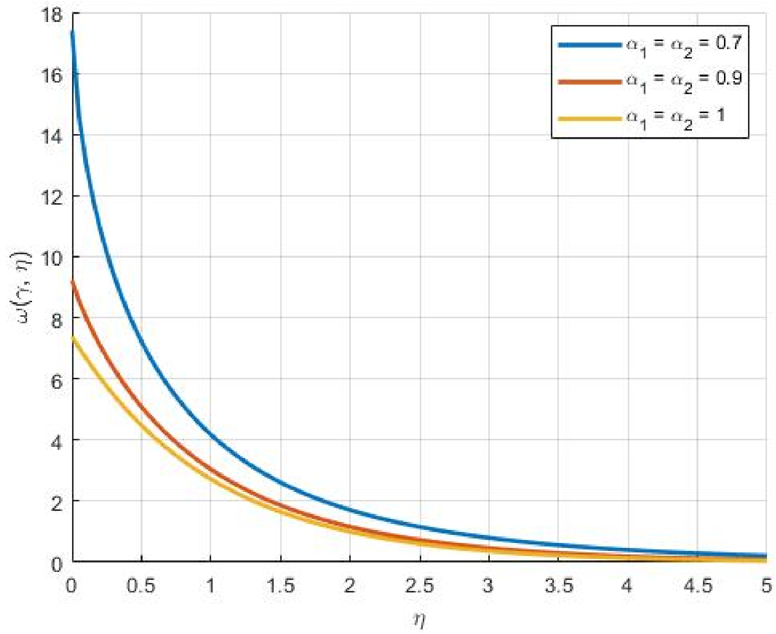
$$\omega(\gamma,\eta)=\left(L_{\gamma}^{\alpha_1}\right)^{-1}\left(W_{\eta}^{\alpha_2}\right)^{-1}\left(\frac{1}{\tau\left(\mu-2\right)\left(1+\tau\right)}\right)=e^{2\frac{\gamma^{\alpha_1}}{\alpha_1}-\frac{\eta^{\alpha_2}}{\alpha_2}}.$$

The following figures show the 3D representation of the solution at $\alpha_1=\alpha_2=0.5,1$.



The following two figures illustrate the 2D graph of the solution with respect to γ *and* η *at* $\alpha_1 = \alpha_2 = 0.7, 0.9, 1.$





Example 2. *Consider the conformable heat equation*

$$2\frac{\partial^{\alpha_2}\omega(\gamma,\eta)}{\partial\eta^{\alpha_2}} + \frac{\partial^{2\alpha_1}\omega(\gamma,\eta)}{\partial\gamma^{2\alpha_1}} = \omega(\gamma,\eta) + 2\frac{\gamma^{\alpha_1}}{\alpha_1}, \text{ where } \gamma,\eta > 0 \tag{6}$$

With IC
$\omega(\gamma,0)=\cos\left(\frac{\gamma^{\alpha_1}}{\alpha_1}\right)-2\frac{\gamma^{\alpha_1}}{\alpha_1}$,
and BCs
$\omega\left(0,\eta\right)=e^{\frac{\eta^{\alpha_1}}{\alpha_1}}$, $\frac{\partial^{\alpha_1}\omega(0,\eta)}{\partial\gamma^{\alpha_1}}=-2$.

Solution 2. *By applying the CL to the IC and the CSW to the BCs, we get*
$L_\gamma^{\alpha_1}\left(\cos\left(\frac{\gamma^{\alpha_1}}{\alpha_1}\right)-2\frac{\gamma^{\alpha_1}}{\alpha_1}\right)=\frac{\mu}{\mu^2+1}-\frac{2}{\mu^2}$, $W_\eta^{\alpha_2}\left(e^{\frac{\eta^{\alpha_1}}{\alpha_1}}\right)=\frac{1}{\tau(1-\tau)}$, $W_\eta^{\alpha_2}\left(-2\right)=\frac{-2}{\tau}$
Apply the CLSW to Equation 6, we get

$$\frac{2}{\tau}\Omega-\frac{2\mu}{\tau^2\left(\mu^2+1\right)}+\frac{4}{\mu^2\tau^2}+\mu^2\Omega-\frac{\mu}{\tau\left(1-\tau\right)}+\frac{2}{\tau}=\Omega+\frac{2}{\mu^2\tau}$$

So,

$$\Omega(\mu,\tau)=\frac{\frac{2\mu}{\tau^2(\mu^2+1)}-\frac{4}{\mu^2\tau^2}+\frac{\mu}{\tau(1-\tau)}-\frac{2}{\tau}+\frac{2}{\mu^2\tau}}{\frac{2}{\tau}+\mu^2-1}$$

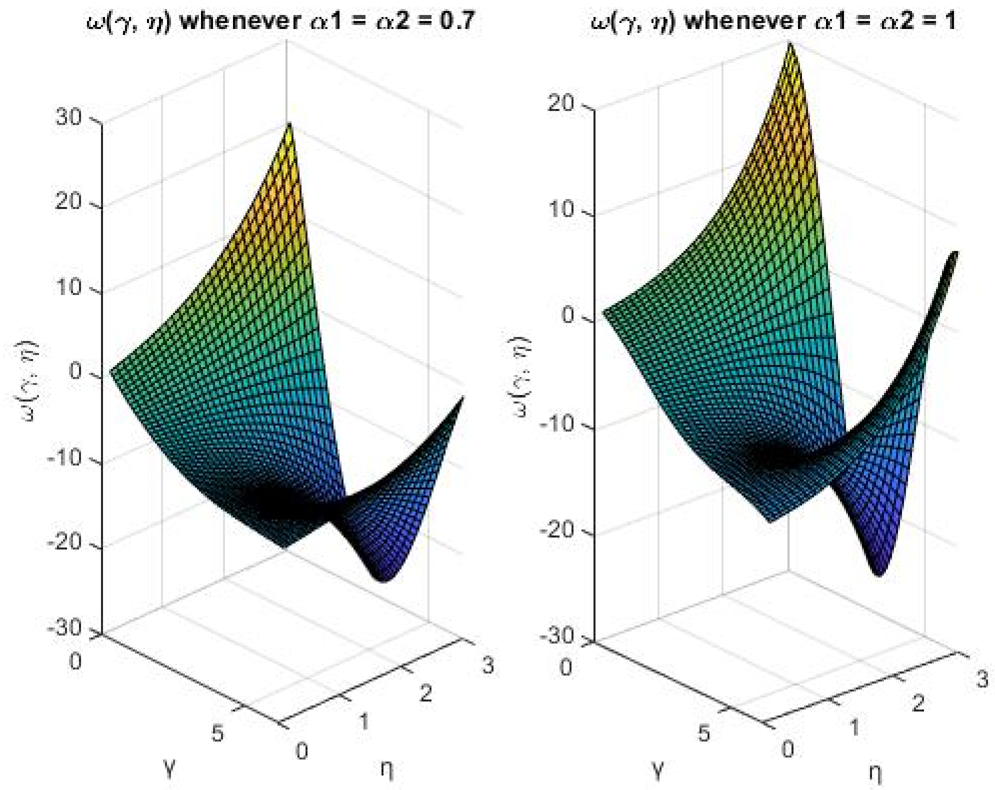
By simplify,

$$\Omega(\mu,\tau)=\frac{\mu}{\tau\left(\mu^2+1\right)\left(1-\tau\right)}-\frac{2}{\mu^2\tau}.$$

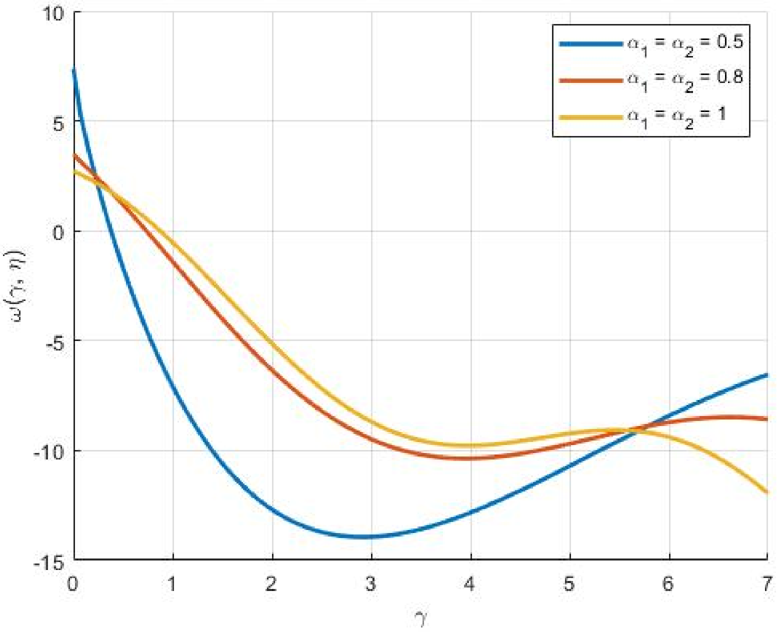
So,

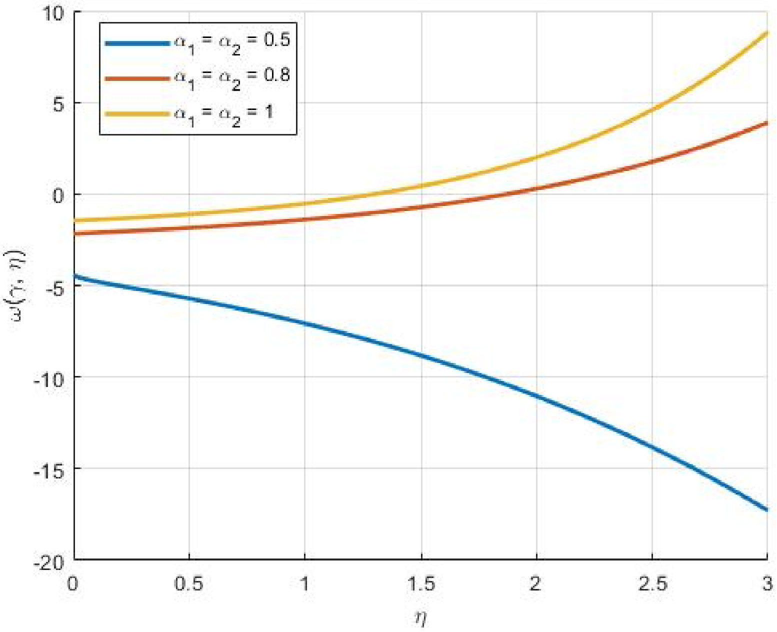
$$\omega(\gamma,\eta)=\left(L_\gamma^{\alpha_1}\right)^{-1}\left(W_\eta^{\alpha_2}\right)^{-1}\left(\frac{\mu}{\tau\left(\mu^2+1\right)\left(1-\tau\right)}-\frac{2}{\mu^2\tau}\right)=e^{\frac{\eta^{\alpha_2}}{\alpha_2}}\cos\left(\frac{\gamma^{\alpha_1}}{\alpha_1}\right)-2\frac{\gamma^{\alpha_1}}{\alpha_1}.$$

The following figures show the 3D representation of the solution at $\alpha_1=\alpha_2=0.7,1$.



The following two figures illustrate the 2D graph of the solution with respect to γ *and* η *at* $\alpha_1 = \alpha_2 = 0.5, 0.8, 1.$





Example 3. *Consider the conformable Klein-Gordon equation*

$$\frac{\partial^{2\alpha_1}\omega(\gamma,\eta)}{\partial\gamma^{2\alpha_1}} + 2\frac{\partial^{2\alpha_2}\omega(\gamma,\eta)}{\partial\eta^{2\alpha_2}} = \omega(\gamma,\eta), \text{ where } \gamma,\eta > 0 \tag{7}$$

With ICs
$\omega(\gamma,0) = \sin\left(\frac{\gamma^{\alpha_1}}{\alpha_1}\right)$, $\frac{\partial^{\alpha_2}\omega(0,\eta)}{\partial\eta^{\alpha_2}} = 0$,
and BCs
$\omega\left(0,\eta\right) = 0$, $\frac{\partial^{\alpha_1}\omega(0,\eta)}{\partial\gamma^{\alpha_1}} = \cosh\left(\frac{\eta^{\alpha_2}}{\alpha_2}\right)$.

Solution 3. *By applying the CL to the ICs and the CSW to the BCs, we get*
$L_\gamma^{\alpha_1}\left(\sin\left(\frac{\gamma^{\alpha_1}}{\alpha_1}\right)\right) = \frac{1}{\mu^2+1}$, $L_\gamma^{\alpha_1}\left(0\right) = 0$, $W_\eta^{\alpha_2}\left(0\right) = 0$, $W_\eta^{\alpha_2}\left(\cosh\left(\frac{\eta^{\alpha_2}}{\alpha_2}\right)\right) = \frac{1}{\tau(1-\tau^2)}$
Apply the CLSW to Equation 7, we get

$$\mu^2\Omega - \frac{1}{\tau\left(1-\tau^2\right)} + \frac{2}{\tau^2}\Omega - \frac{2}{\tau^3\left(\mu^2+1\right)} = \Omega$$

So,

$$\Omega(\mu,\tau) = \frac{\frac{1}{\tau(1-\tau^2)} + \frac{2}{\tau^3(\mu^2+1)}}{\mu^2 + \frac{2}{\tau^2} - 1}.$$

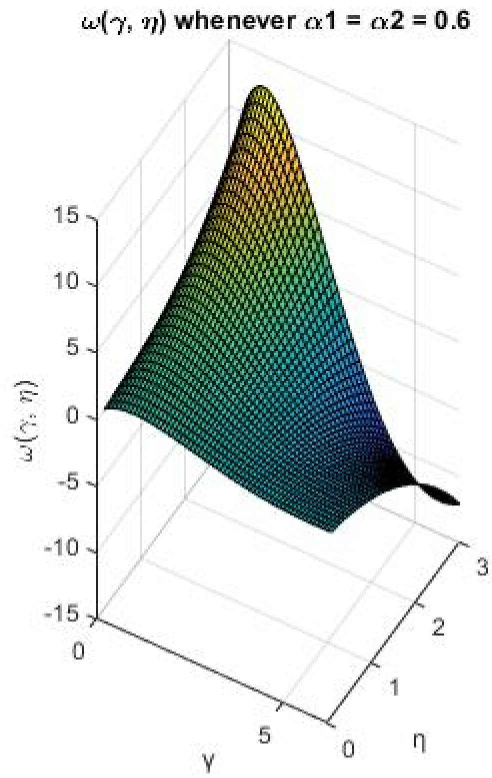
By simplify,

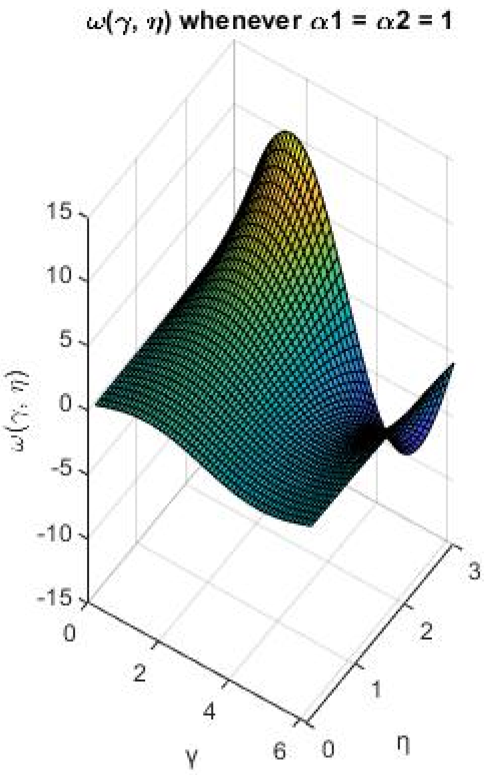
$$\Omega(\mu,\tau) = \frac{1}{\tau\left(\mu^2+1\right)\left(1-\tau^2\right)}.$$

So,

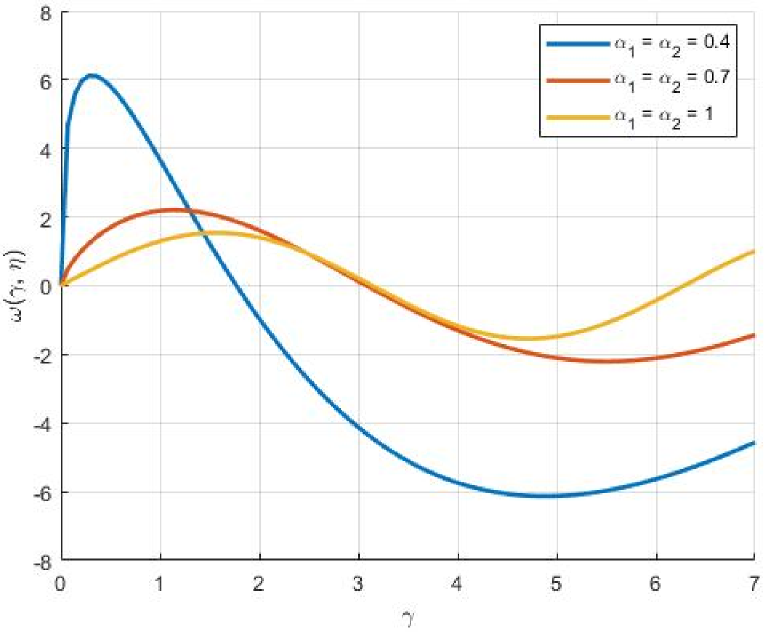
$$\omega(\gamma,\eta) = \left(L_\gamma^{\alpha_1}\right)^{-1}\left(W_\eta^{\alpha_2}\right)^{-1}\left(\frac{1}{\tau\left(\mu^2+1\right)\left(1-\tau^2\right)}\right) = \sin\left(\frac{\gamma^{\alpha_1}}{\alpha_1}\right)\cosh\left(\frac{\eta^{\alpha_2}}{\alpha_2}\right).$$

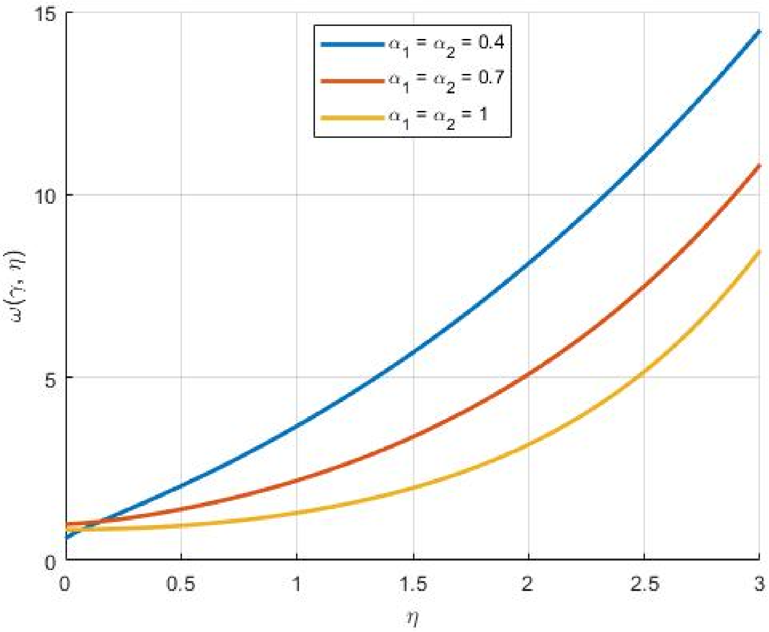
The following figures show the 3D representation of the solution at $\alpha_1 = \alpha_2 = 0.6, 1$.





The following two figures illustrate the 2D graph of the solution with respect to γ *and* η *at* $\alpha_1 = \alpha_2 = 0.4, 0.7, 1.$


$\alpha_1 = \alpha_2 = 0.4$
$\alpha_1 = \alpha_2 = 0.7$
$\alpha_1 = \alpha_2 = 1$
$\omega(\gamma, \eta)$
γ


$\alpha_1 = \alpha_2 = 0.4$
$\alpha_1 = \alpha_2 = 0.7$
$\alpha_1 = \alpha_2 = 1$
$\omega(\gamma, \eta)$
η

5. Conclusion

In this study, we introduced the conformable double Laplace-Sawi transform and explored its application to conformable fractional partial derivatives. We demonstrated its effectiveness by solving fractional partial differential equations. Since the conformable double Laplace-Sawi transform is a newly defined approach, there remain many open problems and potential areas for further research. This transform has the potential to be a powerful tool for solving conformable fractional partial differential equations, making it valuable for modeling various physical and engineering problems.

Author contribution statement
The listed authors have played a key role in developing and writing this article.

Data availability statement
This research did not involve the use of any data.

Conflict of interest
The authors confirm that there are no conflicts of interest.

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