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Independent Edge Domination Topology of Some Graph Families

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Abstract. Let $G = (V(G), E(G))$ be a nonempty graph. An independent edge dominating set is an independent set of edges of G which is also an edge dominating set of G. The family of independent edge dominating sets of G generates a unique topology. In this paper, we formally define the new notion of this topology generated by the family of independent edge dominating sets (IEDS) in a graph called the *independent edge domination topology* on $E(G)$, herein denoted as $\tau^E_{ID}(G)$. Moreover, we characterize the subbasic sets generated from the family of independent edge dominating sets and the independent edge domination topology of some graph families.

2020 Mathematics Subject Classifications: 05C70, 54A05, 54A25, 05C75

Key Words and Phrases: Independent edge domination topology, Independent edge dominating sets, Complete graph, Friendship graph, Complete bipartite graph

1. Introduction

Topology and graph theory are two practically integrated branches of mathematics that overlap in the structural analysis of spaces and the study of networks and connections. From the point of view of topological ideas connected with various graph models, it could look at network properties like continuity, connectedness, and space-containing networks. This lets researchers in both directions make new tools or programs [1]. There are many ways of constructing a topological space from a given graph − undirected or directed graphs − or devising a method of generating a graph from a given (finite) topological space. Some construct models solely based on the set of vertices, while others base their models on the set of edges. However, the most common approach relies solely on the set of vertices. In the study by Macaso and Balingit [2], they introduced a new topology called the block topology, generated by the family of the vertex sets of the blocks of the graph. In 2018, Abdu and Kilicman introduced new topologies generated by edges. These two types of topologies are called edge-compatible topology and edge-incompatible topology.

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They are generated by the edge-compatible sets of directed graphs and edge-incompatible sets of directed graphs [3]. The same association of edge sets is the study of Alsinaia et al. [4], where they associated a new method of topologizing, in which the topology is generated by the edge neighborhoods of the discrete topological graphs.

In a vertex dominating set of a graph, every vertex in the graph is either part of the subset or adjacent to at least one vertex, ensuring that the set dominates or covers every vertex through an edge. This idea is crucial in network design since the nodes in the dominant set function as control points or access nodes that efficiently govern or interact with the whole network [1]. Meanwhile, an independent set is a collection of vertices in which no two vertices are adjacent, indicating the absence of common edges and direct connections. Independent sets are important because they address issues requiring non-interference, such as the allocation of shared resources [5]. In 2020, Hassan and Abed presented another intriguing way of constructing a topological space from the graph that permits isolated vertices. This topology is called the independent topology and is generated by the family of independent sets of vertices of the graph [6]. On the other hand, an edge-dominating set is a subset of edges that covers all edges in a graph, either by inclusion in the subset or by adjacency, thereby ensuring complete domination of the edges. This concept is especially relevant in situations requiring the management or observation of relationships among nodes with a limited number of connections, such as enhancing network coverage or ensuring efficient oversight of infrastructure systems [7]. The notion of an independent set of edges, referred to as matching, is closely associated. A matching is a set of edges that do not share any common vertices. Matchings are fundamental in solving a wide variety of real-world problems, such as job assignments where tasks need to be assigned to workers, ensuring that no worker is assigned more than one task at a time [5]. Moreover, an independent edge-dominating set exemplifies a noteworthy integration of edge domination and independence characteristics. One set of edges dominates the graph and maintains its independence, meaning no two edges are adjacent. This idea makes sure that the chosen edges cover or dominate the graph efficiently and do not overlap. This notion is useful when the need arises to monitor or control edges without having any extra or direct links between them, such as in power grid optimization or routing protocols in telecommunications networks [8].

2. Preliminaries

Definition 1. *[9] A* **graph** $G = (V(G), E(G))$ *is a finite nonempty set* $V(G)$ *of objects called* **vertices** *(the singular is* **vertex***) together with a possibly empty set* $E(G)$ *of 2-element subsets of* $V(G)$ *called* **edges***. Here,* $V(G)$ *is the vertex set of a graph* G *while* $E(G)$ *is the edge set of the graph* G*. The* **order** *of a graph* G *refers to the number of vertices in* G*, while the* **size** *(or* **length***) of a graph* G *refers to the number of edges in* G*. Two distinct vertices* v_1 *and* v_2 *are* **adjacent** *if* $v_1, v_2 \in G$ *and two edges are* **adjacent** *if they have a common vertex. A graph of size* 0 *is called an* **empty graph***. In any empty graph, no two vertices are adjacent. A* **nonempty graph** *then has one or more edges. A graph* G *is* **connected** *if every two vertices of* G *are adjacent, that is, if* G *contains a*

$u - v$ path for every pair u, v of vertices of G. A graph G that is not connected is called **disconnected**.

Notation: Let $G = (V(G), E(G))$ be a simple graph of order $n \in \mathbb{N}$, where $V(G) = \{v_{\alpha_1}, v_{\alpha_2}, \dots, v_{\alpha_n}\}$ for some indices $\alpha_1, \dots, \alpha_n$. Henceforth, as a convention for $n \geq 1$, we denote $[n] = \{1, 2, \dots, n\}$ and the edge $v_{\alpha_i}v_{\alpha_j} = e_{i,j}$, for $i, j \in [n]$.

Observe that the two edges are adjacent if they have a common vertex. With the above notation, the following remark is immediate.

Remark 1. *Let $G = (V(G), E(G))$ be a simple graph. Two edges e_{i_1,j_1} and e_{i_2,j_2} of G are adjacent if and only if $\{i_1, j_1\} \cap \{i_2, j_2\} \neq \varnothing$.*

Illustration: The graph G in Figure 1 is labeled considering the convention for denoting the vertices and edges. Observably, two edges are adjacent if they share a common subscript.

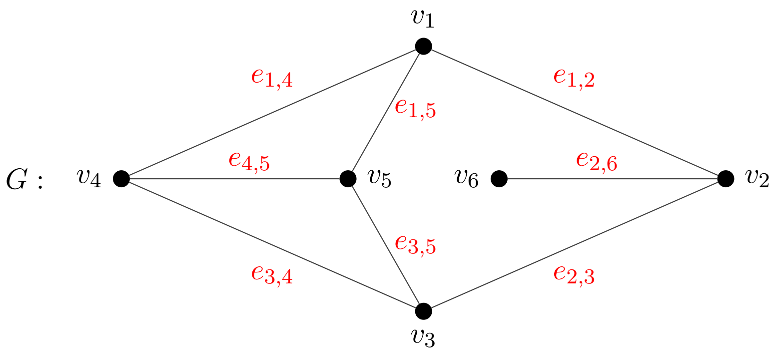

Figure 1: Graph G

Definition 2. *[9] A connected subgraph H of a graph G is a* **component** *of G if H is not a proper subgraph of any connected subgraph of G. The number of components in a graph G is denoted by $k(G)$. Therefore, G is connected if and only if $k(G) = 1$.*

Illustration: The graph H in Figure 2 has a total of 3 components; hence, $k(H) = 3$.

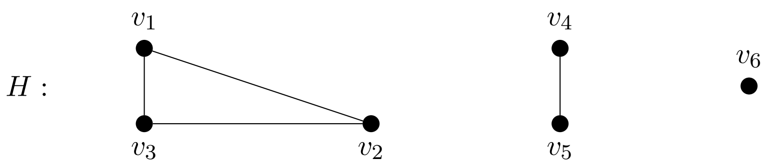

Figure 2: A graph H with $k(H) = 3$

Definition 3. *[9] Two graphs G and H are* **isomorphic**, *denoted by $G \cong H$, if there exists a bijective function $\phi : V(G) \to V(H)$ such that two vertices u and v are adjacent in G if* *and only if $\phi(u)$ and $\phi(v)$ are adjacent in H. The function ϕ is called an* **isomorphism** *from G to H.*

Illustration: The graphs G and H shown in Figure 3 are isomorphic, via the isomorphism $\phi : V(G) \to V(H)$ defined by, $\phi(u_1) = v_2$, $\phi(u_2) = v_3$, $\phi(u_3) = v_1$, $\phi(u_4) = v_4$, $\phi(u_5) = v_5$, $\phi(u_6) = v_6$, and $\phi(u_7) = v_7$.

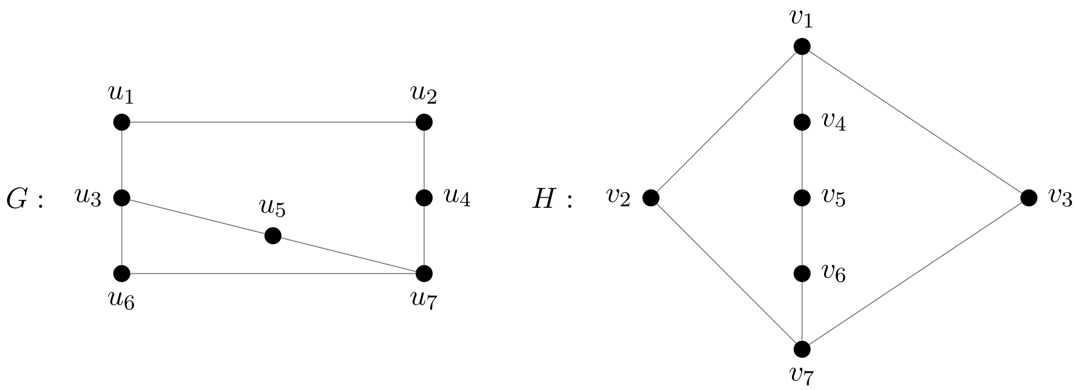

Figure 3: The graph G is isomorphic to graph H

Definition 4. *[10] A set F of edges in a graph G is an* **edge dominating set** *if every edge not in F is adjacent to some edge in F. A set F of edges is an* **independent edge set** *if no two edges in F are adjacent. Consequently, an* **independent edge dominating set (IEDS)** *of G is an independent set of edges which is also an edge dominating set. (The family of all IEDS of G is denoted by ID_G^E).*

Definition 5. *[11] Let X be a set. A* **topology** *on a point set X is a collection τ of subsets of X having the following properties:*

i. $\varnothing$ *and* X *are in* τ.

ii. The union of the elements of any subcollection of τ is in τ; that is, if $\{U_\alpha\}_{\alpha \in A} \subset \tau$ then $\bigcup_{\alpha \in A} U_\alpha \in \tau$.

iii. The intersection of the elements of any finite subcollection of τ is in τ; that is, if $U_1, U_2, \ldots, U_n \in \tau$ then $\bigcap_{i=1}^{n} U_i \in \tau$.

A set X for which a topology τ has been specified is a **topological space***, denoted as the pair (X, τ). A subset of X which is in τ is called a* **τ-open set***. If X is any set and τ_1 is the collection of all subsets of X (that is, τ_1 is the power set of X, $\tau_1 = \mathcal{P}(X)$) then this is a topological space. τ_1 is called the* **discrete topology** *on X. At the other extreme is the topology $\tau_2 = \{\varnothing, X\}$, called the* **indiscrete topology** *or* **trivial topology** *on X.*

Theorem 1. *[11] In a discrete topological space $(X, \mathcal{P}(X))$, every singleton set $\{x\}$ is both open and closed.*

Definition 6. *[12] Given any family $\Sigma = \{A_\alpha : \alpha \in \mathcal{A}\}$ of subsets of X, there always exists a unique, smallest topology $\tau(\Sigma) \supset \Sigma$. The family $\tau(\Sigma)$ can be described as follows: It consists of $\varnothing$, X, all finite intersections of the A_α and all arbitrary unions of these finite intersections. Σ is called a* **subbasis** *for $\tau(\Sigma)$, and $\tau(\Sigma)$ is said to be* **generated by** *Σ.*

Example 1. *Let $X = \{a, b, c, d\}$ and $\Sigma = \{\{a, b\}, \{a, c\}, \{b, d\}, \{c, d\}\}$. By direct application of Definition 6, $\tau(\Sigma) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\} = \mathcal{P}(X)$. This means that $\tau(\Sigma)$ is the discrete topology on X.*

3. Independent Edge Domination Topology

Definition 7. *Let G be a nonempty graph. The* **independent edge domination topology** *of G, denoted by $\tau^E_{ID}(G)$ is the topology generated by the family ID^E_G of all independent edge dominating sets of G. The pair $(E(G), \tau^E_{ID}(G))$ is called the* **independent edge domination topological space** *of G.*

Example 2. *Consider the graph G in Figure 4 with $E(G) = \{e_{1,2}, e_{1,3}, e_{1,4}, e_{2,3}, e_{3,4}\}$. The family of all independent edge dominating sets of G is given by $ID^E_G = \{\{e_{1,3}\}, \{e_{1,2}, e_{3,4}\}, \{e_{1,4}, e_{2,3}\}\}$, and so, by Definition 6 and Definition 7, $\tau^E_{ID}(G) = \{\varnothing, \{e_{1,3}\}, \{e_{1,2}, e_{3,4}\}, \{e_{1,4}, e_{2,3}\}, \{e_{1,2}, e_{1,3}, e_{3,4}\}, \{e_{1,3}, e_{1,4}, e_{2,3}\}, \{e_{1,2}, e_{1,4}, e_{2,3}, e_{3,4}\}, E(G)\}$.*

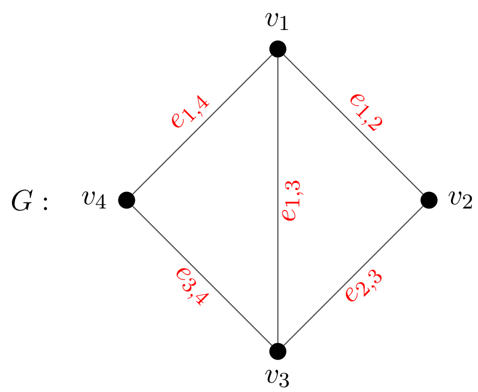

Figure 4: Graph G

Theorem 2. *Let G be a nonempty graph. The topology $\tau^E_{ID}(G)$ is the indiscrete topology on $E(G)$ if and only if G has $k \geq 1$ components, where each component is isomorphic to either P_1 or P_2.*

Proof. Suppose G has a component H that has at least 2 adjacent edges $e_{i_1,j}$ and $e_{i_2,j}$. Then for any IEDS S of G either $e_{i_1,j}, e_{i_2,j} \notin S$ or exactly one of them belongs to S. Both cases imply that there exists $S \in ID^E_G \subseteq \tau^E_{ID}(G)$ such that $S \neq E(G)$. Therefore, $\tau^E_{ID}(G)$ is not the indiscrete topology on $E(G)$.

Conversely, it is easy to see that if G is nonempty and its components are either P_1 or P_2, then $E(G)$ is an IEDS of G, and any proper subset of $E(G)$ is not an edge dominating set of G. Thus, ID_G^E contains only $E(G)$. Consequently, $\tau_{ID}^E(G)$ is the indiscrete topology on $E(G)$. ■

Example 3. *Consider the graph H in Figure 5 with $E(H) = \{e_{1,2}, e_{4,5}\}$. Observe that $ID_H^E = \{\{e_{1,2}, e_{4,5}\}\}$, and so, by Theorem 2, $\tau_{ID}^E(H) = \{\varnothing, E(H)\}$ is the indiscrete topology on $E(H)$.*

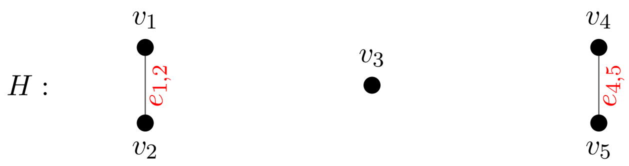

Figure 5: The component of graph H

4. Independent Edge Domination Topology of Complete Graphs

Definition 8. *[9] A graph G is* **complete** *if every two distinct vertices of G are adjacent. A complete graph of order n is denoted by K_n. Therefore, K_n has the maximum possible size of a graph with n vertices. Since every two distinct vertices of K_n are joined by an edge, the number of pairs of vertices in K_n is $\dfrac{n(n-1)}{2}$.*

Notation: Let K_n be a complete graph of order n. We use the following notations:

i. $V(K_n) = \{v_1, v_2, \ldots, v_n\}$; and

ii. $E(K_n) = \{e_{i,j} = v_iv_j : v_i, v_j \in V(K_n)\}$.

Illustration: The complete graph K_5 in Figure 6 is labeled using the abovementioned notation convention.

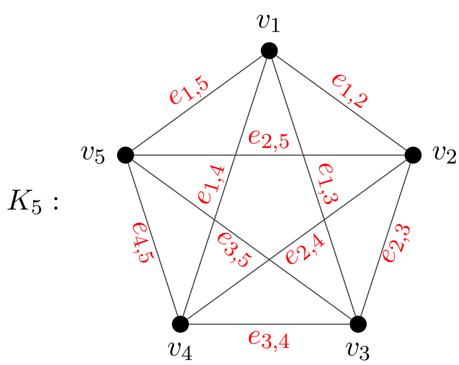

Figure 6: The complete graph K_5

Theorem 3. *Let K_n be a complete graph with $n \geq 2$ and $S \subseteq E(K_n)$. $S \in ID^E_{K_n}$ if and only if $S = \{e_{i_1,j_1}, \ldots, e_{i_k,j_k}\}$ such that $k = |S| = \left\lfloor \frac{n}{2} \right\rfloor$ and $i_1 \neq \ldots \neq i_k \neq j_1 \neq \ldots \neq j_k$.*

Proof. If $S = \{e_{i_1,j_1}, \ldots, e_{i_k,j_k}\}$ is as described, then the condition $i_1 \neq \ldots \neq i_k \neq j_1 \neq \ldots \neq j_k$ implies that S is independent, by Remark 1. Now, let $e_{p,q} \in E(K_n) \setminus S$.

If n is odd, then $k = \frac{n-1}{2}$ and so $|\{i_1, \ldots, i_k, j_1, \ldots, j_k\}| = 2k = n - 1$. This means that, if $p \in \{i_1, \ldots, i_k, j_1, \ldots, j_k\}$, then $e_{p,q}$ is adjacent to one of the edges in S. Also, if $p \notin \{i_1, \ldots, i_k, j_1, \ldots, j_k\}$, then p is the remaining element of $[n]$ not in $\{i_1, \ldots, i_k, j_1, \ldots, j_k\}$, implying that $q \in \{i_1, \ldots, i_k, j_1, \ldots, j_k\}$ since $p \neq q$. Therefore, by Remark 1 $e_{p,q}$ is adjacent to an edge in S.

If n is even, then $k = \frac{n}{2}$ and $|\{i_1, \ldots, i_k, j_1, \ldots, j_k\}| = 2k = n$ this imply that $|\{i_1, \ldots, i_k, j_1, \ldots, j_k\}| = [n]$. Thus, if $p, q \in \{i_1, \ldots, i_k, j_1, \ldots, j_k\}$. By Remark 1, $e_{p,q}$ is adjacent to one of the edges in S. Hence, S is an edge dominating set of K_n. Therefore, $S \in ID^E_{K_n}$.

Conversely, suppose S is not as described. If $|S| > \left\lfloor \frac{n}{2} \right\rfloor$, then $|\{i_1, \ldots, i_k, j_1, \ldots, j_k\}| = 2k \geq n+1$ implying that two of the subscripts in S are equal. By Remark 1, there exists an edge in S that share a common vertex, and so S is not independent. Also, if $|S| < \left\lfloor \frac{n}{2} \right\rfloor$, then $|\{i_1, \ldots, i_k, j_1, \ldots, j_k\}| = 2k \leq n - 2$. This means that there exist $p, q \in [n]$ not appearing in $\{e_{i_1,j_1}, \ldots, e_{i_k,j_k}\}$. So, there exists an edge $e_{p,q}$ in $E(K_n) \setminus S$ which is not adjacent to any of the edges in S. Therefore, S is not an edge dominating set of K_n.

Suppose that S contains elements e_{i_1,j_1} and e_{i_2,j_2} such that $\{i_1, j_1\} \cap \{i_2, j_2\} \neq \varnothing$. Then S is not an independent set. ■

Theorem 4. *For the complete graph K_n with $n \geq 2$,*

$$|ID^E_{K_n}| = \frac{n!}{2^k \cdot k!}, \qquad \text{where } k = \left\lfloor \frac{n}{2} \right\rfloor.$$

Proof. In view of Theorem 3, an IEDS of K_n is formed by choosing the k pairwise disjoint 2-element subsets $\{i_1, j_1\}, \{i_2, j_2\}, \ldots, \{i_k, j_k\}$ of $[n]$ where $k = \left\lfloor \frac{n}{2} \right\rfloor$. There are $\binom{n}{2}$ ways to choose 2 elements from n objects, $\binom{n-2}{2}$ ways to choose 2 elements from $n-2$ objects, ..., and $\binom{n-2k+2}{2}$ ways to select the last 2 elements from the remaining $n - 2k + 2$ objects. In total, given that $n - 2k$ is either 0 or 1,

$$\underbrace{\binom{n}{2}\binom{n-2}{2}\binom{n-2-2}{2}\cdots\binom{n-2k+2}{2}}_{k \text{ factors}} = \frac{n!}{2^k(n-2k)!} = \frac{n!}{2^k}.$$

Since the order of the elements in S does not matter, we divide this quantity by $k!$. Hence,

$$|ID^E_{K_n}| = \frac{n!}{2^k \cdot k!}.$$

■

Example 4. *Consider the complete graph K_5 in Figure 6 with $E(K_5) = \{e_{1,2}, e_{1,3}, e_{1,4}, e_{1,5}, e_{2,3}, e_{2,4}, e_{2,5}, e_{3,4}, e_{3,5}, e_{4,5}\}$. Observe that $ID^E_{K_5} = \{\{e_{1,2}, e_{3,4}\}, \{e_{1,2}, e_{3,5}\}, \{e_{1,2}, e_{4,5}\}, \{e_{1,3}, e_{2,4}\}, \{e_{1,3}, e_{2,5}\}, \{e_{1,3}, e_{4,5}\}, \{e_{1,4}, e_{2,3}\}, \{e_{1,4}, e_{2,5}\}, \{e_{1,4}, e_{3,5}\}, \{e_{1,5}, e_{2,3}\}, \{e_{1,5}, e_{2,4}\}, \{e_{1,5}, e_{3,4}\}, \{e_{2,3}, e_{4,5}\}, \{e_{2,4}, e_{3,5}\}, \{e_{2,5}, e_{3,4}\}\}$. Indeed, with $k = \left\lfloor \frac{5}{2} \right\rfloor = 2$, $|ID^E_{K_4}| = \frac{5!}{2^2 \cdot 2!} = 15$.*

Remark 2. *In view of Theorem 2, $\tau^E_{ID}(K_2)$ is the indiscrete topology on $E(K_2)$. Furthermore, $\tau^E_{ID}(K_3)$ is the discrete topology on $E(K_3)$, given that $ID^E_{K_3} = \{\{e_{1,2}\}, \{e_{1,3}\}, \{e_{2,3}\}\} \subseteq \tau^E_{ID}(K_3)$, by Theorem 1. However, $\tau^E_{ID}(K_4)$ is neither the discrete nor the indiscrete topology on $E(K_4)$. To see this, observe that $E(G) = \{e_{1,2}, e_{1,4}, e_{1,3}, e_{2,3}, e_{2,4}, e_{3,4}\}$, $ID^E_{K_4} = \{\{e_{1,2}, e_{3,4}\}, \{e_{1,4}, e_{2,3}\}, \{e_{1,3}, e_{2,4}\}\}$ and so, $\tau^E_{ID}(K_4) = \{\varnothing, \{e_{1,2}, e_{3,4}\}, \{e_{1,4}, e_{2,3}\}, \{e_{1,3}, e_{2,4}\}, \{e_{1,2}, e_{3,4}, e_{1,4}, e_{2,3}\}, \{e_{1,2}, e_{3,4}, e_{1,3}, e_{2,4}\}, \{e_{1,4}, e_{2,3}, e_{1,3}, e_{2,4}\}, E(K_4)\}$.*

Theorem 5. *Let K_n be a complete graph where $n \geq 5$. Then $\tau^E_{ID}(K_n)$ is the discrete topology on $E(K_n)$.*

Proof. Let $e_{p,q} \in E(K_n)$ where $n > 5$. Then, by Theorem 3, and since $k = \left\lfloor \frac{n}{2} \right\rfloor \geq 3$, $S = \{e_{p,q}, e_{i_1,j_1}, \ldots, e_{i_{k-1},j_{k-1}}\}$ where $k = \left\lfloor \frac{n}{2} \right\rfloor$ and $i_1 \neq i_{k-1} \neq j_1 \neq \ldots \neq j_{k-1} \neq p \neq q$ is an IEDS of K_n. Observe that $S' = \{e_{p,q}, e_{i_1,j_{k-1}}, e_{i_1,j_{k-2}}, \ldots, e_{i_{k-1},j_1}\}$ is an IEDS of K_n such that $S \cap S' = \{e_{p,q}\} \in \tau^E_{ID}(K_n)$.

If $n = 5$, $S = \{e_{p,q}, e_{i_1,j_2}\}$ is an IEDS, for any distinct $i_1, j_1 \in [5] \setminus \{p, q\}$. Putting $r = [5] \setminus \{i_1, j_1, p, q\}$, $S' = \{e_{p,q}, e_{i_1,r}\}$ is an IEDS of K_5 with $S \cap S' = \{e_{p,q}\} \in \tau^E_{ID}(K_5)$, by Theorem 1.

In both cases, $\{e_{p,q}\} \in \tau^E_{ID}(K_n)$ for all $e_{p,q} \in E(K_n)$. Thus, by Theorem 1, $\tau^E_{ID}(K_n)$ is the discrete topology on $E(K_n)$ for all $n \geq 5$. ■

Corollary 1. *For a complete graph K_n of order $n \geq 2$,*

$$|\tau^E_{ID}(K_n)| = \begin{cases} 2 & , \text{ if } n = 2 \\ 8 & , \text{ if } n = 3, 4 \\ 2^{\frac{n(n-1)}{2}} & , \text{ if } n \geq 5 \end{cases}$$

5. Independent Edge Domination Topology of Friendship Graphs

Definition 9. *[13] The* **friendship graph** *of order $n \geq 2$, denoted by Fr_n, is a set of n copies of cycle C_3 having a common vertex v_0.*

Notation: For the friendship graph Fr_n of order $n \geq 2$, we use the following notations:

i. $V(Fr_n) = \{v_0, v_{1_a}, v_{1_b}, v_{2_a}, v_{2_b}, \ldots, v_{n_a}, v_{n_b}\}$, where v_{i_a} is the first vertex of the i^{th} copy of C_3; v_{i_b} is the second vertex of the i^{th} copy of C_3; and v_0 is the common vertex of all copies of C_3

ii. $E(Fr_n) = \{e_{0,1_a}, e_{0,1_b}, e_{1_a,1_b}, e_{0,2_a}, e_{0,2_b}, e_{2_a,2_b}, \ldots, e_{0,n_a}, e_{0,n_b}, e_{n_a,n_b}\}$, where $e_{0,i_a} = v_0 v_{i_a}$, $e_{0,i_b} = v_0 v_{i_b}$, and $e_{i_a,i_b} = v_{i_a} v_{i_b}$.

Illustration: The friendship graph Fr_3 in Figure 7 is labeled using the notation convention.

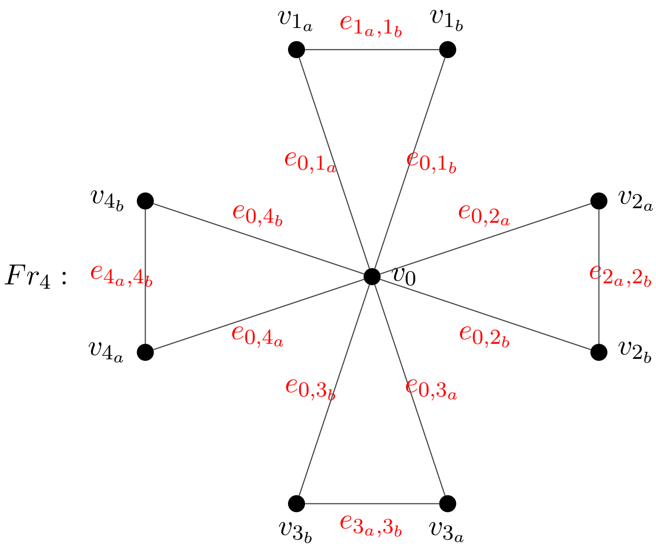

Figure 7: The friendship graph Fr_4

Theorem 6. *Let Fr_n be a friendship graph with $n \geq 2$ and $S \subseteq E(Fr_n)$. $S \in ID^E_{Fr_n}$ if and only if S takes one of the following forms:*

i. $S_1 = \{e_{1_a,1_b}, e_{2_a,2_b}, \ldots, e_{n_a,n_b}\}$

ii. $S^k_a = [\{e_{1_a,1_b}, e_{2_a,2_b}, \ldots, e_{n_a,n_b}\} \setminus \{e_{k_a,k_b}\}] \cup \{e_{0,k_a}\}$ *for some* $k \in [n]$

iii. $S^k_b = [\{e_{1_a,1_b}, e_{2_a,2_b}, \ldots, e_{n_a,n_b}\} \setminus \{e_{k_a,k_b}\}] \cup \{e_{0,k_b}\}$ *for some* $k \in [n]$.

Proof. By Remark 1, $S_1 = \{e_{1_a,1_b}, e_{2_a,2_b}, \ldots, e_{n_a,n_b}\}$ is an independent edge set. Let $e_{p,q} \in E(Fr_n) \setminus S_1$. Then $p = 0$ and q is either k_a or k_b for some $k \in [n]$. Now, observe that, $e_{k_a,k_b} \in S_1$ and is adjacent to $e_{p,q}$, by Remark 1. Since $e_{p,q}$ is arbitrary, S_1 is an edge dominating set of $E(Fr_n)$, and consequently, $S_1 \in ID^E_{Fr_n}$.

In $S^k_a = \{e_{1_a,1_b}, e_{2_a,2_b}, \ldots, e_{n_a,n_b}\} \setminus \{e_{k_a,k_b}\}$, note that by removing e_{k_a,k_b} and replacing it with e_{0,k_a}, which is adjacent only to e_{k_a,k_b}, S^k_a is an independent edge set. Let $e_{p,q} \in E(Fr_n) \setminus S^k_a$. Then either $p = 0$ and $q \in \{r_a, r_b\}$ for some $r \in [n] \setminus \{k\}$, or $e_{p,q} = e_{k_a,k_b}$, and so $e_{p,q}$ is either adjacent to e_{r_a,r_b} or to e_{0,k_a} which are in S^k_a. Hence, S^k_a is an edge dominating set. Therefore, $S^k_a \in ID^E_{Fr_n}$.

Similarly, $S_b^k \in ID_{Fr_n}^E$.

Conversely, let $S \subseteq E(Fr_n)$ such that S is not one of the given forms. If $|S| > n$, then there exists $k \in [n]$ such that two of e_{k_a,k_b}, e_{0,k_a}, or e_{0,k_b} are in S, so that S is not independent. If $|S| < n$, then there exists $k \in [n]$ such that $e_{k_a,k_b}, e_{0,k_a}, e_{0,k_b} \notin S$. This means that e_{k_a,k_b} is not dominated by S, and so S is not an edge dominating set of Fr_n. Now, if $|S| = n$ and there exist distinct $k, k' \in [n]$ such that $e_{0,k_x}, e_{0,k'_y} \in S$ with $x, y \in \{a, b\}$, S is not an independent edge set by Remark 1. ■

Corollary 2. *For the friendship graph Fr_n, $n \geq 2$,*

$$|ID_{Fr_n}^E| = 2n + 1.$$

Example 5. *Consider the friendship graph Fr_4 in Figure 7 with $E(Fr_4) = \{e_{0,1_a}, e_{0,1_b}, e_{1_a,1_b}, e_{0,2_a}, e_{0,2_b}, e_{2_a,2_b}, e_{0,3_a}, e_{0,3_b}, e_{3_a,3_b}, e_{0,4_a}, e_{0,4_b}, e_{4_a,4_b}, \}$. By Theorem 6, observe that $ID_{Fr_4}^E = \{\{e_{1_a,1_b}, e_{2_a,2_b}, e_{3_a,3_b}, e_{4_a,4_b}\}, \{e_{2_a,2_b}, e_{3_a,3_b}, e_{4_a,4_b}, e_{0,1_a}\}, \{e_{1_a,1_b}, e_{3_a,3_b}, e_{4_a,4_b}, e_{0,2_a}\}, \{e_{1_a,1_b}, e_{2_a,2_b}, e_{4_a,4_b}, e_{0,3_a}\}, \{e_{1_a,1_b}, e_{2_a,2_b}, e_{3_a,3_b}, e_{0,4_a}\}, \{e_{2_a,2_b}, e_{3_a,3_b}, e_{4_a,4_b}, e_{0,1_b}\}, \{e_{1_a,1_b}, e_{3_a,3_b}, e_{4_a,4_b}, e_{0,2_b}\}, \{e_{1_a,1_b}, e_{2_a,2_b}, e_{4_a,4_b}, e_{0,3_b}\}, \{e_{1_a,1_b}, e_{2_a,2_b}, e_{3_a,3_b}, e_{0,4_b}\}\}$. Indeed, $|ID_{Fr_4}^E| = 2(4) + 1 = 9$.*

Theorem 7. *Let Fr_n be a friendship graph and $S_1 = \{e_{1_a,1_b}, e_{2_a,2_b}, \ldots, e_{n_a,n_b}\}$. A set $S \subseteq E(Fr_n)$ is $\tau_{ID}^E(Fr_n)$-open if and only if S satisfies any of the following forms:*

i. $S \subseteq S_1$;

ii. $S = S_1 \cup \{e_{0,i_a} : i \in A\} \cup \{e_{0,i_b} : i \in B\}$ *such that* $A, B \in [n]$; *and*

iii. $S = [S_1 \setminus \{e_{k_a,k_b}\}] \cup S^*$ *such that* $S^* \subseteq \{e_{0,k_a}, e_{0,k_b}\}$ *for all* $k \in [n]$.

Proof. Let $S \subseteq E(Fr_n)$. If $S \subseteq S_1$, then for some $A \subseteq [n]$, $S = \{e_{i_a,i_b} : i \in A\} = \bigcap_{i \notin A} S_a^k \in \tau_{ID}^E(Fr_n)$.

Let $A, B \subseteq [n]$. Then, $S = S_1 \cup \{e_{0,i_a} : i \in A\} \cup \{e_{0,i_b} : i \in B\} = S_1 \cup [\bigcup_{i \in A} S_a^i] \cup [\bigcup_{i \in B} S_b^i] \in \tau_{ID}^E(Fr_n)$.

If $k \in [n]$ and $S = [S_1 \setminus \{e_{k_a,k_b}\}] \cup S^*$ where $S^* = \varnothing$, then $S \subseteq S_1$. If S^* is either $\{e_{0,k_a}\}$ or $\{e_{0,k_b}\}$, then S is also either S_a^k or S_b^k. If $S^* = \{e_{0,k_a}, e_{0,k_b}\}$, then $S = S_a^k \cup S_b^k$. In all cases, S is $\tau_{ID}^E(Fr_n)$-open, by Definition 6.

Conversely, suppose $S \subseteq E(Fr_n)$ does not take any of the given forms. Then there exist $k, k', k^* \in [n]$ such that $e_{k_a,k_b}, e_{k'_a,k'_b} \notin S$ and either $e_{0,k^*{}_a} \in S$ or $e_{0,k^*{}_b} \in S$. If S is a $\tau_{ID}^E(Fr_n)$-open set, S can be generated out of the sets in $ID_{Fr_n}^E$ given in the Theorem 6. Necessarily, either $S_a^{k^*} \subseteq S \subseteq S_x^k \cap S_y^{k'}$ or $S_b^{k^*} \subseteq S \subseteq S_x^k \cap S_y^{k'}$ where $x, y \in \{a, b\}$, which are both impossible. Thus, S cannot be $\tau_{ID}^E(Fr_n)$-open. ■

Remark 3. *Let Fr_n be a friendship graph and $S \subseteq E(Fr_n)$ with $n \geq 2$. Then $\tau_{ID}^E(Fr_n)$ is not the indiscrete nor the discrete topology.*

Corollary 3. *For the friendship graph Fr_n with $n \geq 2$,*

$$|\tau_{ID}^E(Fr_n)| = 2^{2n} + 2^n + 3n - 1.$$

Proof. The proof immediately follows by counting the numbers of $\tau_{ID}^{E}(Fr_n)$-open sets in Theorem 7. ■

6. Independent Edge Domination Topology of Complete Bipartite Graphs

Definition 10. *[14] A* **complete bipartite** *is a graph whose vertex set can be partitioned into two disjoint nonempty sets V_a and V_b such that two vertices v_{i_a} and v_{i_b} are adjacent if and only if $v_{i_a} \in V_a$ and $v_{i_b} \in V_b$. If $|V_a| = m$ and $|V_b| = n$, then such a graph is denoted $K_{m,n}$.*

Notation: For the complete bipartite graph $K_{m,n}$, the two partite sets are labeled as

i. $V_a(K_{m,n}) = \{v_{1_a}, v_{2_a}, \ldots, v_{m_a}\}$; and

ii. $V_b(K_{m,n}) = \{v_{1_b}, v_{2_b}, \ldots, v_{n_b}\}$,

so that $E(K_{m,n}) = \{e_{i,j} = v_{i_a}v_{j_b} : v_{i_a} \in V_a(K_{m,n}), v_{j_b} \in V_b(K_{m,n})\}$.

Illustration: The complete bipartite graph $K_{2,3}$ in Figure 8 is labeled using the notation convention.

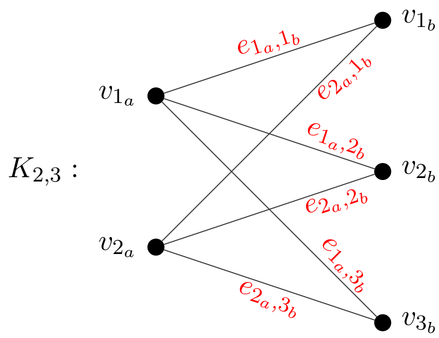

Figure 8: The complete bipartite graph $K_{2,3}$

Theorem 8. *Let $K_{m,n}$ be a complete bipartite graph and $S \subseteq E(K_{m,n})$. $S \in ID^{E}_{K_{m,n}}$ if and only if $S = \{e_{i_1,j_1}, \ldots, e_{i_k,j_k}\}$ such that $|S| = k = \min\{m,n\}$ and $i_1 \neq \ldots \neq i_k, j_1 \neq \ldots \neq j_k$ where $i_1, \ldots, i_k \in \{1_a, \ldots, m_a\}$ and $j_1, \ldots, j_k \in \{1_b, \ldots, n_b\}$.*

Proof. Independence follows immediately by Remark 1 and by the definition of S. Let $e_{p,q} \in E(K_{m,n}) \setminus S$, and assume that $k = \min\{m,n\} = m$. Since $|S| = k$ and S is independent, all the subscripts $1_a, \ldots, m_a$ appear in S. Hence, $p \in \{1_a, \ldots, m_a\}$. This means that there exist $c \in [m]$ and $d \in [n]$ such that $e_{c_a,d_b} \in S$ and $c_a = p$. By Remark 1, e_{c_a,d_b} is adjacent to $e_{p,q}$. Since $e_{p,q}$ is arbitrary, S is an edge dominating set of $K_{m,n}$.

Conversely, assume WLOG that $k = \min\{m, n\} = m$. If $|S| > k = m$, then by the Pigeonhole Principle applied to the subscript p of the edges $e_{p,q}$ in S, there exist at least two edges in S sharing the same first subscripts. By Remark 1, S is not independent.

If $|S| < k = m < n$, then there exists an edge $e_{p,q} \in E(K_{m,n}) \setminus S$ such that p and q do not appear as subscripts of the edges in S. Hence, $e_{p,q}$ is not adjacent to any element of S, by Remark 1. Therefore, S is not an edge dominating set of $K_{m,n}$.

If $|S| = k$ but fails to satisfy the condition $i_1 \neq \ldots \neq i_k, j_1 \neq \ldots \neq j_k$, then, by Remark 1, S is not an independent nor an edge dominating set of $K_{m,n}$. ■

Theorem 9. *For the complete bipartite graph $K_{m,n}$ with $m \leq n$*

$$|ID^E_{K_{m,n}}| = \frac{n!}{(n-m)!}.$$

Proof. Assume that $m \leq n$. Observe that by Theorem 8, all subscripts $1_a, 2_a, \ldots, m_a$ appear in every independent edge dominating set S of $K_{m,n}$. Each of these subscripts is paired to one of the subscripts $1_b, \ldots, n_b$, where the chosen subscripts are all distinct. There are n choices for the subscript paired to 1_a, $n-1$ choices for the subscript paired to 2_a, …, and $n-m+1$ choices for the subscript paired to m_a. This means that there are $n(n-1)(n-2)\ldots(n-m+1) = \dfrac{n!}{(n-m)!}$ such pairs and so $|ID^E_{K_{m,n}}| = \dfrac{n!}{(n-m)!}$. ■

Example 6. *Consider the complete bipartite graph $K_{2,3}$ in Figure 8 with $E(K_{2,3}) = \{e_{1_a,1_b}, e_{1_a,2_b}, e_{1_a,3_b}, e_{2_a,1_b}, e_{2_a,2_b}, e_{2_a,3_b}\}$. Observe that $ID^E_{K_{2,3}} = \{\{e_{1_a,1_b}, e_{2_a,2_b}\}, \{e_{1_a,1_b}, e_{2_a,3_b}\}$, $\{e_{1_a,2_b}, e_{2_a,1_b}\}, \{e_{1_a,2_b}, e_{2_a,3_b}\}, \{e_{1_a,3_b}, e_{2_a,1_a}\}, \{e_{1_a,3_b}, e_{2_a,2_b}\}\}$. Indeed, $|ID^E_{K_{m,n}}| = 6$.*

Remark 4.

i. *$\tau^E_{ID}(K_{1,1})$ is the indiscrete topology on $E(K_{1,1})$, by Theorem 2.*

ii. *$\tau^E_{ID}(K_{2,2})$ is neither the discrete nor the indiscrete topology on $E(K_{2,2})$. To see this, observe that $ID^E_{K_{2,2}} = \{\{e_{1_a,1_b}, e_{2_a,2_b}\}, \{e_{1_a,2_b}, e_{2_a,1_b}\}\}$ and so $\tau^E_{ID}(K_{2,2}) = \{\varnothing$, $\{e_{1_a,1_b}, e_{2_a,2_b}\}, \{e_{1_a,2_b}, e_{2_a,1_b}\}, E(K_{2,2})\}$.*

Theorem 10. *For the complete bipartite graph $K_{1,n}$ (star graph) with $n \geq 2$, $\tau^E_{ID}(K_{1,n})$ is the discrete topology on $E(K_{1,n})$.*

Proof. Let $S \subseteq E(K_{1,n})$ where $n \geq 2$. By Theorem 8, $|S| = \min\{1, n\} = 1$, implying that every singleton subsets of $E(K_{1,n})$ is an independent edge dominating set. Consequently, every subset of $E(K_{m,n})$ is $\tau^E_{ID}(K_{m,n})$-open, thus, by Theorem 1, $\tau^E_{ID}(K_{m,n})$ is the discrete topology. ■

Theorem 11. *Let $K_{m,n}$ be a complete bipartite graph with $m \leq n$. If $m \geq 2$ and $n \geq 3$, then $\tau^E_{ID}(K_{m,n})$ is the discrete topology on $E(K_{m,n})$.*

Proof. Consider the complete bipartite graph $K_{m,n}$ such that $m \leq n$, $m \geq 2$ and $n \geq 3$. Let $e_{p,q} \in E(K_{m,n})$ and $S_1 = \{e_{p,q}, e_{i_1,j_1}, e_{i_2,j_2}, \ldots, e_{i_{m-1},j_{m-1}}\}$ such that $p, i_1, \ldots, i_{m-1} \in \{1_a, 2_a, \ldots, m_a\}$, $q, j_1, \ldots, j_{m-1} \in \{1_b, 2_b, \ldots, n_b\}$ where $p \neq i_1 \neq i_2 \neq \cdots \neq i_{m-1}$ and $q \neq j_1 \neq j_2 \neq \cdots \neq j_{m-1}$. By the preceding theorem, $S_1 \in ID^E_{K_{m,n}}$. Now, if $|S_1| = m = 2$, let $S_2 = \{e_{p,q}, e_{i_1,j_2}\}$ where $j_2 \neq q \neq j_1$; otherwise, let $S_2 = \{e_{p,q}, e_{i_1,j_{m-1}}, e_{i_2,j_{m-2}}, \ldots, e_{i_{m-1},j_1}\}$. Then $S_2 \in ID^E_{K_{m,n}}$ such that $S_1 \cap S_2 = \{e_{p,q}\} \in \tau^E_{ID}(K_{m,n})$. By Theorem 1, $\tau^E_{ID}(K_{m,n})$ is the discrete topology on $E(K_{m,n})$. ■

Corollary 4. *For complete bipartite graph $K_{m,n}$ with $m \leq n$,*

$$|\tau^E_{ID}(K_{m,n})| = \begin{cases} 2 & \text{, if } m = 1, n = 1 \\ 4 & \text{, if } m = 2, n = 2 \\ 2^n & \text{, if } m = 1, n \geq 2 \\ 2^{mn} & \text{, if } m \geq 2, n \geq 3 \end{cases}.$$

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