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The Complete List of Solutions to Möbius's Exponential Equation

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Abstract. Möbius addition is a non-associative binary operation defined on the complex open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ by $a \oplus_M b = \dfrac{a+b}{1+\overline{a}b}$, and Möbius's Exponential Equation is a non-linear functional equation of the form $L(a \oplus_M b) = L(a)L(b)$, where L is a complex-valued function defined on $\mathbb{D}$. In [Aequat. Math. **91** (2017), 491–503], the authors address the problem of determining the solutions to Möbius's Exponential Equation. In this paper, we determine the complete list of solutions to Möbius's Exponential Equation using an algebraic approach.

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1. Introduction

Möbius addition arises naturally in the study of hyperbolic geometry, special relativity, and non-Euclidean structures [1, 2]. In particular, Möbius addition captures key aspects of gyrogroup structures, which have applications in complex analysis and differential geometry, for instance, and serves as a primary motivation for the development of gyrogroup theory. Furthermore, in optimization theory, Möbius transformations and Möbius addition on the complex open unit disk provide tools to formulate and solve problems in non-Euclidean domains, model non-associative dynamics, enforce multiplicative structure constraints, and operate within compact curved spaces. These frameworks are particularly useful for modern applications in machine learning, signal processing, and geometric optimization, where classical Euclidean assumptions are insufficient.

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Recall that Möbius addition is a non-associative binary operation defined on the complex open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ by the formula

$$a \oplus_M b = \frac{a+b}{1+\overline{a}b} \tag{1}$$

for all $a, b \in \mathbb{D}$. It turns out that $(\mathbb{D}, \oplus_M)$ forms a non-associative group-like structure that shares several common properties with groups. In particular, $a \oplus_M b$ belongs to $\mathbb{D}$ for all $a, b \in \mathbb{D}$. In [3], the authors introduce a functional equation in connection with Schur's Lemma of the form

$$L(a \oplus_M b) = L(a)L(b) \tag{ME}$$

called *Möbius's Exponential Equation*, which is in some sense an extension of Cauchy's Exponential Equation, where L is a complex-valued function defined on $\mathbb{D}$. They also give partial solutions on the open interval $(-1, 1)$ and then address the problem of determining the complete solutions to Möbius's Exponential Equation on the whole disk (see Problem 1.3 of [3]). Actually, Möbius's Exponential Equation arises when one attempts to determine the irreducible linear representations of $(\mathbb{D}, \oplus_M)$, as shown in Section 3.2 of [3]. In this paper, we give the complete list of solutions to Möbius's Exponential Equation using the recent notion of associators formulated in [4].

2. The solutions to Möbius's Exponential Equation

Suppose that L is a complex-valued function defined on the disk $\mathbb{D}$. As noted on page 492 of [3], if L satisfies Möbius's Exponential Equation, then L is either everywhere or nowhere zero. Furthermore, if L is assumed to be a Borel measurable function, then either L is the zero function or there exists a complex constant λ for which

$$L(r) = e^{\lambda \tanh^{-1} r}$$

for all $r \in (-1, 1)$ (see Theorem 1.2 of [3]). The authors of [3] establish that Möbius's Exponential Equation does not have any non-trivial holomorphic solutions using the Open Mapping Theorem in complex analysis. Here, we will apply an algebraic tool, together with the well-known Intermediate Value Theorem, to prove the following theorem.

Theorem 1. *The solutions to Möbius's Exponential Equation are only the functions* $L : z \mapsto 0$ *and* $L : z \mapsto 1$, $z \in \mathbb{D}$.

Theorem 1 states that Möbius's Exponential Equation does not have any non-trivial solutions, which refines the result in [3] as the condition of being a holomorphic function is removed. In particular, this implies that the disk $\mathbb{D}$, endowed with Möbius addition $\oplus_M$, has no non-trivial representations (see the remark at the end of [3] for more details). The proof of Theorem 1 is presented in the next section for convenience.

3. The complex Möbius gyrogroup and its associators

It is known that $(\mathbb{D}, \oplus_M)$ forms a non-associative group-like structure. This system is called the *(complex) Möbius gyrogroup* due to the fact that the function τ_a, defined as $\tau_a(b) = a \oplus_M b$, $b \in \mathbb{D}$, is a familiar Möbius transformation on $\mathbb{D}$ for all $a \in \mathbb{D}$. As noted in Section 2, if L satisfies Möbius's Exponential Equation and $L(w) \neq 0$ for some $w \in \mathbb{D}$, then $L(z) \neq 0$ for all $z \in \mathbb{D}$. From this point of view, any non-zero solution to Möbius's Exponential Equation is indeed a homomorphism from the Möbius gyrogroup to the multiplicative group of non-zero complex numbers. It is clear that the functions defined by $L(z) = 0$ for all $z \in \mathbb{D}$ and $L(z) = 1$ for all $z \in \mathbb{D}$ are solutions to Möbius's Exponential Equation. In this section, we prove that these functions are the only solutions to Möbius's Exponential Equation using an algebraic approach. In fact, we prove a stronger result: any homomorphism from the Möbius gyrogroup to a group (which is viewed as a degenerate gyrogroup) is necessarily trivial.

For basic definitions and relevant notations, we refer the reader to [2, 4, 5]. As introduced in [4], for all $a, b, c \in \mathbb{D}$, the *associator* of the triple (a, b, c) in $(\mathbb{D}, \oplus_M)$ is denoted by $[a, b, c]$ and is defined by the formula

$$[a, b, c] = \ominus_M(a \oplus_M (b \oplus_M c)) \oplus_M ((a \oplus_M b) \oplus_M c), \tag{2}$$

where $\ominus_M$ is defined as $\ominus_M a = -a$. For convenience, we define the *associator function* of the Möbius gyrogroup, denoted by $\mathcal{A}_M$, by the formula

$$\mathcal{A}_M(a, b, c) = [a, b, c] \tag{3}$$

for all $a, b, c \in \mathbb{D}$. Note that $(a \oplus_M b) \oplus_M c = a \oplus_M (b \oplus_M c)$ if and only if $[a, b, c] = 0$. We present an alternative useful form of formula (3) in the following proposition.

Proposition 1. *The associator function of the Möbius gyrogroup is given by the formula*

$$\mathcal{A}_M(a, b, c) = \frac{c(\overline{a}b - a\overline{b})(1 + a\overline{b} + a\overline{c} + b\overline{c})}{(1 + a\overline{b} - |c|^2(1 + \overline{a}b))(1 + \overline{a}b + \overline{a}c + \overline{b}c)} \tag{4}$$

for all $a, b, c \in \mathbb{D}$.

Proof. The proof of the proposition can be done by algebraic manipulation using formulas (1), (2), and (3).

To see some aspects of the associator function of the Möbius gyrogroup, we plot a few images under $\mathcal{A}_M$ with specification of a and b in Figures 1 and 2. In light of Proposition 1, we obtain a sufficient and necessary condition for three elements in the disk to be associative with respect to Möbius addition, as shown in Corollary 1. First, let us prove the following theorem, which gives a sufficient and necessary condition for an associator to be zero. Here, $\mathbb{C}$ is viewed as a real vector space in the usual way.

Theorem 2. *Let* $a, b, c \in \mathbb{D}$. *Then* $[a, b, c] = 0$ *if and only if* $c = 0$ *or* a *and* b *are linearly dependent in* $\mathbb{C}$.

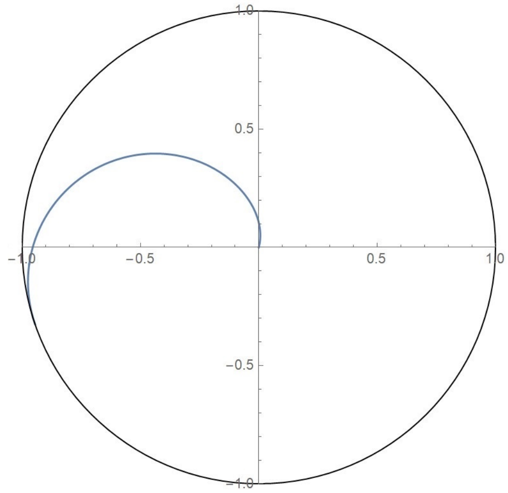

Figure 1: The image of the interval $[0, 1)$ under $\mathcal{A}_M$, where $a = 0.5$ and $b = 0.5i$.

Proof. Suppose that $[a, b, c] = 0$. In view of (4), $c(\overline{a}b - a\overline{b})(1 + a\overline{b} + a\overline{c} + b\overline{c}) = 0$. We claim that $1 + a\overline{b} + a\overline{c} + b\overline{c} \neq 0$. By the closure property of $\oplus_M$, $b \oplus_M c \in \mathbb{D}$. This implies that $1 + \overline{a}(b \oplus_M c) \neq 0$ since otherwise $|a||b \oplus_M c| = 1$, an impossibility. A direct computation shows that

$$1 + \overline{a}(b \oplus_M c) = 1 + \frac{\overline{a}b + \overline{a}c}{1 + \overline{b}c} = \frac{1 + \overline{a}b + \overline{a}c + \overline{b}c}{1 + \overline{b}c},$$

and so $1 + \overline{a}b + \overline{a}c + \overline{b}c \neq 0$. This implies that $1 + a\overline{b} + a\overline{c} + b\overline{c} = \overline{1 + \overline{a}b + \overline{a}c + \overline{b}c} \neq 0$, which proves the claim. Hence, $c = 0$ or $\overline{a}b - a\overline{b} = 0$. In the latter case, we show that a and b are linearly dependent. The case when a and b are both zero is clear. Therefore, we assume that $a \neq 0$ or $b \neq 0$. Now, suppose that $\overline{a}b - a\overline{b} = 0$. Set $a = a_1 + a_2 i$ and $b = b_1 + b_2 i$, where $a_1, a_2, b_1, b_2 \in \mathbb{R}$. By assumption, $a_1 b_2 - a_2 b_1 = 0$. This implies that $b_2 a - a_2 b = 0$. If $b_2 = 0$ and $a_2 = 0$, then $a = a_1$ and $b = b_1$, and furthermore a_1 and b_1 are not simultaneously zero. Hence, $-b_1 a + a_1 b = 0$. This shows that a and b are linearly dependent.

Suppose conversely that $c = 0$ or a and b are linearly dependent. The case when $c = 0$ is clear. Hence, we assume that a and b are linearly dependent. Thus, there are real numbers r and s not simultaneously zero such that $ra + sb = 0$. In the case when $r = 0$, we obtain that $s \neq 0$, and so $sb = 0$ implies $b = 0$, which in turn implies $\overline{a}b - a\overline{b} = 0$. Now, suppose that $r \neq 0$. Set $a = a_1 + a_2 i$ and $b = b_1 + b_2 i$, where $a_1, a_2, b_1, b_2 \in \mathbb{R}$. Then solving the system of equations induced by the equality $ra + sb = 0$ shows that $r(a_1 b_2 - a_2 b_1) = 0$, which implies $a_1 b_2 - a_2 b_1 = 0$. It follows that $\overline{a}b - a\overline{b} = 0$, and so $[a, b, c] = 0$.

Corollary 1. *Let $a, b, c \in \mathbb{D}$. Then $a \oplus_M (b \oplus_M c) = (a \oplus_M b) \oplus_M c$ if and only if $c = 0$ or a and b are linearly dependent in $\mathbb{C}$.*

Proof. The corollary follows from the fact that

$$(a \oplus_M b) \oplus_M c = (a \oplus_M (b \oplus_M c)) \oplus_M [a, b, c]$$

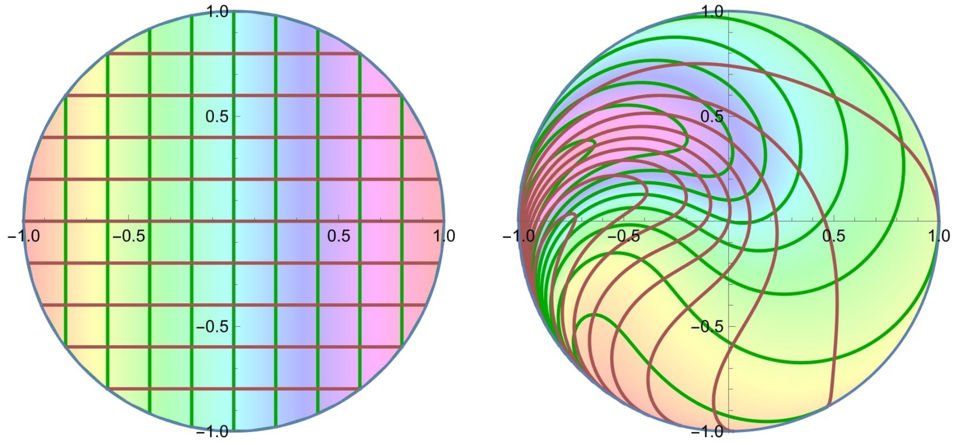

Figure 2: The image of the rectangular grids under $\mathcal{A}_M$, where $a = 0.5$ and $b = 0.5i$.

for all $a, b, c \in \mathbb{D}$.

Next, we prove that any element of the Möbius gyrogroup can be expressed as an associator so that the set of all associators is $\mathbb{D}$ itself. To do so, we need the following lemma, which is important in its own right.

Lemma 1. *If ω is a unimodular complex number (that is, $|\omega| = 1$), then*

$$\mathcal{A}_M(\omega a, \omega b, \omega c) = \omega \mathcal{A}_M(a, b, c)$$

for all $a, b, c \in \mathbb{D}$.

Proof. Suppose that $|\omega| = 1$. Note that if $x, y \in \mathbb{C}$, then

$$\omega x \overline{\omega y} = \omega x \overline{\omega} \overline{y} = |\omega|^2 x \overline{y} = x\overline{y}.$$

Furthermore, $|\omega x| = |x|$ for all $x \in \mathbb{C}$. Hence, the lemma follows directly from making substitutions in formula (4).

We are now in a position to prove that the associator function of the Möbius gyrogroup is surjective. This implies that every element of the Möbius gyrogroup is indeed an associator.

Theorem 3. *The associator function of the Möbius gyrogroup is a surjective function from $\mathbb{D}^3$ onto $\mathbb{D}$.*

Proof. Suppose that $a, b, c \in \mathbb{D}$. First, we show that $1 + a\overline{b} - |c|^2(1 + \overline{a}b) \neq 0$. Note that $1 + \overline{a}b \neq 0$ since otherwise $-1 = \overline{a}b$ implies $|a||b| = 1$, an impossibility. Note also that

$$1 + a\overline{b} - |c|^2(1 + \overline{a}b) = (1 + \overline{a}b)\left(\frac{1 + a\overline{b}}{1 + \overline{a}b} - |c|^2\right).$$

Hence, $1 + a\overline{b} - |c|^2(1 + \overline{a}b) = 0$ would imply $\dfrac{1 + a\overline{b}}{1 + \overline{a}b} - |c|^2 = 0$, which would imply $|c|^2 = \dfrac{1 + a\overline{b}}{1 + \overline{a}b}$, and so $|c| = 1$, a contradiction. Next, $1 + \overline{a}b + \overline{a}c + \overline{b}c \neq 0$, as proved in the proof of Theorem 2.

Fix $a, b \in \mathbb{D}$, and suppose that $a\overline{b} - \overline{a}b \neq 0$. Now, let us define a function $f_{a,b}$ from the close interval $[0, 1]$ to the close disk $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ by the formula

$$f_{a,b}(\lambda) = \frac{\lambda(\overline{a}b - a\overline{b})(1 + a\overline{b} + a\overline{\lambda} + b\overline{\lambda})}{(1 + a\overline{b} - |\lambda|^2(1 + \overline{a}b))(1 + \overline{a}b + \overline{a}\lambda + \overline{b}\lambda)} \tag{5}$$

for all $\lambda \in [0, 1]$. Note that if $\lambda \neq 1$, then $\lambda \in \mathbb{D}$, and so

$$(1 + a\overline{b} - |\lambda|^2(1 + \overline{a}b))(1 + \overline{a}b + \overline{a}\lambda + \overline{b}\lambda) \neq 0,$$

as shown above. In the case when $\lambda = 1$, we obtain that

$$(1 + a\overline{b} - |\lambda|^2(1 + \overline{a}b))(1 + \overline{a}b + \overline{a}\lambda + \overline{b}\lambda) = (a\overline{b} - \overline{a}b)(1 + \overline{a}b + \overline{a} + \overline{b}),$$

which is not zero since

$$1 + \overline{a}b + \overline{a} + \overline{b} = (1 + a\overline{b})\left(\frac{1 + \overline{a}b}{1 + a\overline{b}} + \frac{\overline{a} + \overline{b}}{1 + a\overline{b}}\right) = (1 + a\overline{b})\left(\frac{1 + \overline{a}b}{1 + a\overline{b}} + (\overline{a} \oplus_M \overline{b})\right).$$

This shows that $f_{a,b}$ is well defined. Since $f_{a,b}$ is defined using only addition, multiplication, subtraction, division, and conjugation (which are all continuous functions), it follows that $f_{a,b}$ is continuous.

Fix $a, b \in \mathbb{D}$, and suppose that $a\overline{b} - \overline{a}b \neq 0$. Define a function $g_{a,b}$ by $g_{a,b}(\lambda) = |f_{a,b}(\lambda)|$ for all $\lambda \in [0, 1]$. Since the complex-modulus function is continuous, it follows that $g_{a,b}$ is a continuous function from $[0, 1]$ to $[0, 1]$. Note that $g_{a,b}(0) = 0$ and that $g_{a,b}(1) = 1$ since

$$f_{a,b}(1) = \frac{(\overline{a}b - a\overline{b})(1 + a\overline{b} + a + b)}{(a\overline{b} - \overline{a}b)(1 + \overline{a}b + \overline{a} + \overline{b})} = -\frac{1 + a\overline{b} + a + b}{\overline{1 + a\overline{b} + a + b}}$$

so that $|f_{a,b}(1)| = 1$. By the Intermediate Value Theorem (see, for instance, page 26 of [6]), if $0 < r < 1$, then $g_{a,b}(\lambda_0) = r$ for some $\lambda_0 \in (0, 1)$.

We are now in a position to prove that $\mathcal{A}_M$ is surjective. Let $w \in \mathbb{D}$. In the case when $w = 0$, we obtain that $\mathcal{A}_M(0, 0, 0) = 0$. Now, suppose that $w \neq 0$. Using the polar form, we can write $w = |w|(\cos\alpha + i\sin\alpha)$ for some $\alpha \in \mathbb{R}$. Note that $|w| \neq 0$. Choose $a, b \in \mathbb{D}$ such that $a\overline{b} - \overline{a}b \neq 0$ (for example, $a = 0.5$ and $b = 0.5i$). As above, $g_{a,b}(\lambda_0) = |w|$ for some $\lambda_0 \in (0, 1)$. Note that $|w| = g_{a,b}(\lambda_0) = |f_{a,b}(\lambda_0)|$. Hence, $f_{a,b}(\lambda_0) \neq 0$. Assume that $f_{a,b}(\lambda_0)$ has a polar form as $f_{a,b}(\lambda_0) = r(\cos\beta + i\sin\beta)$, where $0 < r < 1$ and $\beta \in \mathbb{R}$. Thus, $f_{a,b}(\lambda_0) = |w|(\cos\beta + i\sin\beta)$. Set $\omega = \cos(\alpha - \beta) + i\sin(\alpha - \beta)$. Then $|\omega| = 1$. From Lemma 1, it follows that

$$\mathcal{A}_M(\omega a, \omega b, \omega\lambda_0) = \omega\mathcal{A}_M(a, b, \lambda_0)$$

$$\begin{aligned} &= \omega f_{a,b}(\lambda_0) \\ &= (\cos(\alpha-\beta) + i\sin(\alpha-\beta))|w|(\cos\beta + i\sin\beta) \\ &= |w|(\cos\alpha + i\sin\alpha) \\ &= w. \end{aligned}$$

Note that $\omega a, \omega b$, and $\omega\lambda_0$ are in $\mathbb{D}$. This shows that $\mathcal{A}_M$ is surjective, as claimed.

The notion of associators can be used to measure the deviation from associativity of Möbius addition. Following [4], the normal closure of the set of all associators in the Möbius gyrogroup is called the *associator normal subgyrogroup*, denoted by $\mathbb{D}^{\mathsf{a}}$, which is the smallest normal subgyrogroup of $\mathbb{D}$ containing all the associators in $\mathbb{D}$. According to Proposition 3.4 of [4], $\mathbb{D}^{\mathsf{a}}$ is the unique normal subgyrogroup of $\mathbb{D}$ such that $\mathbb{D}/\mathbb{D}^{\mathsf{a}}$ is a group and if φ is a homomorphism from $\mathbb{D}$ to a group, then $\mathbb{D}^{\mathsf{a}}$ lies in the kernel of φ. The quotient $\mathbb{D}/\mathbb{D}^{\mathsf{a}}$ is referred to as the *associativization* of the Möbius gyrogroup. From Theorem 3, we know that every element of the Möbius gyrogroup is an associator. Therefore, we obtain the following corollary immediately.

Corollary 2. *The associator normal subgyrogroup of the Möbius gyrogroup is the Möbius gyrogroup itself.*

We gain a better understanding of the algebraic structure of the Möbius gyrogroup, as stated in the next theorem, which is an application of the previous corollary.

Theorem 4. *There is no non-trivial homomorphism from the Möbius gyrogroup to a group.*

Proof. Suppose that Γ is a group, and suppose that φ is a homomorphism from $\mathbb{D}$ to Γ. By Proposition 3.4 of [4], $\mathbb{D}^{\mathsf{a}} \subseteq \ker\varphi$, and so $\ker\varphi = \mathbb{D}$ because $\mathbb{D}^{\mathsf{a}} = \mathbb{D}$. It follows that φ is trivial, which completes the proof.

As a consequence of Theorem 4, we conclude that a non-trivial homomorphism from the Möbius gyrogroup to the multiplicative group of non-zero complex numbers does not exist. This proves Theorem 1, which fulfills the goal of this paper and completely solves Problem 1.3 of [3]. Moreover, it follows that every representation of the Möbius gyrogroup is trivial. Note that the associativization of the Möbius gyrogroup is trivial, which indicates that the Möbius gyrogroup is, in some sense, most far from being a group. We remark that extending the obtained results to higher dimensions is quite challenging and not straightforward, due to the complexity of the Möbius addition formula. Finally, for the future work, we remark that Möbius addition on the complex open unit disk enables optimization in hyperbolic geometry, where geodesic-based methods replace Euclidean approaches. Its non-associative nature models systems with order-sensitive updates, useful in multi-agent and sequential decision-making. Möbius-based algorithms are suited for learning in non-Euclidean spaces like hyperbolic spaces. As Möbius transformations preserve the disk, they support projection-free optimization in compact domains. Applications may span machine learning, signal processing, and complex domain optimization, where Möbius transformations maintain structural properties vital for constrained and geometric optimization tasks.

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Conflict of interest

The authors declare no conflict of interest.

Data availability

Data sharing not applicable to this paper as no datasets were generated or analyzed during the current study.

Author contributions

R. Maungchang: writing—review and editing, validation, investigation, visualization; W. Atiponrat: writing—review and editing, validation, investigation; T. Suwansri: validation, investigation, visualization; J. Wattanapan: validation, investigation; T. Suksumran: conceptualization, methodology, validation, investigation, writing—original draft preparation, supervision, project administration. All authors have read and agreed to the published version of the manuscript.

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