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Geodetically Undominated Vertices in a Graph

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Abstract. Let $G = (V(G), E(G))$ be a simple undirected graph. If $\gamma_g(G)$ is the geodetic domination number of G and $S \subseteq V(G)$ such that $|S| < \gamma_g(G)$, then definitely, there is at least one vertex of G that is not geodetically dominated by S, that is, not dominated by any vertex in S or not in any geodesic of any two vertices in S. If k is a positive integer with $k \leq \gamma_g(G) - 1$ and $S \subseteq V(G)$ with $|S| = \gamma_g(G) - k$, then the number $\zeta_k^g(S)$ given by $\zeta_k^g(S) = |V(G) \setminus N_G^g[S]|$, where $N_G^g[S] = N_G[S] \cap I_G[S]$, is called the k-geodetic domination defect of S in G. The k-geodetic domination defect of G is denoted and given by $\zeta_k^g(G) = \min\{\zeta_k^g(S) : S \subseteq V(G) \text{ and } |S| = \gamma_g(G) - k\}$. In this paper, we study this newly defined parameter for some known classes of graphs. Moreover, we determine some sharp bounds of the parameter.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: Geodetic set, domination, geodetic domination, k-geodetic domination defect

1. Introduction

Domination is a fundamental concept in graph theory with numerous applications across various fields, including network design, resource allocation, and social network analysis (see [1] and [2]). The domination number $\gamma(G)$ of a graph G refers to the smallest number of vertices required to dominate all the vertices of G. In other words, it is the minimum cardinality of a set S of vertices such that every vertex in the graph is either in the set S or is adjacent to at least one vertex in S. If a set S of vertices has cardinality strictly less than $\gamma(G)$, then there will exist vertices in the graph that are not dominated by any vertex in the set S. Recently, Das et al. [3] introduced and studied the notion of k-domination defect of a graph, where k is a positive integer strictly less than the domination

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number of the graph. In their study, the authors established various bounds on the k-domination defect of a graph, based on its maximum degree, domination number, and other parameters. Subsequent studies on this topic (see [4–6]) focused on characterizing k-domination defect sets and determining the k-domination defect in the join, corona, edge corona, and composition of two graphs. Recently, a new variant of the domination defect was introduced and explored in [7].

Some variations of the standard domination utilize the concept of geodetic set (see [8–12]). The associated parameter, called geodetic number of a graph, was introduced by Harary et al. [13]. Geodetic number and geodetic domination were considered in [14], [15], [16], [17], [18], and [19]. Note that if G is a graph and S is a set of vertices of G with cardinality strictly less than the geodetic domination number $\gamma_g(G)$ of G, then there is at least one vertex outside S that is not geodetically dominated, that is, has no neighbor in S or is not in any shortest path joining any two vertices in S. In this paper, we introduce the notion k-geodetic domination defect and study it for some classes of graphs.

For the motivation of the study, consider a prison facility with numerous prisoners who need to undergo routine assessments such as behavior evaluations or security checks. The warden needs to ensure that every prisoner is evaluated by a jail guard. This situation can be modeled by constructing a graph, where each vertex represents a prisoner or a jail guard, and an edge between a prisoner and a jail guard indicates that the jail guard is responsible for evaluating the prisoner. Moreover, to ensure visibility and security, it is required that every prisoner must be on a shortest path connecting two jail guards. However, due to a lack of personnel and financial support, the required minimum number of jail guards to do the task may not always be met. Furthermore, it may happen that during the assessment, a designated jail guard may be absent and, subsequently, unable to perform his or her task. As a result, some prisoners may not be evaluated in the manner expected. Determining the number of unevaluated prisoners when a designated team of jail guards does not meet the required minimum number of members could assist the management to act accordingly. This situation led us to introduce the concept of geodetic domination defect in a graph.

2. Terminology and Notation

For any two vertices u and v in an undirected connected graph G, the distance $d_G(u,v)$ is the length of a shortest path joining u and v. Any u-v path of length $d_G(u,v)$ is called a *u-v geodesic*. The *diameter* of G, denoted by $diam(G)$, is the maximum distance between any two vertices in G. The distance between two subsets A and B of $V(G)$ is given by $d_G(A,B) = \min\{d_G(a,b) : a \in A \text{ and } b \in B\}$. The *open neighborhood* of a vertex u is the set $N_G(u)$ consisting of all vertices v which are adjacent to u. The *closed neighborhood* of u is $N_G[u] = N_G(u) \cup \{u\}$. For any $A \subseteq V(G)$, $N_G(A) = \bigcup_{v \in A} N_G(v)$ is called the *open neighborhood of A* and $N_G[A] = N_G(A) \cup A$ is called the *closed neighborhood of A*. A vertex v of G is isolated if $|N_G(v)| = 0$. The set containing all the isolated vertices of G is denoted by $I(G)$. If $C \subseteq V(G)$, then the *induced subgraph* $\langle C \rangle$ is the graph with vertex-set

C and $uv \in E(\langle C \rangle)$ whenever $u, v \in C$ and $uv \in E(G)$.

A set $S \subseteq V(G)$ is a *dominating set* of G if $N_G[S] = V(G)$. The smallest cardinality of a dominating set of G, denoted by $\gamma(G)$, is called the *domination number* of G. A dominating set S of G with $|S| = \gamma(G)$, is called a γ-set of G. For every two vertices u and v in G, the symbol $I_G[u, v]$, is the *set interval* containing u, v and all vertices lying in some u-v geodesic. The geodetic closure of a set $S \subseteq V(G)$, denoted by $I_G[S]$, is the union of the intervals $I_G[u, v]$, where $u, v \in S$. The set S is a geodetic set in G if $I_G[S] = V(G)$. The smallest cardinality among all geodetic sets in G, denoted by $g(G)$, is called the geodetic number of G. A geodetic set of cardinality $g(G)$ is called a g-set of G. A set $S \subseteq V(G)$ is a geodetic dominating set in G if it is both a dominating and a geodetic set. The geodetic domination number $\gamma_g(G)$ of G is the minimum cardinality among all geodetic dominating sets in G. Any geodetic dominating set of G with cardinality $\gamma_g(G)$ is called a γ_g-set.

Let G be a non-trivial graph of order n and let $1 \leq k < \gamma_g(G)$. Let $S \subseteq V(G)$ with cardinality $|S| = \gamma_g(G) - k$ and let $N_G^g[S] = N_G[S] \cap I_G[S]$, the set of geodetically dominated set of vertices of G. The set $V(G) \setminus N_G^g[S]$ is called the *k-geodetic domination defect set* of S and the *k-geodetic domination defect* of S in G is $\zeta_k^g(S) = |V(G) \setminus N_G^g[S]| = n - |N_G^g[S]|$. The minimum cardinality of a *k-geodetic domination defect set* in G, denoted by $\zeta_k^g(G)$, is called the *k-geodetic domination defect* of G, i.e.,

$$\zeta_k^g(G) = \min\{\zeta_k^g(S) : S \subseteq V(G) \text{ with } |S| = \gamma_g(G) - k\}.$$

A set $S \subseteq V(G)$ of cardinality $\gamma_g(G) - k$ for which $|V(G) \setminus N_G^g[S]| = \zeta_k^g(G)$ is called a *ζ_k^g-set* of G. Thus, $\langle N_G^g[S] \rangle$, the subgraph induced by $N_G^g[S]$, is a subgraph of G with $n - \zeta_k^g(G)$ vertices and geodetic domination number $\gamma_g - k$.

Consider the graph G in Figure 1. Then $R = \{a, b, x, y\}$ is a γ_g-set of G, i.e., $\gamma_g(G) = 4$. If $k = 1$, then $D_1 = \{a, b, x\}$ is a ζ_1^g-set of G. Since $N_G^g[D_1] = \{a, b, c, d, u, v, x\}$, it follows that

$$\zeta_1^g(G) = \zeta_1^g(D_1) = |V(G)| - |N_G^g[D_1]| = 8 - 7 = 1.$$

The set $\{x, y, c\}$ is not a ζ_1^g-set of G because $N_G^g[\{x, y, c\}] = \{x, y, c, d, u, v\}$, i.e., $\zeta_1^g(\{x, y, c\}) = 8 - 6 = 2$. If $k = 2$, then $D_2 = \{a, x\}$ is a ζ_2^g-set of G and $N_G^g[D_2] = \{a, x, c, d, u, v\}$. Hence, $\zeta_2^g(G) = 8 - 6 = 2$. It is easy to verify that the sets $\{a, y\}$ and $\{c, v\}$ are not ζ_2^g-sets of G. Finally, if $k = 3$, then any 1-element subset D_3 of $V(G)$ is a ζ_3^g-set of G. Since $N_G^g[D_3] = D_3$, it follows that $\zeta_3^g(G) = |V(G)| - |N_G^g[D_3]| = 8 - 1 = 7$.

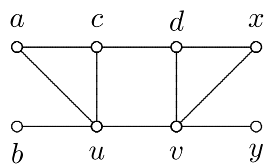

Figure 1: Graph G with $\gamma_g(G) = 4$, $\zeta_1^g(G) = 1$, $\zeta_2^g(G) = 2$, and $\zeta_3^g(G) = 7$

3. Results

Theorem 1 ([20])**.** *Let n be positive integer. Then each of the following holds.*

(i) *For a complete graph K_n, $\gamma_g(K_n) = n$.*

(ii) *For a star graph $K_{1,n-1}$, $\gamma_g(K_{1,n-1}) = n - 1$.*

(iii) *For a complete bipartite graph $K_{m,n}$ with $m, n \geq 2$, $\gamma_g(K_{m,n}) = \min\{m, n, 4\}$.*

(iv) *For a wheel graph W_n, $\gamma_g(W_n) = \lceil \frac{n-1}{2} \rceil,\ n \geq 5$.*

(v) *For a cycle C_n on n vertices, we have $\gamma_g(C_n) = \lceil \frac{n}{3} \rceil,\ n \geq 6$.*

(vi) *For a path P_n on n vertices, $\gamma_g(P_n) = \lceil \frac{n+2}{3} \rceil$.*

(vii) *For the Petersen graph P, $\gamma_g(P) = 4$.*

Remark 1. *Let $G_1, G_2, \cdots, G_r$ be the components of a graph G. Then each of the following holds:*

(i) $\gamma_g(G) = \sum_{j=1}^{r} \gamma_g(G_j)$.

(ii) *If $A_j \subseteq V(G_j)$ for each $j \in [r] = \{1, 2, \cdots, r\}$ and $A = \cup_{j=1}^{r} A_j$, then $N_G^g[A] = \cup_{j=1}^{r} N_G^g[A_j]$ (a disjoint union).*

Theorem 2. *Let $G_1, G_2, \cdots, G_r$ be the components of graph G and let $\zeta_1^g(G_i)$ be the 1-geodetic domination defect of G_i for each $i \in [r] = \{1, 2, \cdots, r\}$. Then*

$$\zeta_1^g(G) = \min\{\zeta_1^g(G_i) : i \in [r]\}.$$

Proof. Let $\gamma_g(G_i)$ and $\gamma_g(G)$ be the geodetic domination numbers of G_i and G, respectively. By Remark 1(i), $\gamma_g(G) = \sum_{j=1}^{r} \gamma_g(G_i)$. For each $i \in [r]$, let D_i be a ζ_1^g-set of G_i. Then $|D_i| = \gamma_g(G_i) - 1$ and $\zeta_1^g(G_i) = |V(G_i) - N_G^g[D_i]|$. Let $j \in [r]$ be such that $\zeta_1^g(G_j) = \min\{\zeta_1^g(G_i) : i \in [r]\}$. Let S_i be a γ_g-set in G_i for each $i \in [r]$ and let $S = (\cup_{i \in [r] \setminus \{j\}} S_i) \cup D_j$. Then

$$|S| = \sum_{i \in [r] \setminus \{j\}} |S_i| + |D_j| = \gamma_g(G) - 1$$

and, by Remark 1(ii),

$$\begin{aligned} |N_G^g[S]| &= |N_{G_j}^g[D_j]| + \sum_{i \in [r] \setminus \{j\}} |N_{G_i}^g[S_i]| \\ &= |V(G_j)| - \zeta_1^g(G_j) + \sum_{i \in [r] \setminus \{j\}} |V(G_i)| \\ &= \sum_{i=1}^{r} |V(G_i)| - \zeta_1^g(G_j). \end{aligned}$$

Thus, in G, $\zeta_1^g(S) = |V(G)| - |N_G^g[S]| = \zeta_1^g(G_j)$. We claim that $\zeta_1^g(S)$ is the minimum among all subsets of $V(G)$ with cardinality $\gamma_g(G) - 1$. To this end, suppose there exists $Q \subseteq V(G)$ such that $|Q| = \gamma_g(G) - 1$ and $\zeta_1^g(Q) < \zeta_1^g(S)$. Let $Q = Q_1 \cup Q_2 \cup \cdots \cup Q_r$ where $Q_i \subseteq V(G_i)$ for each $i \in [r]$. Since $|Q| = \gamma_g(G) - 1$, at least one Q_t is not a geodetic dominating set of G_t by Remark 1(i). Thus, $|Q_t| = \gamma_g(G_t) - 1$ and $\zeta_1^g(Q_t) \geq \zeta_1^g(G_t) \geq \zeta_1^g(G_j)$. Hence,

$$\begin{aligned}\zeta_1^g(Q) = |V(G)| - |N_G^g[Q]| &= \sum_{i=1}^{r}(|V(G_i)| - |N_{G_i}^g[Q_i]|) \\ &\geq |V(G_t)| - |N_{G_t}^g[Q_t]| \\ &= \zeta_1^g(Q_t) \\ &\geq \zeta_1^g(G_j) \\ &= \zeta_1^g(S),\end{aligned}$$

contrary to the assumption that $\zeta_1^g(Q) < \zeta_1^g(S)$. Therefore, $\zeta_1^g(G) = \zeta_1^g(S) = \zeta_1^g(G_j)$. □

Theorem 3. *Let G be a graph with $I(G) \neq \varnothing$ and suppose $|I(G)| = r$. Then $\zeta_j^g(G) = j$ for every $j \in [r] = \{1, 2, \cdots, r\}$ and $\zeta_k^g(G) = r + \zeta_{k-r}^g(G')$ for every $k \in \{r+1, \cdots, \gamma_g(G)-1\}$, where $G' = \langle V(G) \setminus I(G)\rangle$.*

Proof. Let $I(G) = \{v_1, v_2, \cdots, v_r\}$ and let S be a γ_g-set in G. Then $I(G) \subseteq S$. Let $j \in [r]$. Then $D = S \setminus \{v_1, v_2, \cdots, v_j\}$ is a ζ_j^g-set in G and $|N_G^g[D]| = |N_G^g[S]| - |N_G^g[\{v_1, v_2, \cdots, v_j\}]| = |V(G)| - j$. Hence, $\zeta_j^g(G) = |V(G)| - (|V(G)| - j) = j$.

Next, let $k \in \{r+1, \cdots, \gamma_g(G)-1\}$. Then $S_0 = S \setminus I(G)$ is γ_g-set in $G' = \langle V(G) \setminus I(G)\rangle$. Hence, $\gamma_g(G') = \gamma_g(G) - r$. Since $k \leq \gamma_g(G) - 1$, $k - r \leq \gamma_g(G) - (r+1) < \gamma_g(G) - r$. Let S' be a ζ_{k-r}^g-set in G'. Then $|S'| = (\gamma_g(G) - r) - (k - r) = \gamma_g(G) - k$ and $\zeta_{k-r}^g(G') = |V(G')| - |N_{G'}^g[S']| = (|V(G)| - r) - |N_G^g[S']|$. This implies that $|V(G)| - |N_G^g[S']| = r + \zeta_{k-r}^g(G')$. Therefore, since S' is also a ζ_k^g-set in G, $\zeta_k^g(G) = r + \zeta_{k-r}^g(G')$. □

Theorem 4. *If G is a graph of order $n \geq 2$ and $k \leq \gamma_g(G) - 1$, then $1 \leq \zeta_k^g(G) \leq n - 1$.*

Proof. Let S be a ζ_k^g-set in G. Then $|S| = \gamma_g(G) - k$. Hence, $V(G) \setminus N_G^g[S] \neq \varnothing$. It follows that $\zeta_k^g(G) = |V(G)| - |N_G^g[S]| \geq 1$. Also, since $|N_G^g[S]| \geq 1$, $\zeta_k^g(G) = |V(G)| - |N_G^g[S]| \leq n - 1$. This proves the assertion. □

Theorem 5. *Let G be a non-trivial graph of order n. Then $\zeta_1^g(G) = 1$ if and only if there exists $v \in V(G)$ such that $\gamma_g(G - v) = \gamma_g(G) - 1$.*

Proof. Suppose $\zeta_1^g(G) = 1$ and let S be a ζ_1^g-set in G. Then $|S| = \gamma_g(G) - 1$ and $\zeta_k^g(G) = |V(G) \setminus N_G^g[S]| = 1$. Let $v \in V(G) \setminus N_G^g[S]$. Then $N_G^g[S] = V(G) \setminus \{v\}$. Therefore, $\gamma_g(G - v) = \gamma_g(\langle N_G^g[S]\rangle) = \gamma_g(G) - 1$.

Conversely, let $v \in V(G)$ such that $\gamma_g(G - v) = \gamma_g(G) - 1$. Then there exists $D \subseteq V(G)$ with $|D| = \gamma_g(G) - 1$ and $N_G^g[D] = V(G) \setminus \{v\}$. This implies that $\zeta_1^g(G) = |V(G) \setminus N_G^g[D]| = |\{v\}| = 1$. Therefore, $\zeta_1^g(G) = 1$. □

Lemma 1. *Let G be a non-trivial graph of order n. If $k = \gamma_g(G) - 1$, then $\zeta_k^g(G) = n - 1$.*

Proof. Let $k = \gamma_g(G) - 1$ and let S be a ζ_k^g-set of G. Then $|S| = 1$, say, $S = \{x\}$ and $\zeta_k^g(G) = \zeta_k^g(S) = n - |N_G^g[S]|$. Since $x \in N_G[S]$ and $I_G[S] = S$, it follows that $N_G^g[S] = S$. It follows that $\zeta_k^g(G) = n - |N_G^g[S]| = n - 1$. □

Lemma 2. *Let G be a graph and $S \subseteq V(G)$. If every component of $\langle S \rangle$ is complete, then $N_G^g[S] = S$.*

Proof. Let $H_1, H_2, \cdots, H_t$ be the components of $\langle S \rangle$. Then $S = \cup_{i=1}^t V(H_i)$. By Remark 1(ii),

$$N_G^g[S] = \cup_{i=1}^t N_{G_i}^g[V(H_i)] = \cup_{i=1}^t V(H_i) = S.$$

This proves the assertion. □

Theorem 6. *If K_n is a complete graph on n vertices, where $n \geq 2$, and $1 \leq k \leq n - 1$, then $\zeta_k^g(K_n) = k$.*

Proof. Let k be a positive integer with $k \leq \gamma_g(K_n) - 1$. Since $\gamma_g(K_n) = n$, $k \leq n - 1$. Let S be a ζ_k^g-set of K_n. Then S is a clique, $|S| = n - k$ and $\zeta_k^g(G) = n - |N_G^g[S]|$. Therefore, by Lemma 2, $\zeta_k^g(G) = n - (n - k) = k$. □

Let G be a graph and let S be a subset of $V(G)$. Then the set $I_G^2(S)$ is given by

$$I_G^2(S) = \{x \in V(G) \setminus S : x \in I_G(y, z) \text{ for some } y, z \in S \text{ with } d_G(y, z) = 2\}.$$

If G is non-trivial and k is a positive integer with $k \leq \gamma_g(G) - 1$, then the number $\lambda_2^k(G)$ is given by

$$\lambda_2^k(G) = \max\{|I_G^2(S)| : S \subseteq V(G) \text{ and } |S| = \gamma_g(G) - k\}.$$

Remark 2. *Let G be a graph and let k be a postive integer with $k \leq \gamma_g(G) - 1$. If S is a $(\gamma_g(G) - k)$-element subset of $V(G)$ and $\lambda_2^k(G) = |I_G^2(S)|$, then S need not be a ζ_k^g-set in G.*

To see this, consider graph G in Figure 2. The set $D = \{a, b, f\}$ is a γ_g-set in G. Hence, $\gamma_g(G) = 3$. Let $k = 1$. Consider $S = \{b, d\}$ and $S' = \{b, f\}$. Then $N_G^g[S] = \{b, c, d, e\}$ and $N_G^g[S'] = \{b, c, d, e, f\}$. Thus, $\zeta_1^g(S) = 6 - 4 = 2 > 1 = \zeta_1^g(S')$. This implies that S' is a ζ_1-set in G. Since $d_G(b, f) = 3$, it follows that $|I_G^2(S')| = 0$. It is easy to see that $\lambda_2^k(G) = |I_G^2(S)| = 2$ where S is not a ζ_1-set in G.

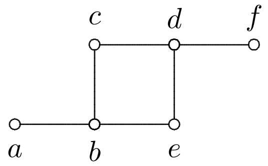

Figure 2: Graph G with $\gamma_g(G) = 3$, $\zeta_1^g(G) = 1$, and $\zeta_2^g(G) = 5$

Theorem 7. *Let G be a non-trivial graph and let $S \subseteq V(G)$. Then the following statements hold:*

(i) $S \cup I_G^2(S) \subseteq N_G^g[S]$.

(ii) *If $diam(G) = 2$, then $N_G^g[S] = S \cup I_G^2(S)$.*

(iii) *If $diam(G) = 2$, k is a positive integer with $k \leq \gamma_g(G) - 1$, and S is a ζ_k^g-set in G, then $\lambda_2^k(G) = |I_G^2(S)|$.*

Proof. (i) Note that if $x \in I_G^2(S)$, then $x \in N_G^g[S]$. Hence, $I_G^2(S) \subseteq N_G^g[S]$. Since $S \subseteq N_G^g[S]$, it follows that $S \cup I_G^2(S) \subseteq N_G^g[S]$.

(ii) Suppose $diam(G) = 2$. Let $x \in N_G^g[S]$. Then $x \in N_G[S] \cap I_G[S]$. If $x \in S$, then $x \in S \cup I_G^2(S)$. If $x \notin S$, then there exist $y, z \in S$ such that $x \in I_G(y, z)$. Since $diam(G) = 2$, it follows that $d_G(y, z) = 2$. Thus, $x \in I_G^2(S)$. Therefore, $N_G^g[S] \subseteq S \cup I_G^2(S)$. With (i), we get the desired equality $N_G^g[S] = S \cup I_G^2(S)$.

(iii) Let S' be a $(\gamma_g(G) - k)$-element subset of $V(G)$. By part (ii), $N_G^g[S'] = S \cup I_G^2(S)$. It follows that $\zeta_k^g(S') = n - \gamma_g(G) + k - |I_G^2(S')|$. Since S is a ζ_k^g-set in G, $\zeta_k^g(S) = n - \gamma_g(G) + k - |I_G^2(S)| \leq \zeta_k^g(S')$. This implies that $|I_G^2(S')| \leq |I_G^2(S)|$. Since S' was arbitrarily chosen, we have $\lambda_2^k(G) = |I_G^2(S)|$. □

Theorem 8. *Let G be a non-trivial graph of order n and let k be a positive integer with $k \leq \gamma_g(G) - 1$. Then*

$$\zeta_k^g(G) \leq n - \gamma_g(G) + k - \lambda_2^k(G)$$

and equality holds if $k = \gamma_g(G) - 1$ or $1 \leq diam(G) \leq 2$.

Proof. Let k be a positive integer with $k \leq \gamma_g(G) - 1$ and let S be a $(\gamma_g(G) - k)$-element subset of $V(G)$ such that $\lambda_2^k(G) = |I_G^2(S)|$. By Theorem 7$(i)$, $S \cup I_G^2(S) \subseteq N_G^g[S]$. Hence,

$$\zeta_k^g(G) \leq \zeta_k^g(S) = n - |N_G^g[S]| \leq n - (|S| + |I_G^2(S)|) = n - \gamma_g(G) + k - \lambda_2^k(G).$$

Suppose $k = \gamma_g(G) - 1$. Then $\zeta_k^g(G) = n - 1$ by Lemma 1. In this case, $\lambda_2^k = 0$. Hence, $\zeta_k^g(G) = n - 1 = n - \gamma_g(G) + k - \lambda_2^k(G)$.

Suppose $1 \leq diam(G) \leq 2$ and let S be a ζ_k^g-set of G. If $diam(G) = 1$, then every component of G is complete and $\gamma_g(G) = n$. Hence, if S' is any $(\gamma_g(G) - k)$-element set in

G, then $|I_G^2(S')| = 0$. It follows that $\lambda_2^k(G) = 0$. Since $N_G^g[S] = S$ by Lemma 2, we have

$$\zeta_k^g(G) = \zeta_k^g(S) = n - (\gamma_g(G) - k) = n - \gamma_g(G) + k - \lambda_2^k(G) = k.$$

Next, suppose that $diam(G) = 2$. Then $N_G^g[S] = S \cup I_G^2(S)$ by Theorem 7(ii) and $\lambda_2^k(G) = |I_G^2(S)|$ by Theorem 7(iii) . Thus,

$$\zeta_k^g(G) = \zeta_k^g(S) = n - \gamma_g(G) + k - |I_G^2(S)| = n - \gamma_g(G) + k - \lambda_2^k(G).$$

This proves the assertion. □

Remark 3. *Strict inequality in Theorem 8 is possible.*

To see this, consider graph $H = P_8$ in Figure 3. By Theorem 1(vi), $\gamma_g(H) = 4$. Let $P_8 = [v_1, v_2, \cdots, v_8]$ and let $k = 1$. Let $S_1 = \{v_1, v_4, v_7\}$. Then $|S| = 3$ and $N_H^g[S_1] = \{v_1, v_2, \cdots, v_7\}$. Hence, $\zeta_1^g(S_1) = 1$. It follows that S_1 is a ζ_1^g-set of H. It can be verified that $\lambda_2^1(G) = 2$. Hence,

$$\zeta_1^g(H) = \zeta_1^g(S_1) = 1 < 3 = 8 - \gamma_g(H) + 1 - \lambda_2^1(G).$$

If $k = 2$, then $S_2 = \{v_1, v_4\}$ is a ζ_2^g-set of H and $N_H^g[S_2] = \{v_1, v_2, v_3, v_4\}$. Also, $\lambda_2^2(H) = 1$. Thus,

$$\zeta_2^g(H) = 4 < 5 = 8 - \gamma_g(H) + 2 - \lambda_2^2(G).$$

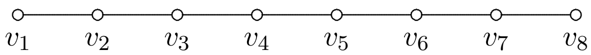

Figure 3: $H = P_8$ and $\gamma_g(H) = 4$

Corollary 1. *For a star graph $K_{1,n}$,*

$$\zeta_k^g(K_{1,n}) = \begin{cases} n & \text{if } k = n-1 \\ k & \text{if } k < n-1. \end{cases}$$

Proof. Let $V(K_{1,n}) = \{v_0, w_1, w_2, \cdots, w_n\}$ where v_0 has degree n. By Theorem 1(ii), $\gamma_g(K_{1,n}) = n$. Thus, $k \leq n-1$. If $k = n-1$, then by Lemma 1, $\zeta_k^g(K_{1,n}) = n+1-1 = n$. Suppose $k < n-1$. Choose an $(n-k)$-element set $S = \{w_1, w_2, \cdots, w_{n-k}\}$. Then S is a ζ_k^g-set of $K_{1,n}$ and, Theorem 7(iii), $\lambda_2^k(K_{1,n}) = |I_{K_{1,n}}^2(S)| = 1$. Hence, $\zeta_k^g(K_{1,n}) = \zeta_k^g(S) = n+1-n+k-1 = k$ by Theorem 8. □

Corollary 2. *For a wheel graph W_n of order n, where $n \geq 5$,*

$$\zeta_k^g(W_n) = \begin{cases} n-1 & \text{if } k = \gamma_g(W_n) - 1 \\ 2k+1 & \text{if } n \text{ is odd} \\ 2k & \text{if } n \text{ is even}. \end{cases}$$

Proof. Let $V(W_n) = \{w_1, w_2, \cdots w_n\}$ where w_1 has degree $n-1$. By Theorem 1(iv), $\gamma_g(W_n) = \lceil \frac{n-1}{2} \rceil$. If $k = \lceil \frac{n-1}{2} \rceil - 1$, then $\zeta_k^g(W_n) = n-1$ by Lemma 1. Let $1 \leq k < \lceil \frac{n-1}{2} \rceil - 1$. If n is odd, then $\gamma_g(W_n) = \frac{n-1}{2}$. Choose the $(\frac{n-1}{2} - k)$-element set $S = \{w_2, w_4, \cdots, w_{n-2k-1}\}$. Then S is a ζ_k^g-set of W_n and $\lambda_2^k(W_n) = \frac{n-2k-1}{2}$. By Theorem 8, $\zeta_k^g(W_n) = \zeta_k^g(S) = n - \frac{n-1}{2} + k - \frac{n-2k-1}{2} = 2k+1$. Hence, $\zeta_k^g(W_n) = 2k+1$. If n is even, then choose the $(\frac{n}{2} - k)$-element set $S = \{w_2, w_4, \cdots, w_{n-2k}\}$. Then S is a ζ_k^g-set of W_n and $\lambda_2^k(W_n) = \frac{n-2k}{2}$. By Theorem 8, $\zeta_k^g(W_n) = \zeta_k^g(S) = n - \frac{n}{2} + k - \frac{n-2k}{2} = 2k$. □

Corollary 3. *For the Petersen graph P,*

$$\zeta_k^g(P) = \begin{cases} 4 & \textit{if} \;\; k = 1 \\ 7 & \textit{if} \;\; k = 2 \\ 9 & \textit{if} \;\; k = 3. \end{cases}$$

Proof. Let $V(P) = \{w_1, w_2, \cdots, w_{10}\}$. By Theorem 1$(vii)$, $\gamma_g(P) = 4$. If $k = 3$, then $\zeta_3^g(P) = 9$ by Lemma 1. Let $1 \leq k \leq 2$. Choose the $(4-k)$-element set $S_k = \{w_1, w_2, w_{4-k}\}$ as shown in Figure 4. Then S_k is a ζ_k^g-set of P. Thus, $\lambda_2^1(P) = |I_P^2(S_1)| = 3$ and $\lambda_2^2(P) = |I_P^2(S_2)| = 1$ by Theorem 7(iii). Therefore, by Theorem 8, $\zeta_1^g(P) = 10 - 4 + k - 3 = 4$

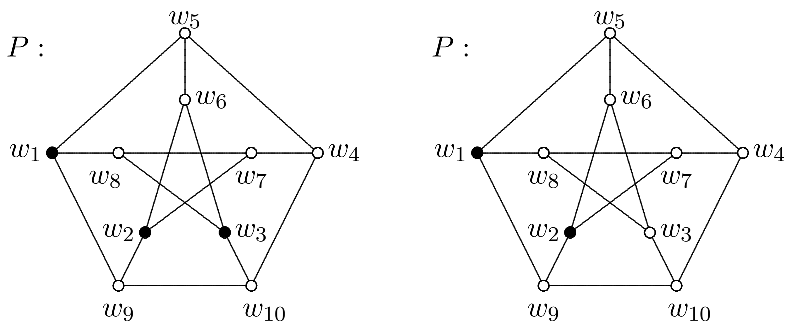

Figure 4: $S_1 = \{w_1, w_2, w_3\}$ and $S_2 = \{w_1, w_2\}$

and $\zeta_2^g(P) = 10 - 4 + 2 - 1 = 7$. This proves the assertion. □

Theorem 9 ([9])**.** *If $G = K_{m_1, m_2, \cdots, m_r}$ is a complete multipartite graph with $2 \leq m_1 \leq m_2 \leq \cdots \leq m_r$, where $r \geq 2$, then*

$$\gamma_g(G) = \begin{cases} 2 & \textit{if} \;\; m_1 = 2 \\ 3 & \textit{if} \;\; m_1 = 3 \\ 4 & \textit{if} \;\; m_1 \geq 4. \end{cases}$$

Theorem 10. *For a complete multipartite graph $G = K_{m_1, m_2, \cdots, m_r}$ where $r \geq 2$ and $2 \leq m_1 \leq m_2 \leq \cdots \leq m_r$, we have*

$$\zeta_k^g(G) = \begin{cases} n-1 & \textit{if } \; k = \gamma_g(G) - 1 \\ 1 & \textit{if } \; m_1 = 3 \; \textit{and} \; \; k = 1 \\ m_1 + k - 4 & \textit{if } \; m_1 \geq 4 \; \textit{and} \; \; 1 \leq k \leq 2. \end{cases}$$

where $n = \sum_{j=1}^{r} m_j$.

Proof. Let $Q_1, Q_2, \cdots, Q_r$ be the partite sets of G with $|Q_j| = m_j$ for each $j \in [r]$. Consider the following cases:

Case 1: $k = \gamma_g(G) - 1$.

By Lemma 1, $\zeta_k^g(G) = n - 1$.

Case 2: $m_1 = 3$ and $k = 1$.

By Theorem 9, $\gamma_g(G) = 3$. Then any 2-element subset S of $V(Q_1)$ is a ζ_k^g-set of G and $\lambda_2^k(G) = \sum_{j=2}^{r} m_j$. By Theorem 8, $\zeta_k^g(G) = \zeta_k^g(S) = \sum_{j=1}^{r} m_j - 3 + 1 - \sum_{j=2}^{r} m_j = 1$.

Case 3: $m_1 \geq 4$ and $1 \leq k \leq 2$.

By Theorem 9, $\gamma_g(G) = 4$. Choose an $(4-k)$-element set $S' = \{\{q_1^1, q_1^2, q_1^{4-k}\}$ where $q_1^t \in Q_1$ and $t \in \{1, 2, 4-k\}$. Then S' is a ζ_k^g-set of G and $\lambda_2^k(G) = \sum_{j=2}^{r} m_j$. By Theorem 8,

$$\zeta_k^g(G) = \zeta_k^g(S) = \sum_{j=1}^{r} m_j - 4 + k - \sum_{j=2}^{r} m_j = m_1 + k - 4.$$

This proves the assertion. □

The next result is a consequence of Theorem 10.

Corollary 4. *For a complete bipartite graph* $K_{m,n}$ *where* $2 \leq m \leq n$,

$$\zeta_k^g(K_{m,n}) = \begin{cases} m+n-1 & \textit{if } \; k = \gamma_g(G) - 1 \\ 1 & \textit{if } \; m = 3 \; \textit{and} \; \; k = 1 \\ m + k - 4 & \textit{if } \; m \geq 4 \; \textit{and} \; \; 1 \leq k \leq 2. \end{cases}$$

Theorem 11. *For a path P_n with $n \geq 2$ vertices,*

$$\zeta_k^g(P_n) = \begin{cases} 3k-2 & \text{if } n = 2 \text{ and } k = 2 \\ & \text{or } n = 3r+2 \text{ and } k \leq r+1 \\ 3k-1 & \text{if } n = 3r \text{ and } k \leq r \\ 3k & \text{if } n = 3r+1 \text{ and } k \leq r. \end{cases}$$

Proof. Let $P_n = [v_1, v_2, \cdots, v_n]$ and let $r \geq 1$. By Theorem 1(vi), $\gamma_g(P_n) = \lceil \frac{n+2}{3} \rceil$. If $n = 2$, then $\gamma_g(P_2) = 2$. Thus, $\zeta_1^g(P_2) = 1$. Let $n \geq 3$. Consider the following cases:

Case 1: $n = 3r$.
By Theorem 1(vi), $\gamma_g(P_n) = \lceil \frac{n+2}{3} \rceil = r+1$. Here, $k \leq r$ and $\gamma_g(P_n) - k = r - k + 1$. Consider an $(r-k+1)$-element set $S_1 = \{v_1, v_4, \cdots, v_{3r-3k+1}\}$. The set S_1 is a ζ_k^g-set of P_n and $N_G^g[S_1] = \{v_1, v_2, v_3, v_4, \cdots, v_{3r-3k+1}\}$. Therefore,

$$\zeta_k^g(P_n) = \zeta_k^g(S_1) = n - |N_G^g[S_1]| = 3r - (3r - 3k + 1) = 3k - 1.$$

Case 2: $n = 3r + 1$.
By Theorem 1(vi), $\gamma_g(P_n) = \lceil \frac{n+2}{3} \rceil = r + 1$. Then $k \leq r$. Choose an $(r-k+1)$-element set $S_2 = \{v_1, v_4, \cdots, v_{3r-3k+1}\}$. Then S_2 is a ζ_k^g-set of P_n and $N_G^g[S_2] = \{v_1, v_2, v_3, v_4, \cdots, v_{3r-3k+1}\}$. Thus,

$$\zeta_k^g(P_n) = \zeta_k^g(S_2) = n - |N_G^g[S_2]| = (3r + 1) - (3r - 3k + 1) = 3k.$$

Case 3: $n = 3r + 2$.
By Theorem 1(vi), $\gamma_g(P_n) = \lceil \frac{n+2}{3} \rceil = r + 2$. Then $k \leq r + 1$ and $\gamma_g(P_n) - k = r - k + 2$. Consider an $(r-k+2)$-element set $S_3 = \{v_1, v_4, \cdots, v_{3r-3k+4}\}$. The set S_3 is a ζ_k^g-set of P_n and $N_G^g[S_3] = \{v_1, v_2, v_3, v_4, \cdots, v_{3r-3k+4}\}$. Therefore,

$$\zeta_k^g(P_n) = \zeta_k^g(S_3) = n - |N_G^g[S_3]| = 3r + 2 - (3r - 3k + 4) = 3k - 2. \quad \square$$

Theorem 12. *If C_n is the cycle with n vertices and $k \leq \gamma_g(C_n) - 1$, then*

$$\zeta_k^g(C_n) = \begin{cases} 3k+2 & \text{if } n = 3r,\ r \geq 2 \text{ and } k = \gamma_g(C_n) - 1 \\ & \text{or } r \text{ is odd},\ r \geq 3, \text{ and } k = \gamma_g(C_n) - 2 \\ 3k+1 & \text{if } n = 3r+2,\ r \geq 2 \text{ and } k = \gamma_g(C_n) - 1 \\ & \text{or } r \text{ is odd},\ r \geq 3, \text{ and } k = \gamma_g(C_n) - 2 \\ 3k & \text{if } n = 3r,\ r \geq 4 \text{ and},\ 1 \leq k \leq \gamma_g(C_n) - 3 \\ & \text{or } n = 3r,\ r \text{ is even},\ r \geq 4, \text{ and } k = \gamma_g(C_n) - 2 \\ & \text{or } n = 3r+1 \text{ and } k = \gamma_g(C_n) - 1 \\ & \text{or } n = 3r+1,\ r \text{ is even}, \text{ and } k = \gamma_g(C_n) - 2 \\ 3k-2 & \text{if } n = 3r+1,\ r \geq 3, \text{ and } 1 \leq k \leq \gamma_g(C_n) - 3 \\ & \text{or } n = 3r+1,\ r \text{ is odd},\ r \geq 3, \text{ and } k = \gamma_g(C_n) - 2 \\ & \text{or } n = 5 \text{ and } k = 2 \\ 3k-1 & \text{if } n = 3r+2,\ r \geq 3 \text{ and } k = 1 \\ & \text{or } n = 3r+2,\ r \text{ is even and } k = \gamma_g(C_n) - 2 \\ & \text{or } n = 3r+2,\ r \geq 4 \text{ and } 2 \leq k \leq \gamma_g(C_n) - 3 \\ & \text{or } n = 5 \text{ and } k = 1 \\ k & \text{if } n = 3. \end{cases}$$

Proof. Let $C_n = [v_1, v_2, \cdots, v_n, v_1]$ and let $k \leq \gamma_g(C_n) - 1$. If $n = 3$, then $\zeta_k^g(C_3) = k$ by Theorem 6. If $n = 5$, then $\zeta_1^g(C_5) = 2 = 3k - 1$ and $\zeta_2^g(C_5) = 4 = 3k - 2$. Let $n = 4$ or $n \geq 6$. Consider the following cases:

Case 1: $n = 3r$ where $r \geq 2$.

By Theorem 1(v), $\gamma_g(C_n) = \lceil \frac{n}{3} \rceil = r$. Thus, $k \leq r-1$. Consider the following subcases:

Subcase 1: $k = \gamma_g(C_n) - 1 = r - 1$.

By Lemma 1, $\zeta_k^g(C_n) = n - 1 = 3r - 1 = 3k + 2$.

Subcase 2: r is odd and $k = \gamma_g(C_n) - 2 = r - 2$.

Let $D_2 = \{v_1, v_4\}$. Then D_2 is a ζ_k^g-set of C_n and $N_G^g[D_2] = \{v_1, v_2, v_3, v_4\}$. Thus,

$$\zeta_k^g(C_n) = \zeta_k^g(D_2) = 3r - |N_G^g[D_2]| = 3r - 4 = 3k + 2.$$

Subcase 3: $r \geq 4$ and $1 \leq k \leq \gamma_g(C_n) - 3$.

Let $k \leq \lfloor \frac{\gamma_g(C_n)-2}{2} \rfloor$ and let $D_3 = \{v_1, v_4, \cdots, v_{3r-3k-2}\}$. Then D_3 is a ζ_k^g-set of

C_n and $N_G^g[D_3] = \{v_1, v_2, v_3, v_4, \cdots, v_{3r-3k-2}, v_{3r-3k-1}, v_{3r}\}$. Thus, $\zeta_k^g(C_n) = \zeta_k^g(D_3) = n - |N_G^g[D_3]| = 3r - [(3r-3k-2)+2] = 3k$. Next, let $\lfloor \frac{\gamma_g(C_n)-2}{2} \rfloor < k \leq \gamma_g(C_n) - 3$. Choose an $(r-k)$-element set $D_4 = \{v_{\lceil \frac{3r+2}{2} \rceil}, v_1, v_4, \cdots, v_{3r-3k-5}\}$. Then D_4 is a ζ_k^g-set of C_n and

$$N_G^g[D_4] = \{v_1, v_2, v_3, v_4, \cdots, v_{3r-3k-5}, v_{3r-3k-4}, v_{\lceil \frac{3r+2}{2} \rceil - 1}, v_{\lceil \frac{3r+2}{2} \rceil}, v_{\lceil \frac{3r+2}{2} \rceil + 1}, v_{3r}\}.$$

This implies that $\zeta_k^g(C_n) = \zeta_k^g(D_4) = n - |N_G^g[D_4]| = 3r - [(3r-3k-4)+4] = 3k$.

Subcase 4: $r \geq 4$ is even and $k = \gamma_g(C_n) - 2 = r - 2$.

Let $D_5 = \{v_1, v_{\frac{3r+2}{2}}\}$. Then D_5 is a ζ_k^g-set of C_n and $N_G^g[D_5] = \{v_1, v_2, v_{\frac{3r}{2}}, v_{\frac{3r+2}{2}}, v_{\frac{3r+4}{2}}, v_{3r}\}$. Hence,

$$\zeta_k^g(C_n) = \zeta_k^g(D_5) = 3r - |N_G^g[D_5]| = 3r - 6 = 3k.$$

Case 2: $n = 3r + 1$.
By Theorem 1(v), $\gamma_g(C_n) = \lceil \frac{n}{3} \rceil = r + 1$. Thus, $k \leq r$. Consider the following subcases:

Subcase 1: $k = \gamma_g(C_n) - 1 = r$.

By Lemma 1, $\zeta_k^g(C_n) = n - 1 = 3r + 1 - 1 = 3r = 3k$.

Subcase 2: r is even and $k = \gamma_g(C_n) - 2 = r - 1$.

Let $D_6 = \{v_1, v_4\}$. Then D_6 is a ζ_k^g-set of C_n and $N_G^g[D_6] = \{v_1, v_2, v_3, v_4\}$. Hence,

$$\zeta_k^g(C_n) = \zeta_k^g(D_6) = 3r + 1 - |N_G^g[D_6]| = 3r + 1 - 4 = 3k.$$

Subcase 3: $r \geq 3$ and $1 \leq k \leq \gamma_g(C_n) - 3$.

Let $k \leq \lfloor \frac{\gamma_g(C_n)-2}{2} \rfloor$ and $D_8 = \{v_1, v_4, \cdots, v_{3r-3k+1}\}$. Then D_8 is a ζ_k^g-set of C_n and

$$N_G^g[D_8] = \{v_1, v_2, v_3, v_4, \cdots, v_{3r-3k+1}, v_{3r-3k+2}, v_{3r+1}\}.$$

This implies that $\zeta_k^g(C_n) = \zeta_k^g(D_8) = n - |N_G^g[D_8]| = 3r + 1 - [(3r-3k+2)+1] = 3k - 2$. Hence, $\zeta_k^g(C_n) = 3k - 2$. Next, let $\lfloor \frac{\gamma_g(C_n)-2}{2} \rfloor < k \leq \gamma_g(C_n) - 3$. Choose an $(r-k+1)$-element set $D_9 = \{v_{\lceil \frac{3r+3}{2} \rceil}, v_1, v_4, \cdots, v_{3r-3k-2}\}$. Then D_9 is a ζ_k^g-set of C_n and

$$N_G^g[D_9] = \{v_1, v_2, v_3, v_4, \cdots, v_{3r-3k-2}, v_{3r-3k-1}, v_{\lceil \frac{3r+3}{2} \rceil - 1}, v_{\lceil \frac{3r+3}{2} \rceil}, v_{\lceil \frac{3r+3}{2} \rceil + 1}, v_{3r+1}\}.$$

This implies that $\zeta_k^g(C_n) = \zeta_k^g(D_9) = n - |N_G^g[D_9]| = 3r + 1 - [(3r-3k-1)+4] = 3k - 2$. Hence, $\zeta_k^g(C_n) = 3k - 2$.

Subcase 4: $r \geq 3$ is odd and $k = \gamma_g(C_n) - 2 = r - 1$.

Let $D_{10} = \{v_1, v_{\frac{3r+3}{2}}\}$. Then D_{10} is a ζ_k^g-set of C_n and $N_G^g[D_{10}] = \{v_1, v_2, v_{\frac{3r+1}{2}}, v_{\frac{3r+3}{2}}, v_{\frac{3r+5}{2}}, v_{3r+1}\}$. Hence,

$$\zeta_k^g(C_n) = \zeta_k^g(D_{10}) = 3r + 1 - |N_G^g[D_{10}]| = 3r + 1 - 6 = 3k - 2.$$

Case 3: $n = 3r + 2$ where $r \geq 2$.

By Theorem 1(v), $\gamma_g(C_n) = \lceil \frac{n}{3} \rceil = r+1$. Thus, $k \leq r$. Consider the following subcases:

Subcase 1: $r \geq 3$ and $k = 1$.

Let $D_{11} = \{v_1, v_4, \cdots, v_{3r-2}\}$ be an (r)-element set. Then $N_G^g[D_{11}] = \{v_1, v_2, \cdots, v_{3r-2}, v_{3r-1}, v_{3r+2}\}$. Thus, D_{11} is a ζ_1^g-set of C_n. Hence,

$$\zeta_1^g(C_n) = \zeta_1^g(D_{11}) = n - |N_G^g[D_{11}]| = 3r + 2 - [(3r - 1) + 1] = 2 = 3k - 1.$$

Subcase 2: $k = \gamma_g(C_n) - 1 = r$.

By Lemma 1, $\zeta_k^g(C_n) = n - 1 = 3r + 2 - 1 = 3r + 1 = 3k + 1$.

Subcase 3: r is odd and $k = \gamma_g(C_n) - 2 = r - 1$.

Let $D_{12} = \{v_1, v_4\}$. Then D_{12} is a ζ_k^g-set of C_n and $N_G^g[D_{12}] = \{v_1, v_2, v_3, v_4\}$. Hence,

$$\zeta_k^g(C_n) = \zeta_k^g(D_{12}) = 3r + 2 - |N_G^g[D_{12}]| = 3r + 2 - 4 = 3k + 1.$$

Subcase 4: r is even and $k = \gamma_g(C_n) - 2 = r - 1$.

Let $D_{14} = \{v_1, v_{\frac{3r+4}{2}}\}$ is a ζ_k^g-set of C_n and $N_G^g[D_{14}] = \{v_1, v_2, v_{\frac{3r+2}{2}}, v_{\frac{3r+4}{2}}, v_{\frac{3r+6}{2}}, v_{3r+2}\}$. Hence,

$$\zeta_k^g(C_n) = \zeta_k^g(D_{14}) = 3r + 2 - |N_G^g[D_{14}]| = 3r + 2 - 6 = 3k - 1.$$

Subcase 5: $r \geq 4$ and $2 \leq k \leq \gamma_g(C_n) - 3$.

Let $k \leq \lfloor \frac{\gamma_g(C_n)-2}{2} \rfloor$ and let $D_{15} = \{v_1, v_4, \cdots, v_{3r-3k+1}\}$. Then D_{15} is a ζ_k^g-set of C_n and $N_G^g[D_{15}] = \{v_1, v_2, v_3, v_4, \cdots, v_{3r-3k+1}, v_{3r-3k+2}, v_{3r+2}\}$. Thus, $\zeta_k^g(C_n) = \zeta_k^g(D_{15}) = n - |N_G^g[D_{15}]| = 3r+2-[(3r-3k+2)+1] = 3k-1$. Next, let $\lfloor \frac{\gamma_g(C_n)-2}{2} \rfloor < k \leq \gamma_g(C_n)-3$. Choose an $(r - k + 1)$-element set $D_{16} = \{v_{\lceil \frac{3r+5}{2} \rceil}, v_1, v_4, \cdots, v_{3r-3k-2}\}$. Then D_{16} is a ζ_k^g-set of C_n and

$$N_G^g[D_{16}] = \{v_1, v_2, v_3, v_4, \cdots, v_{3r-3k-2}, v_{3r-3k-1}, v_{\lceil \frac{3r+5}{2} \rceil - 1}, v_{\lceil \frac{3r+5}{2} \rceil}, v_{\lceil \frac{3r+5}{2} \rceil + 1}, v_{3r+2}\}.$$

This implies that

$$\zeta_k^g(C_n) = \zeta_k^g(D_{16}) = n - |N_G^g[D_{16}]| = 3r + 2 - [(3r - 3k - 1) + 4] = 3k - 1.$$

This shows that the assertion holds. □

4. Conclusion

In this paper, we introduced the concept of k-geodetic domination defect of a graph and computed its values for several well-known graphs. Additionally, we established some sharp bounds of this parameter. It is recommended that further investigation of this newly defined parameter be done especially on graphs resulting from some graph operations.

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