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On Contra-$(\tau_1, \tau_2)p$-continuous Functions

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Abstract. This paper presents a new class of functions called contra-$(\tau_1, \tau_2)p$-continuous functions. Furthermore, several characterizations and some properties concerning contra-$(\tau_1, \tau_2)p$-continuous functions are considered.

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1. Introduction

The notions of contra-continuity and strong S-closedness in topological spaces were introduced by Dontchev [1]. Furthermore, Dontchev [1] obtained very interesting and important results concerning contra-continuity, compactness, S-closedness and strong S-closedness. Dontchev and Noiri [2] introduced and studied the concept of RC-continuity between topological spaces which is weaker than contra-continuity. Jafari and Noiri [3] introduced and investigated a new class of functions called contra-super-continuous functions which lies between classes of RC-continuous functions and contra-continuous functions. In 2002, Jafari and Noiri [4] introduced a new class of function called contra-precontinuous functions which is weaker than contra-continuous functions and studied several basic properties of contra-precontinuous functions. Moreover, the present authors [4] defined contra-preclosed graphs and investigated relations between contra-precontinuity and contra-preclosed graphs. In 2004, Ekici [5] introduced and studied a new class of functions called almost contra-precontinuous functions which generalize classes of regular set-connected functions [6], contra-precontinuous functions [4], contra-continuous functions [1], almost s-continuous functions [7] and perfectly continuous functions [8]. In 2007, Al-Omari and Noorani [9] introduced the concept of almost contra ω-continuous functions via the notion of ω-open sets and investigated several characterizations of contra ω-continuous

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functions and almost contra ω-continuous functions. Noiri and Popa [10] introduced the notion of contra m-continuous functions as functions from a set satisfying some minimal conditions into a topological space and investigated some characterizations and the relationships between contra m-continuity and other related generalized forms of continuity. It turns out that the contra m-continuity is a unified form of several modifications of weak contra-continuity due to Baker [11]. On the other hand, the present authors introduced and studied the notions of (τ_1, τ_2)-continuous functions [12], almost (τ_1, τ_2)-continuous functions [13], weakly (τ_1, τ_2)-continuous functions [14], quasi $\theta(\tau_1, \tau_2)$-continuous functions [15], $\delta(\tau_1, \tau_2)$-continuous functions [16], almost quasi (τ_1, τ_2)-continuous functions [17], weakly quasi (τ_1, τ_2)-continuous functions [18], faintly (τ_1, τ_2)-continuous functions [19] and almost nearly (τ_1, τ_2)-continuous functions [20]. In this paper, we introduce the concept of contra-$(\tau_1, \tau_2)p$-continuous functions. We also investigate some characterizations of contra-$(\tau_1, \tau_2)p$-continuous functions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [21] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [21] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [21] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 1. [21] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2$-Cl(A) and $\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Cl$(A)) = \tau_1\tau_2$-Cl(A).

(2) If $A \subseteq B$, then $\tau_1\tau_2$-Cl$(A) \subseteq \tau_1\tau_2$-Cl$(B)$.

(3) $\tau_1\tau_2$-Cl(A) is $\tau_1\tau_2$-closed.

(4) A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2$-Cl(A).

(5) $\tau_1\tau_2$-Cl$(X - A) = X - \tau_1\tau_2$-Int$(A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$-*open* [22] (resp. $(\tau_1, \tau_2)s$-*open* [23], $(\tau_1, \tau_2)p$-*open* [23], $(\tau_1, \tau_2)\beta$-*open* [23]) if $A = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ (resp. $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$, $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$, $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))))$. The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is said to be $(\tau_1, \tau_2)r$-*closed* (resp. $(\tau_1, \tau_2)s$-*closed*, $(\tau_1, \tau_2)p$-*closed*, $(\tau_1, \tau_2)\beta$-*closed*). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\alpha(\tau_1, \tau_2)$-*open* [24] if $A \subseteq$

$\tau_1\tau_2$-Int($\tau_1\tau_2$-Cl($\tau_1\tau_2$-Int(A))). The complement of an $\alpha(\tau_1, \tau_2)$-open set is said to be $\alpha(\tau_1, \tau_2)$-*closed*. Let A be a subset of a bitopological space (X, τ_1, τ_2). The set

$$\cap\{G \mid A \subseteq G \text{ and } G \text{ is } \tau_1\tau_2\text{-open}\}$$

is called the $\tau_1\tau_2$-*kernel* [21] of A and is denoted by $\tau_1\tau_2$-$ker(A)$.

Lemma 2. [21] *For subsets A, B of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) $A \subseteq \tau_1\tau_2\text{-}ker(A)$.

(2) If $A \subseteq B$, then $\tau_1\tau_2\text{-}ker(A) \subseteq \tau_1\tau_2\text{-}ker(B)$.

(3) If A is $\tau_1\tau_2$-open, then $\tau_1\tau_2\text{-}ker(A) = A$.

(4) $x \in \tau_1\tau_2\text{-}ker(A)$ if and only if $A \cap H \neq \emptyset$ for every $\tau_1\tau_2$-closed set H containing x.

Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $(\tau_1, \tau_2)p$-closed sets of X containing A is called the $(\tau_1, \tau_2)p$-*closure* [25] of A and is denoted by (τ_1, τ_2)-pCl(A). The union of all $(\tau_1, \tau_2)p$-open sets of X contained in A is called the $(\tau_1, \tau_2)p$-*interior* [25] of A and is denoted by (τ_1, τ_2)-pInt(A).

Lemma 3. *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) $(\tau_1, \tau_2)\text{-}pCl(A) = \tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(A)) \cup A$ [25];

(2) $(\tau_1, \tau_2)\text{-}pInt(A) = \tau_1\tau_2\text{-}Int(\tau_1\tau_2\text{-}Cl(A)) \cap A$ [26].

3. Contra-$(\tau_1, \tau_2)p$-continuous functions

In this section, we introduce the concept of contra-$(\tau_1, \tau_2)p$-continuous functions. Moreover, some characterizations of contra-$(\tau_1, \tau_2)p$-continuous functions are discussed.

Definition 1. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be contra-$(\tau_1, \tau_2)p$-continuous if for each $x \in X$ and for each $\sigma_1\sigma_2$-closed set F of Y containing $f(x)$, there exists a $(\tau_1, \tau_2)p$-open set U of X containing x such that $f(U) \subseteq F$.*

Theorem 1. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is contra-$(\tau_1, \tau_2)p$-continuous;

(2) $f^{-1}(F)$ is $(\tau_1, \tau_2)p$-open in X for every $\sigma_1\sigma_2$-closed set F of Y;

(3) $f^{-1}(V)$ is $(\tau_1, \tau_2)p$-closed in X for every $\sigma_1\sigma_2$-open set V of Y;

(4) $f((\tau_1, \tau_2)\text{-}pCl(A)) \subseteq \sigma_1\sigma_2\text{-}ker(f(A))$ for every subset A of X;

(5) (τ_1, τ_2)*-pCl*$(f^{-1}(B)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}ker(B))$ *for every subset* B *of* Y*.*

Proof. (1) ⇒ (2): Let F be any $\sigma_1\sigma_2$-closed set of Y and $x \in f^{-1}(F)$. Then, $f(x) \in F$. Since f is contra-$(\tau_1, \tau_2)p$-continuous, there exists a $(\tau_1, \tau_2)p$-open set U of X containing x such that $f(U) \subseteq F$. Thus, $U \subseteq f^{-1}(F)$ and hence $x \in U \subseteq f^{-1}(F)$. Therefore,

$$x \in (\tau_1, \tau_2)\text{-pInt}(f^{-1}(F)).$$

This implies that $f^{-1}(F) \subseteq (\tau_1, \tau_2)\text{-pInt}(f^{-1}(F))$. Thus, $f^{-1}(F)$ is $(\tau_1, \tau_2)p$-open in X.

(2) ⇔ (3): Let V be any $\sigma_1\sigma_2$-open set of Y. Then, $Y - V$ is $\sigma_1\sigma_2$-closed in Y. By (2), we have $f^{-1}(Y - V) = X - f^{-1}(V)$ is $(\tau_1, \tau_2)p$-open in X and hence $f^{-1}(V)$ is $(\tau_1, \tau_2)p$-closed in X. The converse can be shown easily.

(2) ⇒ (4): Let A be any subset of X. Suppose that $y \notin \sigma_1\sigma_2\text{-}ker(f(A))$. Then by Lemma 2, there exists a $\sigma_1\sigma_2$-closed set K of Y containing y such that $f(A) \cap K = \emptyset$. Thus, $A \cap f^{-1}(K) = \emptyset$ and hence $(\tau_1, \tau_2)\text{-pCl}(A) \cap f^{-1}(K) = \emptyset$. Therefore,

$$f((\tau_1, \tau_2)\text{-pCl}(A)) \cap K = \emptyset$$

and $y \notin f((\tau_1, \tau_2)\text{-pCl}(A))$. This shows that $f((\tau_1, \tau_2)\text{-pCl}(A)) \subseteq \sigma_1\sigma_2\text{-}ker(f(A))$.

(4) ⇒ (5): Let B be any subset of Y. By (4) and Lemma 2, we have

$$\begin{aligned} f((\tau_1, \tau_2)\text{-pCl}(f^{-1}(B))) &\subseteq \sigma_1\sigma_2\text{-}ker(f(f^{-1}(B))) \\ &\subseteq \sigma_1\sigma_2\text{-}ker(B) \end{aligned}$$

and hence $(\tau_1, \tau_2)\text{-pCl}(f^{-1}(B)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}ker(B))$.

(5) ⇒ (3): Let V be any $\sigma_1\sigma_2$-open set of Y. Then by (5) and Lemma 2, we have $(\tau_1, \tau_2)\text{-pCl}(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}ker(V)) = f^{-1}(V)$ and hence $f^{-1}(V)$ is $(\tau_1, \tau_2)p$-closed in X.

(2) ⇒ (1): Let F be any $\sigma_1\sigma_2$-closed set of Y containing $f(x)$. By (2), $f^{-1}(F)$ is $(\tau_1, \tau_2)p$-open in X. Then we have, $x \in (\tau_1, \tau_2)\text{-pInt}(f^{-1}(F))$ and therefore there exists a $(\tau_1, \tau_2)p$-open set U of X such that $x \in U \subseteq f^{-1}(F)$; hence $f(U) \subseteq F$. This shows that f is contra-$(\tau_1, \tau_2)p$-continuous.

Recall that a bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)*-regular* [27] if for each $\tau_1\tau_2$-closed set F and each point $x \in X - F$, there exist disjoint $\tau_1\tau_2$-open sets U and V such that $x \in U$ and $F \subseteq V$.

Definition 2. *A function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is said to be* $(\tau_1, \tau_2)p$*-continuous if for each* $x \in X$ *and for each* $\sigma_1\sigma_2$*-open set* V *of* Y *containing* $f(x)$*, there exists a* $(\tau_1, \tau_2)p$*-open set* U *of* X *containing* x *such that* $f(U) \subseteq V$*.*

Theorem 2. *If a function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is contra-*$(\tau_1, \tau_2)p$*-continuous and* (Y, σ_1, σ_2) *is* (σ_1, σ_2)*-regular, then* f *is* $(\tau_1, \tau_2)p$*-continuous.*

Proof. Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Since (Y, σ_1, σ_2) is (σ_1, σ_2)-regular, there exists a $\sigma_1\sigma_2$-open set W of Y containing $f(x)$ such that

$$\sigma_1\sigma_2\text{-Cl}(W) \subseteq V.$$

Since f is contra-$(\tau_1, \tau_2)p$-continuous, there exists a $(\tau_1, \tau_2)p$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2\text{-Cl}(W)$. Thus, $f(U) \subseteq \sigma_1\sigma_2\text{-Cl}(W) \subseteq V$ and hence f is $(\tau_1, \tau_2)p$-continuous.

Definition 3. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be almost $(\tau_1, \tau_2)p$-continuous if for each $x \in X$ and for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $(\tau_1, \tau_2)p$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))$.*

Definition 4. *A functions $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be $(\tau_1, \tau_2)p$-open if $f(U)$ is $(\sigma_1, \sigma_2)p$-open in Y for every $(\tau_1, \tau_2)p$-open set U of X.*

Theorem 3. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a $(\tau_1, \tau_2)p$-open contra-$(\tau_1, \tau_2)p$-continuous function, then f is almost $(\tau_1, \tau_2)p$-continuous.*

Proof. Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Since f is contra-$(\tau_1, \tau_2)p$-continuous, there exists a $(\tau_1, \tau_2)p$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2\text{-Cl}(V)$. Since f is $(\tau_1, \tau_2)p$-open, $f(U)$ is $\sigma_1\sigma_2$-open in Y. Therefore, $f(U) \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(f(U))) \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$. This shows that f is almost $(\tau_1, \tau_2)p$-continuous.

Definition 5. [26] *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be almost weakly (τ_1, τ_2)-continuous if for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$,*

$$x \in \tau_1\tau_2\text{-}Int(\tau_1\tau_2\text{-}Cl(f^{-1}(\sigma_1\sigma_2\text{-}Cl(V)))).$$

Lemma 4. [26] *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is almost weakly (τ_1, τ_2)-continuous;

(2) $f^{-1}(V) \subseteq \tau_1\tau_2\text{-}Int(\tau_1\tau_2\text{-}Cl(f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))))$ for every $\sigma_1\sigma_2$-open set V of Y;

(3) $\tau_1\tau_2\text{-}Cl(\tau_1\tau_2\text{-}Int(f^{-1}(V))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(4) $(\tau_1, \tau_2)\text{-}pCl(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(V))$ for every $\sigma_1\sigma_2$-open set V of Y;

(5) $f^{-1}(V) \subseteq (\tau_1, \tau_2)\text{-}pInt(f^{-1}(\sigma_1\sigma_2\text{-}Cl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(6) for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $(\tau_1, \tau_2)p$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2\text{-}Cl(V)$.

Theorem 4. *If a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is contra-$(\tau_1, \tau_2)p$-continuous, then f is almost weakly (τ_1, τ_2)-continuous.*

Proof. Let V be any $\sigma_1\sigma_2$-open set of Y. Since f is contra-$(\tau_1, \tau_2)p$-continuous and $\sigma_1\sigma_2\text{-Cl}(V)$ is $\sigma_1\sigma_2$-closed in Y, by Theorem 1 we have $f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$ is $(\tau_1, \tau_2)p$-open in X. Thus, $f^{-1}(V) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)) \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))))$. By Lemma 4(2), f is almost weakly (τ_1, τ_2)-continuous.

The (τ_1, τ_2)*p-frontier* [25] of a subset A of a bitopological space (X, τ_1, τ_2), denoted by (τ_1, τ_2)-pfr(A), is defined by

$$\begin{aligned}(\tau_1, \tau_2)\text{-pfr}(A) &= (\tau_1, \tau_2)\text{-pCl}(A) \cap (\tau_1, \tau_2)\text{-pCl}(X - A)\\ &= (\tau_1, \tau_2)\text{-pCl}(A) - (\tau_1, \tau_2)\text{-pInt}(A).\end{aligned}$$

Theorem 5. *The set of all points x of X at which a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is not contra-(τ_1, τ_2)p-continuous is identical with the union of the (τ_1, τ_2)p-frontier of the inverse images of $\sigma_1\sigma_2$-closed sets of Y containing $f(x)$.*

Proof. Suppose that f is not contra-$(\tau_1, \tau_2)p$-continuous at $x \in X$. Then, there exists a $\sigma_1\sigma_2$-closed set F of Y containing $f(x)$ such that $f(U) \cap (Y - F) \neq \emptyset$ for every $(\tau_1, \tau_2)p$-open set U of X containing x. This implies that $U \cap f^{-1}(Y - F) \neq \emptyset$. Therefore, $x \in (\tau_1, \tau_2)\text{-pCl}(f^{-1}(Y - F)) = (\tau_1, \tau_2)\text{-pCl}(X - f^{-1}(F))$. On the other hand, we have $x \in f^{-1}(F) \subseteq (\tau_1, \tau_2)\text{-pCl}(f^{-1}(F))$ and hence $x \in (\tau_1, \tau_2)\text{-pfr}(f^{-1}(F))$.

Conversely, suppose that $x \in (\tau_1, \tau_2)\text{-pfr}(f^{-1}(F))$ for some $\sigma_1\sigma_2$-closed set F of Y containing $f(x)$. Now, we assume that f is contra-$(\tau_1, \tau_2)p$-continuous at x. Then, there exists a $(\tau_1, \tau_2)p$-open set U of X containing x such that $f(U) \subseteq F$. Thus, $U \subseteq f^{-1}(F)$ and hence $x \in (\tau_1, \tau_2)\text{-pInt}(f^{-1}(F)) \subseteq X - (\tau_1, \tau_2)\text{-pfr}(f^{-1}(F))$. This is a contradiction. This means that f is not contra-$(\tau_1, \tau_2)p$-continuous.

Definition 6. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called contra-$\alpha(\tau_1, \tau_2)$-continuous if $f^{-1}(V)$ is $\alpha(\tau_1, \tau_2)$-closed in X for each $\sigma_1\sigma_2$-open set V of Y.*

Definition 7. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called contra-(τ_1, τ_2)s-continuous if $f^{-1}(V)$ is (τ_1, τ_2)s-closed in X for each $\sigma_1\sigma_2$-open set V of Y.*

Lemma 5. *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties are equivalent:*

(1) A is $\alpha(\tau_1, \tau_2)$-open;

(2) A is (τ_1, τ_2)p-open and (τ_1, τ_2)s-open.

Proof. (1) $\Rightarrow$ (2): Let A be $\alpha(\tau_1, \tau_2)$-open. Then, $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A)))$. Therefore, $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$ and $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$. This shows that A is $(\tau_1, \tau_2)p$-open and $(\tau_1, \tau_2)s$-open.

(2) $\Rightarrow$ (1): Let A be $(\tau_1, \tau_2)p$-open and $(\tau_1, \tau_2)s$-open. Then, $A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))$ and $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$. Thus,

$$A \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)) \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A)))$$

and hence A is $\alpha(\tau_1, \tau_2)$-open.

Theorem 6. *For a $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is contra-$\alpha(\tau_1, \tau_2)$-continuous;

(2) f is contra-$(\tau_1, \tau_2)p$-continuous and contra-$(\tau_1, \tau_2)s$-continuous.

Proof. This is an immediate consequence of Lemma 5.

Definition 8. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be RC-(τ_1, τ_2)-continuous if $f^{-1}(V)$ is $(\tau_1, \tau_2)r$-closed in X for each $\sigma_1\sigma_2$-open set V of Y.*

Definition 9. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be $(\tau_1, \tau_2)s$-continuous if for $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $(\tau_1, \tau_2)s$-open set U of X containing x such that $f(U) \subseteq V$.*

Definition 10. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be $(\tau_1, \tau_2)\beta$-continuous if for $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $(\tau_1, \tau_2)\beta$-open set U of X containing x such that $f(U) \subseteq V$.*

Lemma 6. *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties are equivalent:*

(1) A is $(\tau_1, \tau_2)r$-closed;

(2) A is $(\tau_1, \tau_2)p$-closed and $(\tau_1, \tau_2)s$-open;

(3) A is $\alpha(\tau_1, \tau_2)$-closed and $(\tau_1, \tau_2)\beta$-open.

Proof. (1) $\Rightarrow$ (2): Let A be $(\tau_1, \tau_2)r$-closed. Then, we have $A = \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$. Thus, A is $(\tau_1, \tau_2)p$-closed and $(\tau_1, \tau_2)s$-open.

(2) $\Rightarrow$ (3): Let A be $(\tau_1, \tau_2)p$-closed and $(\tau_1, \tau_2)s$-open. Then, $\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A)) \subseteq A$ and $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A))$. Thus, $\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A)) = \tau_1\tau_2\text{-Cl}(A)$ and hence

$$\begin{aligned}\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))) &= \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A)))) \\ &= \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A)) \subseteq A.\end{aligned}$$

This shows that A is $\alpha(\tau_1, \tau_2)$-closed. It is obvious that A is $(\tau_1, \tau_2)\beta$-open.

(3) $\Rightarrow$ (1): Let A be $\alpha(\tau_1, \tau_2)$-closed and $(\tau_1, \tau_2)\beta$-open. Then, we have

$$\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))) \subseteq A$$

and $A \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))$. Thus, $A = \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A)))$ and hence

$$\begin{aligned}\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(A)) &= \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))))) \\ &= \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(A))) = A.\end{aligned}$$

Therefore, A is $(\tau_1, \tau_2)r$-closed.

As a consequence of Lemma 6, we have the following result:

Theorem 7. *For a $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) f is RC-(τ_1, τ_2)-continuous;

(2) f is contra-$(\tau_1, \tau_2)p$-continuous and $(\tau_1, \tau_2)s$-continuous;

(3) f is contra-$\alpha(\tau_1, \tau_2)$-continuous and $(\tau_1, \tau_2)\beta$-continuous.

Definition 11. [28] *A bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-Urysohn if for each pair of distinct points x and y in X, there exist $\tau_1\tau_2$-open sets U and V such that $x \in U$, $y \in V$ and $\tau_1\tau_2$-$Cl(U) \cap \tau_1\tau_2$-$Cl(V) = \emptyset$.*

Definition 12. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to have a contra-$(\tau_1, \tau_2)p$-closed graph if for each $(x, y) \in (X \times Y) - G(f)$, there exists a $(\tau_1, \tau_2)p$-open set U of X containing x and a $\sigma_1\sigma_2$-closed set K of Y containing y such that $(U \times K) \cap G(f) = \emptyset$.*

Lemma 7. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ has a contra-$(\tau_1, \tau_2)p$-closed graph if and only if for each $(x, y) \in (X \times Y) - G(f)$, there exists a $(\tau_1, \tau_2)p$-open set U of X containing x and a $\sigma_1\sigma_2$-closed set K of Y containing y such that $f(U) \cap K = \emptyset$.*

Theorem 8. *If a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is contra-$(\tau_1, \tau_2)p$-continuous and (Y, σ_1, σ_2) is $\sigma_1\sigma_2$-Urysohn, then $G(f)$ is contra-$(\tau_1, \tau_2)p$-closed.*

Proof. Let $(x, y) \in (X \times Y) - G(f)$. Then, $y \neq f(x)$. Since (Y, σ_1, σ_2) is $\sigma_1\sigma_2$-Urysohn, there exist $\sigma_1\sigma_2$-open sets V and W of Y containing y and $f(x)$, respectively, such that $\sigma_1\sigma_2$-Cl$(V) \cap \sigma_1\sigma_2$-Cl$(W) = \emptyset$. Since f is contra-$(\tau_1, \tau_2)p$-continuous, there exists a $(\tau_1, \tau_2)p$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2$-Cl$(W)$. Thus, $f(U) \cap \sigma_1\sigma_2$-Cl$(V) = \emptyset$ and hence by Lemma 7, $G(f)$ is contra-$(\tau_1, \tau_2)p$-closed.

Recall that a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-*connected* [21] if X cannot be written as the union of two nonempty disjoint $\tau_1\tau_2$-open sets.

Definition 13. *A bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)p$-connected if X cannot be written as the union of two nonempty disjoint $(\tau_1, \tau_2)p$-open sets.*

Theorem 9. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a contra-$(\tau_1, \tau_2)p$-continuous surjection and (X, τ_1, τ_2) is $(\tau_1, \tau_2)p$-connected, then (Y, σ_1, σ_2) is $\sigma_1\sigma_2$-connected.*

Proof. Suppose that (Y, σ_1, σ_2) is not $\sigma_1\sigma_2$-connected. Then, there exist nonempty $\sigma_1\sigma_2$-open sets V and W such that $Y = V \cup W$. Therefore, V and W are $\sigma_1\sigma_2$-clopen in Y. Since f is contra-$(\tau_1, \tau_2)p$-continuous, $f^{-1}(V)$ and $f^{-1}(W)$ are $(\tau_1, \tau_2)p$-open in X. Moreover, $f^{-1}(V)$ and $f^{-1}(W)$ are nonempty disjoint and $X = f^{-1}(V) \cup f^{-1}(W)$. This shows that (X, τ_1, τ_2) is not $(\tau_1, \tau_2)p$-connected. This is a contradiction. This means that (Y, σ_1, σ_2) is $\sigma_1\sigma_2$-connected.

Definition 14. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called perfectly (τ_1, τ_2)-continuous if $f^{-1}(V)$ is $\tau_1\tau_2$-clopen in X for each $\sigma_1\sigma_2$-open set V of Y.*

Definition 15. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called $\alpha(\tau_1, \tau_2)$-continuous if $f^{-1}(V)$ is $\alpha(\tau_1, \tau_2)$-open in X for each $\sigma_1\sigma_2$-open set V of Y.*

Theorem 10. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is perfectly (τ_1, τ_2)-continuous if and only if f is contra-$(\tau_1, \tau_2)p$-continuous and $\alpha(\tau_1, \tau_2)$-continuous.*

Proof. This is obvious.

Conversely, let V be any $\sigma_1\sigma_2$-open set of Y. Since f is contra-$(\tau_1, \tau_2)p$-continuous and $\alpha(\tau_1, \tau_2)$-continuous, $f^{-1}(V)$ is $(\tau_1, \tau_2)p$-closed and $\alpha(\tau_1, \tau_2)$-open in X. Therefore, we have $\tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(f^{-1}(V)))) \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(f^{-1}(V))) \subseteq f^{-1}(V) \subseteq \tau_1\tau_2\text{-Int}(\tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(f^{-1}(V)))) \subseteq \tau_1\tau_2\text{-Cl}(\tau_1\tau_2\text{-Int}(f^{-1}(V)))$. This implies that $f^{-1}(V)$ is $\tau_1\tau_2$-clopen in X. Thus, f is perfectly (τ_1, τ_2)-continuous.

Recall that a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-*compact* [21] if every cover of X by $\tau_1\tau_2$-open sets has a finite subcover.

Definition 16. *A bitopological space (X, τ_1, τ_2) is said to be mildly $\tau_1\tau_2$-compact if every $\tau_1\tau_2$-clopen cover of X has a finite subcover.*

Theorem 11. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a perfectly (τ_1, τ_2)-continuous surjection and (X, τ_1, τ_2) is mildly $\tau_1\tau_2$-compact, then (Y, σ_1, σ_2) is $\sigma_1\sigma_2$-compact.*

Proof. Let $\{V_\gamma \mid \gamma \in \nabla\}$ be any $\sigma_1\sigma_2$-open cover of Y. Since f is perfectly (τ_1, τ_2)-continuous, we have $\{f^{-1}(V_\gamma) \mid \gamma \in \nabla\}$ is a $\tau_1\tau_2$-clopen cover of X. Since (X, τ_1, τ_2) is mildly $\tau_1\tau_2$-compact, there exists a finite subset ∇_0 of ∇ such that

$$X = \cup\{f^{-1}(V_\gamma) \mid \gamma \in \nabla_0\}.$$

Since f is surjective, $Y = \cup\{V_\gamma \mid \gamma \in \nabla_0\}$ and (Y, σ_1, σ_2) is $\sigma_1\sigma_2$-compact.

Definition 17. *A bitopological space (X, τ_1, τ_2) is said to be:*

(1) $(\tau_1, \tau_2)p$-irreducible if every pair of nonempty $(\tau_1, \tau_2)p$-closed sets of X has a nonempty intersection;

(2) $\tau_1\tau_2$-hyperconnected if $\tau_1\tau_2$-Cl$(V) = X$ for every nonempty $\tau_1\tau_2$-open set V of X.

Theorem 12. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a contra-$(\tau_1, \tau_2)p$-continuous surjection and (X, τ_1, τ_2) is $(\tau_1, \tau_2)p$-irreducible, then (Y, σ_1, σ_2) is $\sigma_1\sigma_2$-hyperconnected.*

Proof. Suppose that (Y, σ_1, σ_2) is not $\sigma_1\sigma_2$-hyperconnected. Then, $\sigma_1\sigma_2\text{-Cl}(V) \neq Y$ for some nonempty $\sigma_1\sigma_2$-open set V of Y. Therefore, there exists a point $y \notin \sigma_1\sigma_2\text{-Cl}(V)$ and $V \cap W = \emptyset$ for some $\sigma_1\sigma_2$-open set W of Y containing y. Since f is contra-$(\tau_1, \tau_2)p$-continuous, $f^{-1}(V)$ and $f^{-1}(W)$ are disjoint nonempty $(\tau_1, \tau_2)p$-closed sets of X. Thus, (X, τ_1, τ_2) is not $(\tau_1, \tau_2)p$-irreducible.

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