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Characterizations of Weakly Contra-(τ_1, τ_2)-continuous Functions

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Abstract. This paper presents a new class of functions called weakly contra-(τ_1, τ_2)-continuous functions. Moreover, several characterizations and some properties concerning weakly contra-(τ_1, τ_2)-continuous functions are established.

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Key Words and Phrases: $\tau_1\tau_2$-open set, weakly contra-(τ_1, τ_2)-continuous function

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1. Introduction

It is well-known that the branch of mathematics called topology is concerned with all questions directly or indirectly related to continuity. Stronger and weaker forms of open sets play an important role in the generalization of different forms of continuity. Using different forms of open sets, many authors have introduced and studied various types of continuity for functions. In [1], the present authors investigated several characterizations of (Λ, sp)-continuous functions by utilizing the notions of (Λ, sp)-open sets and (Λ, sp)-closed sets due to Boonpok and Khampakdee [2]. Dungthaisong et al. [3] introduced and investigated the concept of $g_{(m,n)}$-continuous functions. Duangphui et al. [4] introduced and studied the notion of $(\mu, \mu')^{(m,n)}$-continuous functions. Furthermore, several characterizations of almost (Λ, p)-continuous functions, strongly $\theta(\Lambda, p)$-continuous functions, almost strongly $\theta(\Lambda, p)$-continuous functions, $\theta(\Lambda, p)$-continuous functions, weakly (Λ, b)-continuous functions, $\theta(\star)$-precontinuous functions, $(\Lambda, p(\star))$-continuous functions, $\star$-continuous functions, θ-$\mathscr{I}$-continuous functions, almost (g, m)-continuous functions, pairwise almost M-continuous functions, faintly (τ_1, τ_2)-continuous functions, $\delta(\tau_1, \tau_2)$-continuous functions and almost nearly (τ_1, τ_2)-continuous functions were presented in [5],

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[6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17] and [18], respectively. In 1996, Dontchev [19] introduced the notion of contra-continuous functions. Jafari and Noiri [20] introduced and investigated the concept of contra-precontinuous functions. Moreover, Jafari and Noiri [21] introduced and studied the notion of contra-α-continuous functions. In 1999, Dontchev and Noiri [22] introduced and investigated the concept of contra-semicontinuous functions. Caldas and Jafari [23] studied some properties of contra-β-continuous functions. In 2007, Baker [24] introduced and investigated the notion of weakly contra-continuous functions. Baker [25] introduced and studied the concept of weakly contra β-continuous functions. Noiri and Popa [26] introduced the notion of contra-m-continuous functions as functions from a set satisfying some minimal conditions into a topological space and investigated some characterizations and the relationships between contra-m-continuity and other related generalized forms of continuity. In 2011, Noiri and Popa [27] introduced a new class of functions called weakly contra-m-continuous functions as functions from a set satisfying some minimal conditions into a topological space and obtained some characterizations and several properties of such functions. It turns out that the weak contra-m-continuity is a unified form of several modifications of weak contra-continuity due to Baker [24]. On the other hand, the present authors introduced and studied the concepts of (τ_1, τ_2)-continuous functions [28], almost (τ_1, τ_2)-continuous functions [29], weakly (τ_1, τ_2)-continuous functions [30], quasi $\theta(\tau_1, \tau_2)$-continuous functions [31], almost quasi (τ_1, τ_2)-continuous functions [32], weakly quasi (τ_1, τ_2)-continuous functions [33], almost weakly (τ_1, τ_2)-continuous functions [34] and almost contra-(Λ, sp)-continuous functions [35]. In this paper, we introduce the concept of weakly contra-(τ_1, τ_2)-continuous functions. We also investigate some characterizations of weakly contra-(τ_1, τ_2)-continuous functions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [36] if $A = \tau_1\text{-Cl}(\tau_2\text{-Cl}(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [36] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [36] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 1. [36] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2\text{-}Cl(A)$ and $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Cl(A)) = \tau_1\tau_2\text{-}Cl(A)$.

(2) If $A \subseteq B$, then $\tau_1\tau_2\text{-}Cl(A) \subseteq \tau_1\tau_2\text{-}Cl(B)$.

(3) $\tau_1\tau_2\text{-}Cl(A)$ is $\tau_1\tau_2$-closed.

(4) A *is* $\tau_1\tau_2$*-closed if and only if* $A = \tau_1\tau_2\text{-}Cl(A)$.

(5) $\tau_1\tau_2\text{-}Cl(X - A) = X - \tau_1\tau_2\text{-}Int(A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is called $(\tau_1, \tau_2)r$*-open* [37] (resp. $(\tau_1, \tau_2)s$*-open* [38], $(\tau_1, \tau_2)p$*-open* [38], $(\tau_1, \tau_2)\beta$*-open* [38]) if $A = \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$ (resp. $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$, $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$, $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))$). The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is said to be $(\tau_1, \tau_2)r$*-closed* (resp. $(\tau_1, \tau_2)s$*-closed*, $(\tau_1, \tau_2)p$*-closed*, $(\tau_1, \tau_2)\beta$*-closed*). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\alpha(\tau_1, \tau_2)$*-open* [39] if $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A)))$. The complement of an $\alpha(\tau_1, \tau_2)$-open set is said to be $\alpha(\tau_1, \tau_2)$*-closed*. Let A be a subset of a bitopological space (X, τ_1, τ_2). The set

$$\cap\{G \mid A \subseteq G \text{ and } G \text{ is } \tau_1\tau_2\text{-open}\}$$

is called the $\tau_1\tau_2$*-kernel* [36] of A and is denoted by $\tau_1\tau_2\text{-}ker(A)$.

Lemma 2. [36] *For subsets A, B of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) $A \subseteq \tau_1\tau_2\text{-}ker(A)$.

(2) If $A \subseteq B$, *then* $\tau_1\tau_2\text{-}ker(A) \subseteq \tau_1\tau_2\text{-}ker(B)$.

(3) If A *is* $\tau_1\tau_2$*-open, then* $\tau_1\tau_2\text{-}ker(A) = A$.

(4) $x \in \tau_1\tau_2\text{-}ker(A)$ *if and only if* $A \cap H \neq \emptyset$ *for every* $\tau_1\tau_2$*-closed set* H *containing* x.

Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $(\tau_1, \tau_2)p$-closed sets of X containing A is called the $(\tau_1, \tau_2)p$*-closure* [40] of A and is denoted by $(\tau_1, \tau_2)\text{-pCl}(A)$. The union of all $(\tau_1, \tau_2)p$-open sets of X contained in A is called the $(\tau_1, \tau_2)p$*-interior* [40] of A and is denoted by $(\tau_1, \tau_2)\text{-pInt}(A)$.

Lemma 3. *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) $(\tau_1, \tau_2)\text{-}pCl(A) = \tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A)) \cup A$ [40];

(2) $(\tau_1, \tau_2)\text{-}pInt(A) = \tau_1\tau_2\text{-}Int(\tau_1\tau_2\text{-}Cl(A)) \cap A$ [34].

3. Characterizations of weakly contra-(τ_1, τ_2)-continuous functions

In this section, we introduce the concept of weakly contra-(τ_1, τ_2)-continuous functions. Furthermore, several characterizations of weakly contra-(τ_1, τ_2)-continuous functions are discussed.

Definition 1. *A function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is said to be weakly contra-(τ_1, τ_2)-continuous if for each* $\sigma_1\sigma_2$*-open set* V *of* Y *and each* $\sigma_1\sigma_2$*-closed set* K *of* Y *such that* $K \subseteq V$, $\tau_1\tau_2\text{-}Cl(f^{-1}(K)) \subseteq f^{-1}(V)$.

Definition 2. [41] *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be contra-(τ_1, τ_2)-continuous if $f^{-1}(V)$ is $\tau_1\tau_2$-closed in X for every $\sigma_1\sigma_2$-open set V of Y.*

Lemma 4. [41] *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is contra-(τ_1, τ_2)-continuous;

(2) $f^{-1}(K)$ is $\tau_1\tau_2$-open in X for every $\sigma_1\sigma_2$-closed set K of Y;

(3) for each $x \in X$ and each $\sigma_1\sigma_2$-closed set K of Y containing $f(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq K$;

(4) $f(\tau_1\tau_2\text{-}Cl(A)) \subseteq \sigma_1\sigma_2\text{-}ker(f(A))$ for every subset A of X;

(5) $\tau_1\tau_2\text{-}Cl(f^{-1}(B)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}ker(B))$ for every subset B of Y.

Theorem 1. *If a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is contra-(τ_1, τ_2)-continuous, then f is weakly contra-(τ_1, τ_2)-continuous.*

Proof. Let V be any $\sigma_1\sigma_2$-open set of Y and K be any $\sigma_1\sigma_2$-closed set of Y such that $K \subseteq V$. Since f is contra-(τ_1, τ_2)-continuous, by Lemma 4 we have $f^{-1}(V)$ is $\tau_1\tau_2$-closed in X and hence $\tau_1\tau_2\text{-Cl}(f^{-1}(K)) \subseteq \tau_1\tau_2\text{-Cl}(f^{-1}(V)) = f^{-1}(V)$. This shows that f is weakly contra-(τ_1, τ_2)-continuous.

Definition 3. [28] *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called (τ_1, τ_2)-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq V$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called (τ_1, τ_2)-continuous if f has this property at each point of X.*

Lemma 5. [28] *For a function $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is (τ_1, τ_2)-continuous;

(2) $f^{-1}(V)$ is $\tau_1\tau_2$-open in X for every $\sigma_1\sigma_2$-open set V of Y;

(3) $f(\tau_1\tau_2\text{-}Cl(A)) \subseteq \sigma_1\sigma_2\text{-}Cl(f(A))$ for every subset A of X;

(4) $\tau_1\tau_2\text{-}Cl(f^{-1}(B)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;

(5) $f^{-1}(\sigma_1\sigma_2\text{-}Int(B)) \subseteq \tau_1\tau_2\text{-}Int(f^{-1}(B))$ for every subset B of Y;

(6) $f^{-1}(K)$ is $\tau_1\tau_2$-closed in X for every $\sigma_1\sigma_2$-closed set K of Y.

Theorem 2. *If a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is (τ_1, τ_2)-continuous, then f is weakly contra-(τ_1, τ_2)-continuous.*

Proof. Let V be any $\sigma_1\sigma_2$-open set of Y and K be any $\sigma_1\sigma_2$-closed set of Y such that $K \subseteq V$. Since f is (τ_1, τ_2)-continuous, by Lemma 5 we have $f^{-1}(K)$ is $\tau_1\tau_2$-closed in X and so $\tau_1\tau_2\text{-Cl}(f^{-1}(K)) = f^{-1}(K) \subseteq f^{-1}(V)$. Thus, f is weakly contra-(τ_1, τ_2)-continuous.

Definition 4. [42] *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be slightly (τ_1, τ_2)-continuous if for each $x \in X$ and each $\sigma_1\sigma_2$-clopen set V of Y containing $f(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq V$.*

Lemma 6. [42] *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is slightly (τ_1, τ_2)-continuous;

(2) $f^{-1}(V)$ is $\tau_1\tau_2$-open in X for each $\sigma_1\sigma_2$-clopen set V of Y;

(3) $f^{-1}(V)$ is $\tau_1\tau_2$-closed in X for each $\sigma_1\sigma_2$-clopen set V of Y;

(4) $f^{-1}(V)$ is $\tau_1\tau_2$-clopen in X for each $\sigma_1\sigma_2$-clopen set V of Y.

Theorem 3. *If a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is weakly contra-(τ_1, τ_2)-continuous, then f is slightly (τ_1, τ_2)-continuous.*

Proof. Let V be any $\sigma_1\sigma_2$-clopen set of Y. If we put $K = V$, then by the weak contra-(τ_1, τ_2)-continuity we have $\tau_1\tau_2\text{-Cl}(f^{-1}(V)) \subseteq f^{-1}(V)$ and hence $f^{-1}(V)$ is $\tau_1\tau_2$-closed in X. It follows from Lemma 6 that f is slightly (τ_1, τ_2)-continuous.

Definition 5. [30] *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be weakly (τ_1, τ_2)-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2\text{-}Cl(V)$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be weakly (τ_1, τ_2)-continuous if f has this property at each point of X.*

Lemma 7. [30] *For a function $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is weakly (τ_1, τ_2)-continuous;

(2) $f^{-1}(V) \subseteq \tau_1\tau_2\text{-}Int(f^{-1}(\sigma_1\sigma_2\text{-}Cl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(3) $\tau_1\tau_2\text{-}Cl(f^{-1}(\sigma_1\sigma_2\text{-}Int(K))) \subseteq f^{-1}(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $\tau_1\tau_2\text{-}Cl(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B)))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;

(5) $f^{-1}(\sigma_1\sigma_2\text{-}Int(B)) \subseteq \tau_1\tau_2\text{-}Int(f^{-1}(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(B))))$ for every subset B of Y;

(6) $\tau_1\tau_2\text{-}Cl(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set V of Y.

Definition 6. [39] *A bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)-extremally disconnected if the $\tau_1\tau_2$-closure of every $\tau_1\tau_2$-open set U of X is $\tau_1\tau_2$-open.*

Theorem 4. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is weakly contra-(τ_1, τ_2)-continuous and (Y, σ_1, σ_2) is (σ_1, σ_2)-extremally disconnected, then f is weakly (τ_1, τ_2)-continuous.*

Proof. Let V be any $\sigma_1\sigma_2$-open set of Y. Since (Y, σ_1, σ_2) is (σ_1, σ_2)-extremally disconnected, $\sigma_1\sigma_2$-Cl(V) is $\sigma_1\sigma_2$-open. Since f weakly contra-(τ_1, τ_2)-continuous,

$$\tau_1\tau_2\text{-Cl}(f^{-1}(V)) \subseteq \tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)).$$

Thus, $\tau_1\tau_2$-Cl$(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$ for every $\sigma_1\sigma_2$-open set V of Y. It follows from Lemma 7 that f is weakly (τ_1, τ_2)-continuous.

Recall that a subset A of a bitopological space (X, τ_1, τ_2) is said to be *generalized (τ_1, τ_2)-closed* (briefly, *g-(τ_1, τ_2)-closed*) [43] if $\tau_1\tau_2$-Cl$(A) \subseteq U$ whenever $A \subseteq U$ and U is $\tau_1\tau_2$-open.

Definition 7. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be g-(τ_1, τ_2)-continuous if $f^{-1}(K)$ is g-(τ_1, τ_2)-closed in X for every $\sigma_1\sigma_2$-closed set K of Y.*

Definition 8. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be approximately (τ_1, τ_2)-continuous if $\tau_1\tau_2$-Cl$(K) \subseteq f^{-1}(V)$ whenever V is $\sigma_1\sigma_2$-open in Y and K is g-(τ_1, τ_2)-closed in X such that $K \subseteq f^{-1}(V)$.*

Theorem 5. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is g-(τ_1, τ_2)-continuous and approximately (τ_1, τ_2)-continuous, then f is weakly contra-(τ_1, τ_2)-continuous.*

Proof. Let V be any $\sigma_1\sigma_2$-open set of Y and K be any $\sigma_1\sigma_2$-closed set of Y such that $K \subseteq V$. Since f is g-(τ_1, τ_2)-continuous, $f^{-1}(K)$ is g-(τ_1, τ_2)-closed in X. Since $f^{-1}(K) \subseteq f^{-1}(V)$ and f is approximately (τ_1, τ_2)-continuous, $\tau_1\tau_2$-Cl$(K) \subseteq f^{-1}(V)$. This shows that f is weakly contra-(τ_1, τ_2)-continuous.

Theorem 6. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is weakly contra-(τ_1, τ_2)-continuous and $f(K)$ is $\sigma_1\sigma_2$-closed in Y for every g-(τ_1, τ_2)-closed set K of X, then f is approximately (τ_1, τ_2)-continuous.*

Proof. Let V be any $\sigma_1\sigma_2$-open set of Y and K be any g-(τ_1, τ_2)-closed set of X such that $K \subseteq f^{-1}(V)$. Then, $f(K)$ is $\sigma_1\sigma_2$-closed and $f(K) \subseteq V$. Since f is weakly contra-(τ_1, τ_2)-continuous, $\tau_1\tau_2$-Cl$(f^{-1}(f(K))) \subseteq f^{-1}(V)$ and hence $\tau_1\tau_2$-Cl$(K) \subseteq f^{-1}(V)$. This shows that f is approximately (τ_1, τ_2)-continuous.

Recall that a bitopological space (X, τ_1, τ_2) is said to be *$\tau_1\tau_2$-compact* [36] if every cover of X by $\tau_1\tau_2$-open sets of X has a finite subcover.

Definition 9. [41] *A bitopological space (X, τ_1, τ_2) is said to be strongly S-$\tau_1\tau_2$-closed if every cover of X by $\tau_1\tau_2$-closed sets of X has a finite subcover.*

Definition 10. *A bitopological space (X, τ_1, τ_2) is called a $\mathscr{C}$-(τ_1, τ_2)-space if for every $\tau_1\tau_2$-open set U of X and each $x \in U$, there exists a $\tau_1\tau_2$-closed set F of X such that $x \in F \subseteq U$.*

Theorem 7. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a weakly contra-(τ_1, τ_2)-continuous function and (Y, σ_1, σ_2) be a $\mathscr{C}$-(σ_1, σ_2)-space. If (X, τ_1, τ_2) is strongly S-$\tau_1\tau_2$-closed, then $f(X)$ is $\sigma_1\sigma_2$-compact.*

Proof. Suppose that (X, τ_1, τ_2) is strongly S-$\tau_1\tau_2$-closed. Let $\{V_\gamma \in \nabla\}$ be any cover of $f(X)$ by $\sigma_1\sigma_2$-open sets of Y. For each $x \in X$, there exists $\gamma(x) \in \nabla$ such that $f(x) \in V_{\gamma(x)}$. Since (Y, σ_1, σ_2) be a $\mathscr{C}$-(σ_1, σ_2)-space, there exists a $\sigma_1\sigma_2$-closed set $F_{\gamma(x)}$ of Y such that $f(x) \in F_{\gamma(x)} \subseteq V_{\gamma(x)}$. Since f is weakly contra-(τ_1, τ_2)-continuous, $\tau_1\tau_2\text{-Cl}(f^{-1}(F_{\gamma(x)})) \subseteq f^{-1}(V_{\gamma(x)})$. The family $\{\tau_1\tau_2\text{-Cl}(f^{-1}(F_{\gamma(x)})) \mid x \in X\}$ is a $\tau_1\tau_2$-closed cover of X. Since (X, τ_1, τ_2) is strongly S-$\tau_1\tau_2$-closed, there exists a finite number of pints, say, $x_1, x_2, x_3, ..., x_n$ in X such that

$$X = \cup\{\tau_1\tau_2\text{-Cl}(f^{-1}(F_{\gamma(x_k)})) \mid x_k \in X; 1 \leq k \leq n\}.$$

Thus,

$$\begin{aligned} f(X) &= \cup\{f(\tau_1\tau_2\text{-Cl}(f^{-1}(F_{\gamma(x_k)}))) \mid x_k \in X; 1 \leq k \leq n\} \\ &\subseteq \cup\{V_{\gamma(x_k)} \mid x_k \in X; 1 \leq k \leq n\}. \end{aligned}$$

This shows that $f(X)$ is $\sigma_1\sigma_2$-compact.

Definition 11. [44] *A bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)-T_1 if for any pair of distinct points x, y in X, there exist $\tau_1\tau_2$-open sets U and V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.*

Lemma 8. [44] *For a bitopological space (X, τ_1, τ_2), the following properties are equivalent:*

(1) (X, τ_1, τ_2) is (τ_1, τ_2)-T_1;

(2) for each $x \in X$, the singleton $\{x\}$ is $\tau_1\tau_2$-closed in X;

(3) for each $x \in X$, the singleton $\{x\}$ is a $\Lambda_{(\tau_1,\tau_2)}$-set.

Theorem 8. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a weakly contra-(τ_1, τ_2)-continuous injection and (Y, σ_1, σ_2) is (σ_1, σ_2)-T_1, (X, τ_1, τ_2) is (τ_1, τ_2)-T_1.*

Proof. Let x and x' be any distinct points of X. Since f is injective, $f(x) \neq f(x')$. Moreover, since (Y, σ_1, σ_2) is (σ_1, σ_2)-T_1, there exists a $\sigma_1\sigma_2$-open set V of Y such that $f(x) \in V$ and $f(x') \notin V$. By Lemma 8, $\{f(x)\}$ is $\sigma_1\sigma_2$-closed in Y. Since f is weakly contra-(τ_1, τ_2)-continuous, $\tau_1\tau_2\text{-Cl}(f^{-1}(\{f(x)\})) \subseteq f^{-1}(V)$. Since $x' \notin f^{-1}(V)$, we have $x' \notin \tau_1\tau_2\text{-Cl}(f^{-1}(\{f(x)\}))$. Then by Lemma 1, $\tau_1\tau_2\text{-Cl}(f^{-1}(\{f(x)\}))$ is $\tau_1\tau_2$-closed and hence $X - \tau_1\tau_2\text{-Cl}(f^{-1}(\{f(x)\}))$ is a $\tau_1\tau_2$-open set of X containing x' but not x. This shows that (X, τ_1, τ_2) is (τ_1, τ_2)-T_1.

Definition 12. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to have a contra-$\mathscr{C}$-closed graph if for each $(x, y) \in (X \times Y) - G(f)$, there exist a $\tau_1\tau_2$-closed set F of X containing x and a $\sigma_1\sigma_2$-closed set F' of Y containing y such that $(F \times F') \cap G(f) = \emptyset$.*

Theorem 9. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is weakly contra-(τ_1, τ_2)-continuous and (Y, σ_1, σ_2) is (σ_1, σ_2)-T_1, then $G(f)$ is contra-$\mathscr{C}$-closed.*

Proof. Let $(x, y) \in (X \times Y) - G(f)$. Then, $y \neq f(x)$. Since (Y, σ_1, σ_2) is (σ_1, σ_2)-T_1, there exists a $\sigma_1\sigma_2$-open set V of Y such that $y \notin V$ and $f(x) \notin V$. By Lemma 8, we have $\{f(x)\}$ is $\sigma_1\sigma_2$-closed in Y. Since f is weakly contra-(τ_1, τ_2)-continuous, $\tau_1\tau_2$-$\text{Cl}(f^{-1}(\{f(x)\})) \subseteq f^{-1}(V)$ and hence

$$(x, y) \in \tau_1\tau_2\text{-Cl}(f^{-1}(\{f(x)\})) \times (Y - V) \subseteq (X \times Y) - G(f).$$

This shows that $G(f)$ is contra-$\mathscr{C}$-closed.

Definition 13. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to have a contra-$\mathscr{C}_R$-closed graph if for each $(x, y) \in (X \times Y) - G(f)$, there exist a $\tau_1\tau_2$-closed set F of X containing x and a $(\sigma_1, \sigma_2)r$-closed set F' of Y containing y such that $(F \times F') \cap G(f) = \emptyset$.*

Definition 14. [45] *A bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-Urysohn if for each pair of distinct points x and y in X, there exist $\tau_1\tau_2$-open sets U and V such that $x \in U$, $y \in V$ and $\tau_1\tau_2$-Cl$(U) \cap \tau_1\tau_2$-Cl$(V) = \emptyset$.*

Theorem 10. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is weakly contra-(τ_1, τ_2)-continuous and (Y, σ_1, σ_2) is $\sigma_1\sigma_2$-Urysohn, then $G(f)$ is contra-$\mathscr{C}_R$-closed.*

Proof. Let $(x, y) \in (X \times Y) - G(f)$. Then, $y \neq f(x)$. Since (Y, σ_1, σ_2) is $\sigma_1\sigma_2$-Urysohn, there exist $\sigma_1\sigma_2$-open sets V and W of Y containing y and $f(x)$, respectively, such that $\sigma_1\sigma_2$-$\text{Cl}(V) \cap \sigma_1\sigma_2$-$\text{Cl}(W) = \emptyset$; hence $\sigma_1\sigma_2$-$\text{Cl}(V) \subseteq Y - \sigma_1\sigma_2$-$\text{Cl}(W)$. Since f is weakly contra-(τ_1, τ_2)-continuous, $\tau_1\tau_2$-$\text{Cl}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))) \subseteq f^{-1}(Y - \sigma_1\sigma_2\text{-Cl}(W))$. Thus, $(x, y) \in \tau_1\tau_2$-$\text{Cl}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))) \times \sigma_1\sigma_2$-$\text{Cl}(W) \subseteq (X \times Y) - G(f)$ and hence $G(f)$ is contra-$\mathscr{C}_R$-closed.

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