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Exponential Fuzzy Sets and Applications of AI-Powered Investment Decision-Making Using the Weighted Mean Method

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Abstract. The exponential fuzzy set ($\mathcal{EFS}$) is a new modification that allows for a more flexible representation of uncertainty by using an exponential function to define membership degree. In this work, we define basic operations on $\mathcal{EFS}$, such as complement, union, intersection, simple difference, and limited difference functions. The equivalency formula, symmetrical difference formula, disjoint sets, disjoint sum, and disjunctive sum are further important qualities that we examine. We analyse essential laws in the exponential fuzzy framework, such as the idempotent law of union. In addition, we present a few theorems that govern the relational and algebraic structures of $\mathcal{EFS}$. $\mathcal{EFS}$ has been compared against traditional approaches and the resulting studies showcase its advantages in modeling uncertainty, artificial intelligence, and decision making. This paper studies the use of exponential fuzzy sets in AI driven investment decision processes using the weighted mean method of multifactor investment analysis.

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Key Words and Phrases: Exponential Fuzzy Set, Operations of Exponential Fuzzy Set, Decision-Making, Weighted Mean Method

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1. Introduction

Zadeh's [1] fuzzy sets have proved to be beneficial in a number of fields of mathematical modeling and decision making. This concept has resulted in the creation of such sets as the intuitionistic fuzzy sets [2], neutrosophic fuzzy sets [3], and even pythagorean fuzzy sets (Yager, 2013). All of which were created to solve some form of ambiguity. One of these extensions is the exponential fuzzy set, which enables the simulation of

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systems where there is a significant level of uncertainty and provides robust mathematical support. Exponential fuzzy sets have great potential for use in optimization, control systems, as well as decision analysis. The theoretical basis for exponential fuzzy sets is the need to define uncertainty in dynamic environments with extreme rates of change. For example, the effectiveness of exponential membership functions was proved by Wu et al. [4], who employed them in a fuzzy control approach to examine the stability of nonlinear parabolic systems. Furthermore, the predicted value of exponential fuzzy numbers and their use in inventory models for degrading objects were investigated by Garai et al. [5]. These works reflect the practical use of fuzzy sets with exponential-type membership functions for information systems that are rapidly evolving and have a strong time constraint. One of the crucial advantages of exponential fuzzy sets is that unlike traditional fuzzy models, they can represent abrupt changes in uncertainty more effective.

The quantitative evaluation of differences of information in uncertain situations has been enhanced by an extreme divergence measure of Tomar and Ohlan [6]. Also, Bustince et al. [7] provide an account of the history of all types of fuzzy sets and give justification for the new novel exponential fuzzy sets from a research perspective. In the mathematical formulation of exponential fuzzy sets, the degree of membership is implemented in the form of exponent, which is the basic parameter of softening the set. The use of exponential functions in fuzzy set theories makes it possible to more accurately model a wide range of real life problems such as intelligent decision making systems, environmental, economic and sociological forecasts (Hadi-Vencheh & Mirjaberi [8]). In addition Liang & Xu [9], the growing use of fuzzy logic in practice is backed by new advances in multiple-attribute decision-making methods, e.g., hesitant Pythagorean fuzzy sets and exponential fuzzy TOPSIS. In [10] and [11] discussed group decision making problems. The aim of this work is to provide an extensive overview of exponential fuzzy sets, such as their theory and numerous applications. We survey the existing material exponential membership functions enhance fuzzy modeling techniques. Finally, we emphasize the advantages of exponential fuzzy sets in dealing with dynamic and exponentially changing uncertainty and compare their efficiencies with other fuzzy extensions. From the above literature we found the research gap and exponential fuzzy concept we introduced.

This paper is organized as: The basic definitions in section 2, The basic definitions and characteristics of exponential fuzzy sets are covered in Section 3, Main results of exponential fuzzy sets are shown in Section 4, application in section 5 and future research possibilities are discussed in Section 6.

1.1. Motivation

- Traditional $\mathfrak{F}$-sets provide a foundation for handling uncertainty, but they have limitations in capturing rapid variations in membership values.
- $\mathcal{EFS}$s increase the flexibility of membership functions by introducing a non-linear transformation.
- $\mathcal{EFS}$ increases decision-making sensitivity, especially in situations where little

changes in input have a big influence on results.

- By adding exponential functions, it expands on traditional fuzzy logic and is appropriate for more complex uses like pattern identification and risk assessment.
- EFS better captures uncertainty and hesitancy in expert opinions, leading to improved diagnostic accuracy.
- EFS improves edge detection and noise reduction in image analysis, leading to clearer and more accurate image segmentation.

1.2. Need of $\mathcal{EFS}$

Exponential fuzzy sets address this issue by incorporating an exponential function into the membership structure. This allows for a more flexible and precise representation of uncertainty, especially in situations where small changes in input values lead to significant variations in membership degrees. For example, in medical diagnosis, financial risk assessment, and engineering problems, uncertainty often behaves in a nonlinear manner, making exponential fuzzy sets a better choice.

1.3. Advantages of $\mathcal{EFS}$

- Suitable for situations where uncertainty follows an exponential pattern rather than a linear one.
- Ensures gradual changes in membership values, preventing abrupt shifts.
- More responsive to small variations in data, improving accuracy in decision-making.
- Enhances accuracy and robustness in multi-criteria decision-making (MCDM) and expert systems.

1.4. Novelty

- Describes a new membership function transformation for an $\mathcal{EFS}$.
- Incorporates exponential scaling to improve the representation of uncertainty in conventional fuzzy set theory.
- Enhances similarity and divergence metrics to help make better decisions.

2. Preliminaries

Definition 1. *[1] A fuzzy set $\mathcal{A}$ in a universe of discourse $\mathfrak{Z}$ is characterized by a membership function $\aleph_{\mathcal{A}}$ which takes the value in the unit interval* $[0,1]$,

$$\aleph_{\mathcal{A}}(\mathfrak{s}) = \mathfrak{Z} \to [0,1].$$

The value of $\aleph_{\mathcal{A}}(\mathfrak{s})$ represents the grade of membership of $\mathfrak{Z}$ in $\mathring{\mathcal{A}}$ and is a point in $[0,1]$.

Definition 2. *[1] The complement $\mathcal{A}'$ is defined by $\aleph_{\mathcal{A}'}(\mathfrak{s}) = 1 - \aleph_{\mathcal{A}}(\mathfrak{s})$. Where $\mathcal{A}$ is a $\mathfrak{F}$-set in $\mathfrak{Z}$.*

Definition 3. *[12] If $\aleph_{\mathcal{A}}$ and $\aleph_{\mathcal{B}}$ are membership functions of $\mathfrak{F}$-sets $\mathcal{A}$ and $\mathcal{B}$, then the union and intersection of two $\mathfrak{F}$-sets is*

$$\aleph_{\mathcal{A}\cup\mathcal{B}}(\mathfrak{s}) = max\left\{\aleph_{\mathcal{A}}(\mathfrak{s}), \aleph_{\mathcal{B}}(\mathfrak{s})\right\}, \forall \mathfrak{s} \in \mathfrak{Z}.$$

$$\aleph_{\mathcal{A}\cap\mathcal{B}}(\mathfrak{s}) = min\left\{\aleph_{\mathcal{A}}(\mathfrak{s}), \aleph_{\mathcal{B}}(\mathfrak{s})\right\}, \forall \mathfrak{s} \in \mathfrak{Z}.$$

Definition 4. *[12] The difference of two $\mathfrak{F}$-sets $\mathcal{A}$ and $\mathcal{B}$ is given by $\mathcal{A} - \mathcal{B} = \mathcal{A} \cap \mathcal{B}^c$.*

Definition 5. *[12] A bounded difference of two $\mathfrak{F}$-sets $\mathcal{A}$ and $\mathcal{B}$ is given by*

$$\aleph_{AoB}(\mathfrak{s}) = max\left[o, \aleph_{\mathcal{A}}(\mathfrak{s}), \aleph_{\mathcal{B}}(\mathfrak{s})\right]$$

Definition 6. *[12] The disjoint sum of two $\mathfrak{F}$-sets is given by*

$$\aleph_{\mathcal{A}\otimes\mathcal{B}}(\mathfrak{s}) = |\otimes, \aleph_{\mathcal{A}}(\mathfrak{s}), \aleph_{\mathcal{B}}(\mathfrak{s})|$$

where $\aleph_{\mathcal{A}}(\mathfrak{s})$ and $\aleph_{\mathcal{B}}(\mathfrak{s})$ are membership function of $\mathcal{A}$ and $\mathcal{B}$.

Definition 7. *Let $\mathcal{A}$ and $\mathcal{B}$ be any two $\mathfrak{F}$-sets of $\mathfrak{Z}$ then the disjunctive sum is given by:*

$$\aleph_{\mathcal{A}\Delta\mathcal{B}}(\mathfrak{s}) = (\mathcal{A} \cap \mathcal{B}^c) \cup (\mathcal{A}^c \cap \mathcal{B}) = (A * \mathcal{B}^c) \oplus (\mathcal{A}^c * \mathcal{B}).$$

Definition 8. *Let $\mathcal{A}$ and $\mathcal{B}$ be any two $\mathfrak{F}$-sets of $\mathfrak{Z}$ then the equivalence formula is*

$$(\mathcal{A}^c \cup \mathcal{B}) \cap (\mathcal{A} \cup \mathcal{B}^c) = (\mathcal{A}^c \cap \mathcal{B}^c) \cup (\mathcal{A} \cap \mathcal{B}).$$

Definition 9. *Symmetrical difference formula for two fuzzy sets $\mathcal{A}$ and $\mathcal{B}$ is given by*

$$(\mathcal{A}^c \cap \mathcal{B}) \cup (\mathcal{A} \cap \mathcal{B}^c) = (\mathcal{A}^c \cup \mathcal{B}^c) \cap (\mathcal{A} \cup \mathcal{B}).$$

3. Exponential Fuzzy Sets

Definition 10. *If $\mathfrak{Z}$ is a universe discourse and $\mathfrak{s}$ be any particular element of $\mathfrak{Z}$. The $\mathcal{EFS}$ $\mathcal{E}_{\mathcal{A}}$ defined on $\mathfrak{Z}$ is a collection of ordered pairs, $\mathcal{E}_{\mathcal{A}} = \left\{\left(\mathfrak{s}, \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})}\right) | \mathfrak{s} \in \mathfrak{Z}, \mathsf{T} > 0\right\}$, where $\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} : \mathfrak{Z} \to [0,1]$ is called the membership function. The degree of membership function $0 \leq \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} \leq 1$.*

Example 1. *Let $\mathfrak{Z} = \{1, 2, 3, 4, 5\}$ be the universal set and fuzzy membership values of $\mathfrak{Z}$ is $\aleph_A(\mathfrak{s}) = \{0.9, 0.7, 0.5, 0.4, 0.3\}$, the decay parameter $\mathsf{T} = 0.02$. The exponential fuzzy membership function is given by: $\mathcal{E}_{\mathcal{A}}(\mathfrak{s}) = \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})}$. The exponential fuzzy membership values*
$\mathcal{E}_{\mathcal{A}}(\mathfrak{s}) = \{(1, 0.8839), (2, 0.6903), (3, 0.4950), (4, 0.3968), (5, 0.2982)\}$.

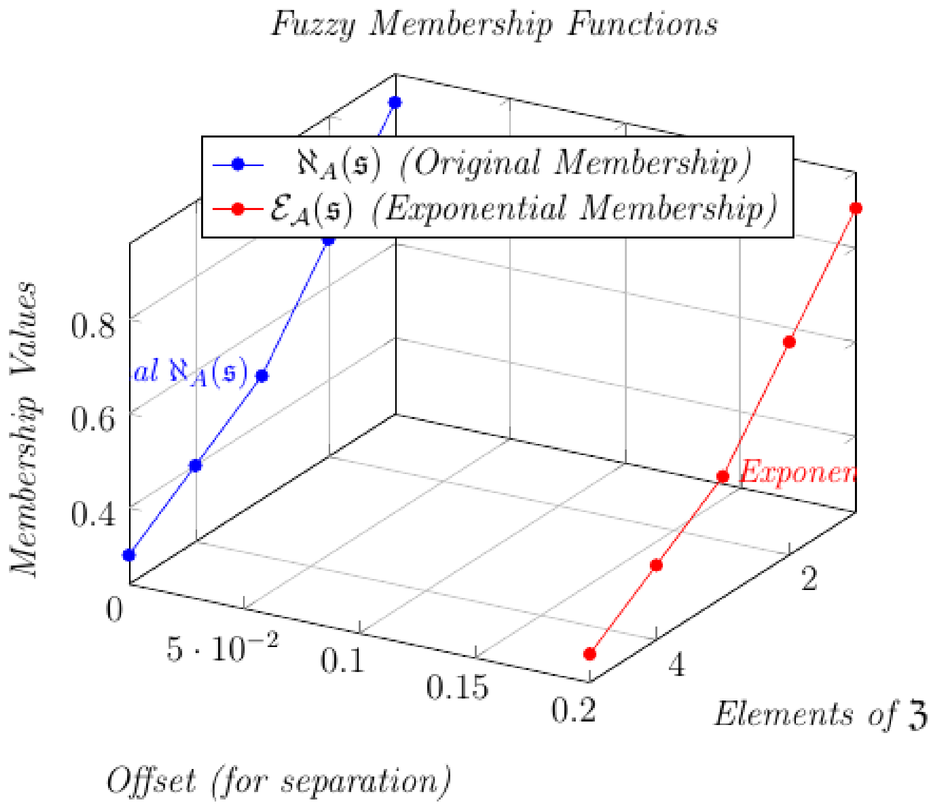

Figure 1: Exponential Fuzzy Set

Definition 11. *Let* $\mathcal{E}_{\mathcal{A}}$ *and* $\mathcal{E}_{\mathcal{B}}$ *be two* $\mathcal{EFS}s$ *on* $\mathfrak{Z}$ *with their grade values given by:*

$$\aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) = \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})}$$

$$\aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) = \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}$$

The exponential fuzzy intersection of $\mathcal{E}_{\mathcal{A}}$ *and* $\mathcal{E}_{\mathcal{B}}$ *is defined as:*

$$\begin{aligned}\aleph_{\mathcal{E}_{\mathcal{A}}\cap\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) &= min\left\{\aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}), \aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s})\right\}\\ &= min\left\{\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})}, \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right\} \qquad (3.1)\end{aligned}$$

Similarly, the exponential fuzzy union of $\mathcal{E}_{\mathcal{A}}$ *and* $\mathcal{E}_{\mathcal{B}}$ *is defined as:*

$$\begin{aligned}\aleph_{\mathcal{E}_{\mathcal{A}}\cup\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) &= max\left\{\aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}), \aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s})\right\}\\ &= max\left\{\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})}, \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right\} \qquad (3.2)\end{aligned}$$

Example 2. *Let* $\mathfrak{Z} = \{\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4, \mathfrak{s}_5\}$ *be a finite universe, and the membership functions of two exponential* $\mathfrak{F}$*-sets* $\mathcal{E}_{\mathcal{A}}$ *and* $\mathcal{E}_{\mathcal{B}}$ *are*

$$\aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) = \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})}$$

$$\aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) = \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph_{\mathcal{B}}(\mathfrak{s})}$$

Now, we compute the corresponding $\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}$ and $\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}$ membership values:

$\mathfrak{s}$	$\aleph_A(\mathfrak{s})$	$\mathcal{E}_{\mathcal{A}}(\mathfrak{s})$	$\aleph_B(\mathfrak{s})$	$\mathcal{E}_{\mathcal{B}}(\mathfrak{s})$	$\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}$	$\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}$
$\mathfrak{s}_1$	0.8	0.7873	0.7	0.6903	0.6903	0.7873
$\mathfrak{s}_2$	0.6	0.5928	0.5	0.4950	0.4950	0.5928
$\mathfrak{s}_3$	0.4	0.3968	0.3	0.2982	0.2982	0.3968
$\mathfrak{s}_4$	0.2	0.1992	0.2	0.1992	0.1992	0.1992
$\mathfrak{s}_5$	0.1	0.0998	0.1	0.0998	0.0998	0.0998

Definition 12. *Consider two $\mathcal{EFS}$s $\mathcal{E}_{\mathcal{A}}, \mathcal{E}_{\mathcal{B}}$, and $\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph_{\mathcal{A}}(\mathfrak{s})}, \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph_{\mathcal{B}}(\mathfrak{s})}$ denotes the membership functions $\mathcal{E}_{\mathcal{A}}$ and $\mathcal{E}_{\mathcal{B}}$. The simple difference $\mathcal{E}_{\mathcal{A}} - \mathcal{E}_{\mathcal{B}}$ of these two $\mathcal{EFS}$s $\mathcal{E}_{\mathcal{A}}$ and $\mathcal{E}_{\mathcal{B}}$ is given by*

$$\begin{aligned}\mathcal{E}_{\mathcal{A}} - \mathcal{E}_{\mathcal{B}} &= \mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}{}^{c} \\ &= \aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) * \aleph^{c}_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \\ &= \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph_{\mathcal{A}}(\mathfrak{s})} * \aleph_{\mathcal{B}}(\mathfrak{s})^{c}\mathfrak{e}^{-\intercal \aleph^{c}_{\mathcal{B}}(\mathfrak{s})}\end{aligned}$$

Example 3. *Let $\mathcal{E}_{\mathcal{A}} = \left\{\frac{0.8\mathfrak{e}^{-\intercal 0.8}}{\mathfrak{s}_1} + \frac{0.5\mathfrak{e}^{-\intercal 0.5}}{\mathfrak{s}_2} + \frac{0.6\mathfrak{e}^{-\intercal 0.6}}{\mathfrak{s}_3}\right\}$ and $\mathcal{E}_{\mathcal{B}} = \left\{\frac{0.4\mathfrak{e}^{-\intercal 0.4}}{\mathfrak{s}_1} + \frac{0.3\mathfrak{e}^{-\intercal 0.3}}{\mathfrak{s}_2} + \frac{0.3\mathfrak{e}^{-\intercal 0.3}}{\mathfrak{s}_3}\right\}$ be two two $\mathcal{EFS}$s. The simple difference is*

$$\begin{aligned}\mathcal{E}_{\mathcal{A}} - \mathcal{E}_{\mathcal{B}} &= \mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}{}^{c} \\ &= \left\{\frac{0.8\mathfrak{e}^{-\intercal 0.8}}{\mathfrak{s}_1} + \frac{0.5\mathfrak{e}^{-\intercal 0.5}}{\mathfrak{s}_2} + \frac{0.6\mathfrak{e}^{-\intercal 0.6}}{\mathfrak{s}_3}\right\} * \left\{\frac{0.6\mathfrak{e}^{-\intercal 0.6}}{\mathfrak{s}_1} + \frac{0.7\mathfrak{e}^{-\intercal 0.7}}{\mathfrak{s}_2} + \frac{0.7\mathfrak{e}^{-\intercal 0.7}}{\mathfrak{s}_3}\right\} \\ &= \left\{\frac{0.6\mathfrak{e}^{-\intercal 0.6}}{\mathfrak{s}_1} + \frac{0.5\mathfrak{e}^{-\intercal 0.5}}{\mathfrak{s}_2} + \frac{0.6\mathfrak{e}^{-\intercal 0.6}}{\mathfrak{s}_3}\right\}.\end{aligned}$$

Definition 13. *Let $\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph_{\mathcal{A}}(\mathfrak{s})}$ and $\aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph_{\mathcal{B}}(\mathfrak{s})}$ be the membership functions of $\mathcal{EFS}$ $\mathcal{E}_{\mathcal{A}}$ and $\mathcal{E}_{\mathcal{B}}$. The bounded difference is*

$$\mathcal{E}_{\mathcal{A}} \circ \mathcal{E}_{\mathcal{B}}(\mathfrak{s}) = max\left[\circ, \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph_{\mathcal{A}}(\mathfrak{s})}, \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph_{\mathcal{B}}(\mathfrak{s})}\right].$$

Example 4. *Let $\mathcal{E}_{\mathcal{A}} = \left(\frac{0.9\mathfrak{e}^{-\intercal 0.9}}{\mathfrak{s}_1} + \frac{0.8\mathfrak{e}^{-\intercal 0.8}}{\mathfrak{s}_2} + \frac{0.6\mathfrak{e}^{-\intercal 0.6}}{\mathfrak{s}_3}\right)$ and $\mathcal{E}_{\mathcal{B}} = \left(\frac{0.1\mathfrak{e}^{-\intercal 0.1}}{\mathfrak{s}_1} + \frac{0.4\mathfrak{e}^{-\intercal 0.4}}{\mathfrak{s}_2} + \frac{0.5\mathfrak{e}^{-\intercal 0.5}}{\mathfrak{s}_3}\right)$ be two two $\mathcal{EFS}$s. The bounded difference of these two $\mathcal{EFS}$s is:*

$$\mathcal{E}_{\mathcal{A}} \circ \mathcal{E}_{\mathcal{B}}(\mathfrak{s}) = \left(\frac{0.8\mathfrak{e}^{-\intercal 0.8}}{\mathfrak{s}_1} + \frac{0.4\mathfrak{e}^{-\intercal 0.4}}{\mathfrak{s}_2} + \frac{0.1\mathfrak{e}^{-\intercal 0.1}}{\mathfrak{s}_3}\right)$$

Definition 14. *A disjoint sum of $\mathcal{E}_{\mathcal{A}}$ and $\mathcal{E}_{\mathcal{B}}$ is as follows*

$$\mathcal{E}_{\mathcal{A}} \otimes \mathcal{E}_{\mathcal{B}}(\mathfrak{s}) = \left|\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph_{\mathcal{A}}(\mathfrak{s})} - \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph_{\mathcal{B}}(\mathfrak{s})}\right|$$

Example 5. *Let* $\mathcal{E}_{\mathcal{A}} = \left(\frac{0.2\mathfrak{e}^{-\mathsf{T}0.2}}{\mathfrak{s}_1} + \frac{0.3\mathfrak{e}^{-\mathsf{T}0.3}}{\mathfrak{s}_2} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_3}\right)$ *and* $\mathcal{E}_{\mathcal{B}} = \left(\frac{0.15\mathfrak{e}^{-\mathsf{T}0.15}}{\mathfrak{s}_1} + \frac{0.35\mathfrak{e}^{-\mathsf{T}0.35}}{\mathfrak{s}_2} + \frac{0.65\mathfrak{e}^{-\mathsf{T}0.65}}{\mathfrak{s}_3}\right)$ *be two two* $\mathcal{EFS}$*s. Using the max function for calculating the phase term, the disjoint sum of theses two two* $\mathcal{EFS}$*s is:*

$$\mathcal{E}_{\mathcal{A}} \otimes \mathcal{E}_{\mathcal{B}}(\mathfrak{s}) = \left(\frac{0.05\mathfrak{e}^{-\mathsf{T}0.05}}{\mathfrak{s}_1} + \frac{0.05\mathfrak{e}^{-\mathsf{T}0.05}}{\mathfrak{s}_2} + \frac{0.05\mathfrak{e}^{-\mathsf{T}0.05}}{\mathfrak{s}_3}\right)$$

Definition 15. *Let* $\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})}$ *and* $\aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}$ *denotes the membership functions* $\mathcal{E}_{\mathcal{A}}$ *and* $\mathcal{E}_{\mathcal{B}}$*. The disjunctive sum is*

$$\mathcal{E}_{\mathcal{A}}\Delta\mathcal{E}_{\mathcal{B}}(\mathfrak{s}) = (\mathcal{E}_{\mathcal{A}} \cap {\mathcal{E}_{\mathcal{B}}}^c) \cup ({\mathcal{E}_{\mathcal{A}}}^c \cap \mathcal{E}_{\mathcal{B}}).$$

The membership function of $\mathcal{E}_{\mathcal{A}}\Delta\mathcal{E}_{\mathcal{B}}(\mathfrak{s})$ *is:*

$$\begin{aligned}\aleph_{\mathcal{E}_{\mathcal{A}}\Delta\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) &= [\aleph_{\mathcal{E}_{\mathcal{A}}\cap{\mathcal{E}_{\mathcal{B}}}^c}(\mathfrak{s}) \oplus \aleph_{{\mathcal{E}_{\mathcal{A}}}^c\cap\mathcal{E}_{\mathcal{B}}}(\mathfrak{s})]\\ &= \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} * \aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] \oplus \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{A}}(\mathfrak{s})} * \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right].\end{aligned}$$

Example 6. *Suppose* $\mathcal{E}_{\mathcal{A}} = \left(\frac{0.6\mathfrak{e}^{-\mathsf{T}0.6}}{\mathfrak{s}_1} + \frac{0.7\mathfrak{e}^{-\mathsf{T}0.7}}{\mathfrak{s}_2} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_3}\right)$ *and* $\mathcal{E}_{\mathcal{B}} = \left(\frac{0.3\mathfrak{e}^{-\mathsf{T}0.3}}{\mathfrak{s}_1} + \frac{0.4\mathfrak{e}^{-\mathsf{T}0.4}}{\mathfrak{s}_2} + \frac{0.7\mathfrak{e}^{-\mathsf{T}0.7}}{\mathfrak{s}_3}\right)$. *Then the disjunctive sum of these two* $\mathcal{EFS}$*s is define as*

$$\begin{aligned}\aleph_{\mathcal{E}_{\mathcal{A}}\Delta\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) &= [\aleph_{\mathcal{E}_{\mathcal{A}}\cap{\mathcal{E}_{\mathcal{B}}}^c}(\mathfrak{s}) \oplus \aleph_{{\mathcal{E}_{\mathcal{A}}}^c\cap\mathcal{E}_{\mathcal{B}}}(\mathfrak{s})]\\ &= \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} * \aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] \oplus \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{A}}(\mathfrak{s})} * \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right].\\ \aleph_{\mathcal{E}_{\mathcal{A}}\Delta\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) &= \left(\frac{0.6\mathfrak{e}^{-\mathsf{T}0.6}}{\mathfrak{s}_1} + \frac{0.6\mathfrak{e}^{-\mathsf{T}0.6}}{\mathfrak{s}_2} + \frac{0.3\mathfrak{e}^{-\mathsf{T}0.3}}{\mathfrak{s}_3}\right) \oplus \left(\frac{0.3\mathfrak{e}^{-\mathsf{T}0.3}}{\mathfrak{s}_1} + \frac{0.3\mathfrak{e}^{-\mathsf{T}0.3}}{\mathfrak{s}_2} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_3}\right).\\ \aleph_{\mathcal{E}_{\mathcal{A}}\Delta\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) &= \left(\frac{0.6\mathfrak{e}^{-\mathsf{T}0.6}}{\mathfrak{s}_1} + \frac{0.6\mathfrak{e}^{-\mathsf{T}0.6}}{\mathfrak{s}_2} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_3}\right).\end{aligned}$$

Definition 16. *For any two* $\mathcal{EFS}$*s* $\mathcal{E}_{\mathcal{A}}$ *and* $\mathcal{E}_{\mathcal{B}}$*, the equivalence formula is;* $({\mathcal{E}_{\mathcal{A}}}^c \cup \mathcal{E}_{\mathcal{B}}) \cap (\mathcal{E}_{\mathcal{A}} \cup {\mathcal{E}_{\mathcal{B}}}^c) = ({\mathcal{E}_{\mathcal{A}}}^c \cap {\mathcal{E}_{\mathcal{B}}}^c) \cup (\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}})$*. The membership function of two* $\mathcal{EFS}$*s* $\mathcal{E}_{\mathcal{A}}$ *and* $\mathcal{E}_{\mathcal{B}}$ *are given below*

$$[\aleph_{{\mathcal{E}_{\mathcal{A}}}^c\cup\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \cap \aleph_{\mathcal{E}_{\mathcal{A}}\cup{\mathcal{E}_{\mathcal{B}}}^c}(\mathfrak{s})] = \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] * \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right]$$

$$[\aleph_{{\mathcal{E}_{\mathcal{A}}}^c\cap{\mathcal{E}_{\mathcal{B}}}^c}(\mathfrak{s}) \cup \aleph_{\mathcal{E}_{\mathcal{A}}\cap\mathcal{E}_{\mathcal{B}}}(\mathfrak{s})] = \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{A}}(\mathfrak{s})} * \aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] \oplus \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} * \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right]$$

Example 7. *Suppose* $\mathcal{E}_{\mathcal{A}} = \left(\frac{0.6\mathfrak{e}^{-\mathsf{T}0.6}}{\mathfrak{s}_1} + \frac{0.8\mathfrak{e}^{-\mathsf{T}0.8}}{\mathfrak{s}_2} + \frac{0.7\mathfrak{e}^{-\mathsf{T}0.7}}{\mathfrak{s}_3}\right)$ *and* $\mathcal{E}_{\mathcal{B}} = \left(\frac{0.8\mathfrak{e}^{-\mathsf{T}0.8}}{\mathfrak{s}_1} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_2} + \frac{0.4\mathfrak{e}^{-\mathsf{T}0.4}}{\mathfrak{s}_3}\right)$. *The equivalence formula is;* $({\mathcal{E}_{\mathcal{A}}}^c \cup \mathcal{E}_{\mathcal{B}}) \cap (\mathcal{E}_{\mathcal{A}} \cup {\mathcal{E}_{\mathcal{B}}}^c) = ({\mathcal{E}_{\mathcal{A}}}^c \cap {\mathcal{E}_{\mathcal{B}}}^c) \cup (\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}})$.

$$\begin{aligned}({\mathcal{E}_{\mathcal{A}}}^c \cup \mathcal{E}_{\mathcal{B}}) \cap (\mathcal{E}_{\mathcal{A}} \cup {\mathcal{E}_{\mathcal{B}}}^c) &= \left(\frac{0.8\mathfrak{e}^{-\mathsf{T}0.8}}{\mathfrak{s}_1} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_2} + \frac{0.4\mathfrak{e}^{-\mathsf{T}0.4}}{\mathfrak{s}_3}\right) * \left(\frac{0.6\mathfrak{e}^{-\mathsf{T}0.6}}{\mathfrak{s}_1} + \frac{0.8\mathfrak{e}^{-\mathsf{T}0.8}}{\mathfrak{s}_2} + \frac{0.7\mathfrak{e}^{-\mathsf{T}0.7}}{\mathfrak{s}_3}\right)\\ &= \left(\frac{0.6\mathfrak{e}^{-\mathsf{T}0.6}}{\mathfrak{s}_1} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_2} + \frac{0.4\mathfrak{e}^{-\mathsf{T}0.4}}{\mathfrak{s}_3}\right) \quad (3.3)\end{aligned}$$

$$(\mathcal{E_A}^c \cap \mathcal{E_B}^c) \cup (\mathcal{E_A} \cap \mathcal{E_B}) = \left(\frac{0.2\mathfrak{e}^{-\mathsf{T}0.2}}{\mathfrak{s}_1} + \frac{0.2\mathfrak{e}^{-\mathsf{T}0.2}}{\mathfrak{s}_2} + \frac{0.3\mathfrak{e}^{-\mathsf{T}0.3}}{\mathfrak{s}_3}\right) \oplus \left(\frac{0.6\mathfrak{e}^{-\mathsf{T}0.6}}{\mathfrak{s}_1} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_2} + \frac{0.4\mathfrak{e}^{-\mathsf{T}0.4}}{\mathfrak{s}_3}\right)$$

$$= \left(\frac{0.6\mathfrak{e}^{-\mathsf{T}0.6}}{\mathfrak{s}_1} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_2} + \frac{0.4\mathfrak{e}^{-\mathsf{T}0.4}}{\mathfrak{s}_3}\right) \quad (3.4)$$

From equation 3.3 and 3.4, we have

$$(\mathcal{E_A}^c \cup \mathcal{E_B}) \cap (\mathcal{E_A} \cup \mathcal{E_B}^c) = (\mathcal{E_A}^c \cap \mathcal{E_B}^c) \cup (\mathcal{E_A} \cap \mathcal{E_B}).$$

Definition 17. *The symmetrical difference formula for two $\mathcal{EFS}s$ $\mathcal{E_A}$ and $\mathcal{E_B}$ are denoted by*

$$(\mathcal{E_A}^c \cap \mathcal{E_B}) \cup (\mathcal{E_A} \cap \mathcal{E_B}^c) = (\mathcal{E_A}^c \cup \mathcal{E_B}^c) \cap (\mathcal{E_A} \cap \mathcal{E_B}).$$

The symmetrical difference formula two $\mathcal{EFS}s$ $\mathcal{E_A}$ and $\mathcal{E_B}$ are given below

$$[\aleph_{\mathcal{E_A}^c \cap \mathcal{E_B}}(\mathfrak{s}) \cup \aleph_{\mathcal{E_A} \cap \mathcal{E_B}^c}(\mathfrak{s})] = \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{A}}(\mathfrak{s})} * \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] \oplus \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} * \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right]$$

$$[\aleph_{\mathcal{E_A}^c \cup \mathcal{E_B}^c}(\mathfrak{s}) \cap \aleph_{\mathcal{E_A} \cup \mathcal{E_B}}(\mathfrak{s})] = \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] * \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right]$$

Example 8. *Suppose* $\mathcal{E_A} = \left(\frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_1} + \frac{0.7\mathfrak{e}^{-\mathsf{T}0.7}}{\mathfrak{s}_2} + \frac{0.8\mathfrak{e}^{-\mathsf{T}0.8}}{\mathfrak{s}_3}\right)$ *and* $\mathcal{E_B} = \left(\frac{0.3\mathfrak{e}^{-\mathsf{T}0.3}}{\mathfrak{s}_1} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_2} + \frac{0.2\mathfrak{e}^{-\mathsf{T}0.2}}{\mathfrak{s}_3}\right)$. *The symmetrical difference formula is*

$$(\mathcal{E_A}^c \cap \mathcal{E_B}) \cup (\mathcal{E_A} \cap \mathcal{E_B}^c) = (\mathcal{E_A}^c \cup \mathcal{E_B}^c) \cap (\mathcal{E_A} \cap \mathcal{E_B}).$$

$$(\mathcal{E_A}^c \cap \mathcal{E_B}) \cup (\mathcal{E_A} \cap \mathcal{E_B}^c) = \left(\frac{0.3\mathfrak{e}^{-\mathsf{T}0.3}}{\mathfrak{s}_1} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_2} + \frac{0.2\mathfrak{e}^{-\mathsf{T}0.2}}{\mathfrak{s}_3}\right) \oplus \left(\frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_1} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_2} + \frac{0.8\mathfrak{e}^{-\mathsf{T}0.8}}{\mathfrak{s}_3}\right)$$

$$= \left(\frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_1} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_2} + \frac{0.8\mathfrak{e}^{-\mathsf{T}0.8}}{\mathfrak{s}_3}\right) \quad (3.5)$$

$$(\mathcal{E_A}^c \cup \mathcal{E_B}^c) \cap (\mathcal{E_A} \cap \mathcal{E_B}) = \left(\frac{0.7\mathfrak{e}^{-\mathsf{T}0.7}}{\mathfrak{s}_1} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_2} + \frac{0.8\mathfrak{e}^{-\mathsf{T}0.8}}{\mathfrak{s}_3}\right) \oplus \left(\frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_1} + \frac{0.7\mathfrak{e}^{-\mathsf{T}0.7}}{\mathfrak{s}_2} + \frac{0.8\mathfrak{e}^{-\mathsf{T}0.8}}{\mathfrak{s}_3}\right)$$

$$= \left(\frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_1} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_2} + \frac{0.8\mathfrak{e}^{-\mathsf{T}0.8}}{\mathfrak{s}_3}\right) \quad (3.6)$$

From equation 3.5 and 3.6, we have

$$(\mathcal{E_A}^c \cap \mathcal{E_B}) \cup (\mathcal{E_A} \cap \mathcal{E_B}^c) = (\mathcal{E_A}^c \cup \mathcal{E_B}^c) \cap (\mathcal{E_A} \cap \mathcal{E_B}).$$

Definition 18. *Let $\mathcal{E_A}$ and $\mathcal{E_B}$ and $\mathcal{EFS}s$ be three $\mathcal{EFS}s$ then, the distributive law are*

$$\mathcal{E_A} \cup (\mathcal{E_B} \cap \mathcal{E_C}) = (\mathcal{E_A} \cup \mathcal{E_B}) \cap (\mathcal{E_A} \cup \mathcal{E_C})$$
$$\mathcal{E_A} \cap (\mathcal{E_B} \cup \mathcal{E_C}) = (\mathcal{E_A} \cap \mathcal{E_B}) \cup (\mathcal{E_A} \cap \mathcal{E_C}).$$

Theses two law are said to be distributive law of union over intersection and intersection over union. If $\mathcal{E_A} = \aleph_{\mathcal{E_A}}(\mathfrak{s}) = \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})}, \mathcal{E_B} = \aleph_{\mathcal{E_B}}(\mathfrak{s}) = \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}$ and $\mathcal{E_C} = \aleph_{\mathcal{E_C}}(\mathfrak{s}) = \aleph_{C}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{C}(\mathfrak{s})}$, the distributive law of union intersection become:

$$[\aleph_{\mathcal{E_A}}(\mathfrak{s}) \oplus (\aleph_{\mathcal{E_B}}(\mathfrak{s}) * \aleph_{\mathcal{E_C}}(\mathfrak{s}))] = \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \left[\aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})} * \aleph_{C}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{C}(\mathfrak{s})}\right]\right]$$

$$[\aleph_{\mathcal{E_A}}(\mathfrak{s}) \oplus \aleph_{\mathcal{E_B}}(\mathfrak{s})] * [\aleph_{\mathcal{E_A}}(\mathfrak{s}) \oplus \aleph_{\mathcal{E_C}}(\mathfrak{s})] = \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] \oplus \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{C}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{C}(\mathfrak{s})}\right].$$

Example 9.
$\mathcal{E}_{\mathcal{A}} = \left(\frac{0.9\mathfrak{e}^{-\mathsf{T}0.9}}{\mathfrak{s}_1} + \frac{0.6\mathfrak{e}^{-\mathsf{T}0.6}}{\mathfrak{s}_2} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_3}\right), \mathcal{E}_{\mathcal{B}} = \left(\frac{0.7\mathfrak{e}^{-\mathsf{T}0.7}}{\mathfrak{s}_1} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_2} + \frac{0.4\mathfrak{e}^{-\mathsf{T}0.4}}{\mathfrak{s}_3}\right)$ *and*
$\mathcal{E}_{\mathcal{C}} = \left(\frac{0.6\mathfrak{e}^{-\mathsf{T}0.6}}{\mathfrak{s}_1} + \frac{0.3\mathfrak{e}^{-\mathsf{T}0.3}}{\mathfrak{s}_2} + \frac{1\mathfrak{e}^{-\mathsf{T}1}}{\mathfrak{s}_3}\right)$ *be the three exponential fuzzysets.*

$$\begin{aligned}
\mathcal{E}_{\mathcal{A}} \cup (\mathcal{E}_{\mathcal{B}} \cap \mathcal{E}_{\mathcal{C}}) &= \left[\frac{0.9\mathfrak{e}^{-\mathsf{T}0.9}}{\mathfrak{s}_1} + \frac{0.6\mathfrak{e}^{-\mathsf{T}0.6}}{\mathfrak{s}_2} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_3}\right] \\
&\oplus \left[\left[\frac{0.7\mathfrak{e}^{-\mathsf{T}0.7}}{\mathfrak{s}_1} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_2} + \frac{0.4\mathfrak{e}^{-\mathsf{T}0.4}}{\mathfrak{s}_3}\right] * \left[\frac{0.6\mathfrak{e}^{-\mathsf{T}0.6}}{\mathfrak{s}_1} + \frac{0.3\mathfrak{e}^{-\mathsf{T}0.3}}{\mathfrak{s}_2} + \frac{1\mathfrak{e}^{-\mathsf{T}1}}{\mathfrak{s}_3}\right]\right] \\
&= \left[\frac{0.9\mathfrak{e}^{-\mathsf{T}0.9}}{\mathfrak{s}_1} + \frac{0.6\mathfrak{e}^{-\mathsf{T}0.6}}{\mathfrak{s}_2} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_3}\right] \oplus \left[\frac{0.6\mathfrak{e}^{-\mathsf{T}0.6}}{\mathfrak{s}_1} + \frac{0.3\mathfrak{e}^{-\mathsf{T}0.3}}{\mathfrak{s}_2} + \frac{0.4\mathfrak{e}^{-\mathsf{T}0.4}}{\mathfrak{s}_3}\right] \\
&= \left[\frac{0.9\mathfrak{e}^{-\mathsf{T}0.9}}{\mathfrak{s}_1} + \frac{0.6\mathfrak{e}^{-\mathsf{T}0.6}}{\mathfrak{s}_2} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_3}\right] \qquad (3.7)
\end{aligned}$$

$$\begin{aligned}
(\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}) \cap (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{C}}) &= \left[\left[\frac{0.9\mathfrak{e}^{-\mathsf{T}0.9}}{\mathfrak{s}_1} + \frac{0.6\mathfrak{e}^{-\mathsf{T}0.6}}{\mathfrak{s}_2} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_3}\right] \oplus \left[\frac{0.7\mathfrak{e}^{-\mathsf{T}0.7}}{\mathfrak{s}_1} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_2} + \frac{0.4\mathfrak{e}^{-\mathsf{T}0.4}}{\mathfrak{s}_3}\right]\right] \\
&* \left[\left[\frac{0.9\mathfrak{e}^{-\mathsf{T}0.9}}{\mathfrak{s}_1} + \frac{0.6\mathfrak{e}^{-\mathsf{T}0.6}}{\mathfrak{s}_2} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_3}\right] \oplus \left[\frac{0.6\mathfrak{e}^{-\mathsf{T}0.6}}{\mathfrak{s}_1} + \frac{0.3\mathfrak{e}^{-\mathsf{T}0.3}}{\mathfrak{s}_2} + \frac{1\mathfrak{e}^{-\mathsf{T}1}}{\mathfrak{s}_3}\right]\right] \\
&= \left[\left[\frac{0.9\mathfrak{e}^{-\mathsf{T}0.9}}{\mathfrak{s}_1} + \frac{0.6\mathfrak{e}^{-\mathsf{T}0.6}}{\mathfrak{s}_2} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_3}\right] * \left[\frac{0.9\mathfrak{e}^{-\mathsf{T}0.9}}{\mathfrak{s}_1} + \frac{0.6\mathfrak{e}^{-\mathsf{T}0.6}}{\mathfrak{s}_2} + \frac{1\mathfrak{e}^{-\mathsf{T}1}}{\mathfrak{s}_3}\right]\right] \\
&= \left[\frac{0.9\mathfrak{e}^{-\mathsf{T}0.9}}{\mathfrak{s}_1} + \frac{0.6\mathfrak{e}^{-\mathsf{T}0.6}}{\mathfrak{s}_2} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_3}\right] \qquad (3.8)
\end{aligned}$$

From equation 3.7 and 3.8 we have $\mathcal{E}_{\mathcal{A}} \cup (\mathcal{E}_{\mathcal{B}} \cap \mathcal{E}_{\mathcal{C}}) = (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}) \cap (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{C}})$.
Next, we see the distributive condition of intersection over union is

$$\begin{aligned}
\mathcal{E}_{\mathcal{A}} \cap (\mathcal{E}_{\mathcal{B}} \cup \mathcal{E}_{\mathcal{C}}) &= \left[\frac{0.9\mathfrak{e}^{-\mathsf{T}0.9}}{\mathfrak{s}_1} + \frac{0.6\mathfrak{e}^{-\mathsf{T}0.6}}{\mathfrak{s}_2} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_3}\right] \\
&* \left[\left[\frac{0.7\mathfrak{e}^{-\mathsf{T}0.7}}{\mathfrak{s}_1} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_2} + \frac{0.4\mathfrak{e}^{-\mathsf{T}0.4}}{\mathfrak{s}_3}\right] \oplus \left[\frac{0.6\mathfrak{e}^{-\mathsf{T}0.6}}{\mathfrak{s}_1} + \frac{0.3\mathfrak{e}^{-\mathsf{T}0.3}}{\mathfrak{s}_2} + \frac{1\mathfrak{e}^{-\mathsf{T}1}}{\mathfrak{s}_3}\right]\right] \\
&= \left[\frac{0.9\mathfrak{e}^{-\mathsf{T}0.9}}{\mathfrak{s}_1} + \frac{0.6\mathfrak{e}^{-\mathsf{T}0.6}}{\mathfrak{s}_2} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_3}\right] * \left[\frac{0.7\mathfrak{e}^{-\mathsf{T}0.7}}{\mathfrak{s}_1} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_2} + \frac{1\mathfrak{e}^{-\mathsf{T}1}}{\mathfrak{s}_3}\right] \\
&= \left[\frac{0.7\mathfrak{e}^{-\mathsf{T}0.7}}{\mathfrak{s}_1} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_2} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_3}\right] \qquad (3.9)
\end{aligned}$$

$$\begin{aligned}
(\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}) \cup (\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{C}}) &= \left[\left[\frac{0.9\mathfrak{e}^{-\mathsf{T}0.9}}{\mathfrak{s}_1} + \frac{0.6\mathfrak{e}^{-\mathsf{T}0.6}}{\mathfrak{s}_2} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_3}\right] * \left[\frac{0.7\mathfrak{e}^{-\mathsf{T}0.7}}{\mathfrak{s}_1} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_2} + \frac{0.4\mathfrak{e}^{-\mathsf{T}0.4}}{\mathfrak{s}_3}\right]\right] \\
&\oplus \left[\left[\frac{0.9\mathfrak{e}^{-\mathsf{T}0.9}}{\mathfrak{s}_1} + \frac{0.6\mathfrak{e}^{-\mathsf{T}0.6}}{\mathfrak{s}_2} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_3}\right] * \left[\frac{0.6\mathfrak{e}^{-\mathsf{T}0.6}}{\mathfrak{s}_1} + \frac{0.3\mathfrak{e}^{-\mathsf{T}0.3}}{\mathfrak{s}_2} + \frac{1\mathfrak{e}^{-\mathsf{T}1}}{\mathfrak{s}_3}\right]\right] \\
&= \left[\left[\frac{0.7\mathfrak{e}^{-\mathsf{T}0.7}}{\mathfrak{s}_1} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_2} + \frac{0.4\mathfrak{e}^{-\mathsf{T}0.4}}{\mathfrak{s}_3}\right] \oplus \left[\frac{0.6\mathfrak{e}^{-\mathsf{T}0.6}}{\mathfrak{s}_1} + \frac{0.3\mathfrak{e}^{-\mathsf{T}0.3}}{\mathfrak{s}_2} + \frac{0.5\mathfrak{e}^{-\mathsf{T}0.5}}{\mathfrak{s}_3}\right]\right]
\end{aligned}$$

$$= \left[\frac{0.7\mathfrak{e}^{-\intercal 0.7}}{\mathfrak{s}_1} + \frac{0.5\mathfrak{e}^{-\intercal 0.5}}{\mathfrak{s}_2} + \frac{0.5\mathfrak{e}^{-\intercal 0.5}}{\mathfrak{s}_3}\right] \tag{3.10}$$

From equation 3.9 and 3.10 we have $\mathcal{E}_{\mathcal{A}} \cap (\mathcal{E}_{\mathcal{B}} \cup \mathcal{E}_{\mathcal{C}}) = (\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}) \cup (\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{C}})$.

Definition 19. *The union of idempotent law in a* $\mathcal{EFS}s$ $\mathcal{E}_{\mathcal{A}}$ *is* $\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{A}} = \mathcal{E}_{\mathcal{A}}$ *and the idempotent law of intersection is* $\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{A}} = \mathcal{E}_{\mathcal{A}}$. *If a membership value of* $\mathcal{E}_{\mathcal{A}}$ *is* $\aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) = \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph_{\mathcal{A}}(\mathfrak{s})}$ *the idempotent law of union becomes* $\aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) = \aleph_{\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{A}}}(\mathfrak{s})$. *This prove this, we have*

$$\begin{aligned}
\aleph_{\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) &= [\aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) \oplus \aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s})] \\
&= \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph_{\mathcal{A}}(\mathfrak{s})}\right] \\
&= \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph_{\mathcal{A}}(\mathfrak{s})} \\
&= \aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}).
\end{aligned}$$

Similarly

$$\aleph_{\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) = \aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}).$$

Example 10. *Let* $\mathcal{E}_{\mathcal{A}} = \left(\frac{0.8\mathfrak{e}^{-\intercal 08}}{\mathfrak{s}_1} + \frac{0.9\mathfrak{e}^{-\intercal 0.9}}{\mathfrak{s}_2} + \frac{0.7\mathfrak{e}^{-\intercal 0.7}}{\mathfrak{s}_3}\right)$, *be a* $\mathcal{EFS}$, *the idempotent law of union is,*

$$\begin{aligned}
\aleph_{\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) &= \left(\frac{0.8\mathfrak{e}^{-\intercal 08}}{\mathfrak{s}_1} + \frac{0.9\mathfrak{e}^{-\intercal 0.9}}{\mathfrak{s}_2} + \frac{0.7\mathfrak{e}^{-\intercal 0.7}}{\mathfrak{s}_3}\right) \oplus \left(\frac{0.8\mathfrak{e}^{-\intercal 08}}{\mathfrak{s}_1} + \frac{0.9\mathfrak{e}^{-\intercal 0.9}}{\mathfrak{s}_2} + \frac{0.7\mathfrak{e}^{-\intercal 0.7}}{\mathfrak{s}_3}\right) \\
&= \left(\frac{0.8\mathfrak{e}^{-\intercal 08}}{\mathfrak{s}_1} + \frac{0.9\mathfrak{e}^{-\intercal 0.9}}{\mathfrak{s}_2} + \frac{0.7\mathfrak{e}^{-\intercal 0.7}}{\mathfrak{s}_3}\right) \\
&= \aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}).
\end{aligned}$$

The idempotent law of intersection is ,

$$\begin{aligned}
\aleph_{\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) &= \left(\frac{0.8\mathfrak{e}^{-\intercal 08}}{\mathfrak{s}_1} + \frac{0.9\mathfrak{e}^{-\intercal 0.9}}{\mathfrak{s}_2} + \frac{0.7\mathfrak{e}^{-\intercal 0.7}}{\mathfrak{s}_3}\right) * \left(\frac{0.8\mathfrak{e}^{-\intercal 08}}{\mathfrak{s}_1} + \frac{0.9\mathfrak{e}^{-\intercal 0.9}}{\mathfrak{s}_2} + \frac{0.7\mathfrak{e}^{-\intercal 0.7}}{\mathfrak{s}_3}\right) \\
&= \left(\frac{0.8\mathfrak{e}^{-\intercal 08}}{\mathfrak{s}_1} + \frac{0.9\mathfrak{e}^{-\intercal 0.9}}{\mathfrak{s}_2} + \frac{0.7\mathfrak{e}^{-\intercal 0.7}}{\mathfrak{s}_3}\right) \\
&= \aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}).
\end{aligned}$$

The union and intersection law of idempotent laws are hold.

Definition 20. $\mathcal{EFS}$ *satisfied the involution law using standard complement function. The involution law for a* $\mathcal{EFS}$ $\mathcal{E}_{\mathcal{A}}$ *is* $(\mathcal{E}_{\mathcal{A}}{}^c)^c = \mathcal{E}_{\mathcal{A}}$. *If a membership value of* $\mathcal{E}_{\mathcal{A}}$ *is* $\aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) = \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph_{\mathcal{A}}(\mathfrak{s})}$ *the involution law is*

$$\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph_{\mathcal{A}}(\mathfrak{s})} = (\aleph^c_{\mathcal{A}}(\mathfrak{s}))^c \mathfrak{e}^{-\intercal (\aleph^c_{\mathcal{A}}(\mathfrak{s}))^c}.$$

Example 11. *Let* $\mathcal{E}_{\mathcal{A}} = \left(\frac{0.7\mathfrak{e}^{-\mathsf{T}0.7}}{\mathfrak{s}_1} + \frac{0.6\mathfrak{e}^{-\mathsf{T}0.6}}{\mathfrak{s}_2} + \frac{0.4\mathfrak{e}^{-\mathsf{T}0.4}}{\mathfrak{s}_3}\right)$, *be a* $\mathcal{EFS}$. *By using standard complement function, the involution law valid.*

$$\begin{aligned}
{\mathcal{E}_{\mathcal{A}}}^{c} &= \frac{0.3\mathfrak{e}^{-\mathsf{T}0.3}}{\mathfrak{s}_1} + \frac{0.7\mathfrak{e}^{-\mathsf{T}0.7}}{\mathfrak{s}_2} + \frac{0.6\mathfrak{e}^{-\mathsf{T}0.6}}{\mathfrak{s}_3}\\
{\mathcal{E}_{\mathcal{A}}}^{c^c} &= \frac{0.7\mathfrak{e}^{-\mathsf{T}0.7}}{\mathfrak{s}_1} + \frac{0.6\mathfrak{e}^{-\mathsf{T}0.6}}{\mathfrak{s}_2} + \frac{0.4\mathfrak{e}^{-\mathsf{T}0.4}}{\mathfrak{s}_3}\\
&= \mathcal{E}_{\mathcal{A}}.
\end{aligned}$$

4. Main Results of Exponential Fuzzy Sets

Theorem 1. *Let* $\mathcal{E}_{\mathcal{A}}$ *and* $\mathcal{E}_{\mathcal{B}}$ *be* $\mathcal{EFS}$*s over the classical set* $\mathfrak{Z}$*, the symmetrical difference condition is satisfied for union, intersection and their complement functions of phase term.*

Proof. $\mathcal{E}_{\mathcal{A}}$ and $\mathcal{E}_{\mathcal{B}}$ be two $\mathcal{EFS}$s. To demonstrate the formula for symmetrical difference

$$({\mathcal{E}_{\mathcal{A}}}^{c} \cap \mathcal{E}_{\mathcal{B}}) \cup (\mathcal{E}_{\mathcal{A}} \cap {\mathcal{E}_{\mathcal{B}}}^{c}) = ({\mathcal{E}_{\mathcal{A}}}^{c} \cup {\mathcal{E}_{\mathcal{B}}}^{c}) \cap (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}),$$

To determine the phase term, the max function.

Case 1 . $\aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}), \aleph^{c}_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}), \aleph^{c}_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s})$ and $\aleph^{c}_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \leq \aleph^{c}_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s})$.

$$\begin{aligned}
({\mathcal{E}_{\mathcal{A}}}^{c} \cap \mathcal{E}_{\mathcal{B}}) \cup (\mathcal{E}_{\mathcal{A}} \cap {\mathcal{E}_{\mathcal{B}}}^{c}) &= \left[\aleph^{c}_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^{c}_{\mathcal{A}}(\mathfrak{s})} * \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] \oplus \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} * \aleph^{c}_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^{c}_{\mathcal{B}}(\mathfrak{s})}\right]\\
&= \left[\aleph^{c}_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^{c}_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph^{c}_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^{c}_{\mathcal{B}}(\mathfrak{s})}\right]\\
&= \left[\aleph^{c}_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^{c}_{\mathcal{A}}(\mathfrak{s})}\right] \qquad (4.1)\\
({\mathcal{E}_{\mathcal{A}}}^{c} \cup {\mathcal{E}_{\mathcal{B}}}^{c}) \cap (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}) &= \left[\aleph^{c}_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^{c}_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph^{c}_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^{c}_{\mathcal{B}}(\mathfrak{s})}\right] * \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right]\\
&= \left[\aleph^{c}_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^{c}_{\mathcal{A}}(\mathfrak{s})} * \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right]\\
&= \left[\aleph^{c}_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^{c}_{\mathcal{A}}(\mathfrak{s})}\right] \qquad (4.2)
\end{aligned}$$

From equation 4.1 and 4.2

$$({\mathcal{E}_{\mathcal{A}}}^{c} \cap \mathcal{E}_{\mathcal{B}}) \cup (\mathcal{E}_{\mathcal{A}} \cap {\mathcal{E}_{\mathcal{B}}}^{c}) = ({\mathcal{E}_{\mathcal{A}}}^{c} \cup {\mathcal{E}_{\mathcal{B}}}^{c}) \cap (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}).$$

Case 2 . $\aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}), \aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \leq \aleph^{c}_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}), \aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) \leq \aleph^{c}_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s})$ and $\aleph^{c}_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \leq \aleph^{c}_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s})$.

$$\begin{aligned}
({\mathcal{E}_{\mathcal{A}}}^{c} \cap \mathcal{E}_{\mathcal{B}}) \cup (\mathcal{E}_{\mathcal{A}} \cap {\mathcal{E}_{\mathcal{B}}}^{c}) &= \left[\aleph^{c}_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^{c}_{\mathcal{A}}(\mathfrak{s})} * \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] \oplus \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} * \aleph^{c}_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^{c}_{\mathcal{B}}(\mathfrak{s})}\right]\\
&= \left[\aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})} \oplus \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})}\right]\\
&= \left[\aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] \qquad (4.3)
\end{aligned}$$

$$\begin{aligned}
(\mathcal{E}_{\mathcal{A}}{}^{c} \cup \mathcal{E}_{\mathcal{B}}{}^{c}) \cap (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}) &= \left[\aleph^{c}_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^{c}_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph^{c}_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^{c}_{\mathcal{B}}(\mathfrak{s})}\right] * \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] \\
&= \left[\aleph^{c}_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^{c}_{\mathcal{A}}(\mathfrak{s})} * \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] \\
&= \left[\aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] \qquad (4.4)
\end{aligned}$$

From equation 4.3 and 4.4

$$(\mathcal{E}_{\mathcal{A}}{}^{c} \cap \mathcal{E}_{\mathcal{B}}) \cup (\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}{}^{c}) = (\mathcal{E}_{\mathcal{A}}{}^{c} \cup \mathcal{E}_{\mathcal{B}}{}^{c}) \cap (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}).$$

Case 3 . $\aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) \le \aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}), \aleph^{c}_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) \le \aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}), \aleph^{c}_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \le \aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s})$ and $\aleph^{c}_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \le \aleph^{c}_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s})$.

$$\begin{aligned}
(\mathcal{E}_{\mathcal{A}}{}^{c} \cap \mathcal{E}_{\mathcal{B}}) \cup (\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}{}^{c}) &= \left[\aleph^{c}_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^{c}_{\mathcal{A}}(\mathfrak{s})} * \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] \oplus \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} * \aleph^{c}_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^{c}_{\mathcal{B}}(\mathfrak{s})}\right] \\
&= \left[\aleph^{c}_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^{c}_{\mathcal{B}}(\mathfrak{s})} \oplus \aleph^{c}_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^{c}_{\mathcal{A}}(\mathfrak{s})}\right] \\
&= \left[\aleph^{c}_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^{c}_{\mathcal{B}}(\mathfrak{s})}\right] \qquad (4.5) \\
(\mathcal{E}_{\mathcal{A}}{}^{c} \cup \mathcal{E}_{\mathcal{B}}{}^{c}) \cap (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}) &= \left[\aleph^{c}_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^{c}_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph^{c}_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^{c}_{\mathcal{B}}(\mathfrak{s})}\right] * \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] \\
&= \left[\aleph^{c}_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^{c}_{\mathcal{A}}(\mathfrak{s})} * \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] \\
&= \left[\aleph^{c}_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^{c}_{\mathcal{B}}(\mathfrak{s})}\right] \qquad (4.6)
\end{aligned}$$

From equation 4.5 and 4.6

$$(\mathcal{E}_{\mathcal{A}}{}^{c} \cap \mathcal{E}_{\mathcal{B}}) \cup (\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}{}^{c}) = (\mathcal{E}_{\mathcal{A}}{}^{c} \cup \mathcal{E}_{\mathcal{B}}{}^{c}) \cap (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}).$$

Case 4 . $\aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) \le \aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}), \aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \le \aleph^{c}_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}), \aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) \le \aleph^{c}_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s})$ and $\aleph^{c}_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \le \aleph^{c}_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s})$.

$$\begin{aligned}
(\mathcal{E}_{\mathcal{A}}{}^{c} \cap \mathcal{E}_{\mathcal{B}}) \cup (\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}{}^{c}) &= \left[\aleph^{c}_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^{c}_{\mathcal{A}}(\mathfrak{s})} * \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] \oplus \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} * \aleph^{c}_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^{c}_{\mathcal{B}}(\mathfrak{s})}\right] \\
&= \left[\aleph^{c}_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^{c}_{\mathcal{B}}(\mathfrak{s})} \oplus \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})}\right] \\
&= \left[\aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] \qquad (4.7) \\
(\mathcal{E}_{\mathcal{A}}{}^{c} \cup \mathcal{E}_{\mathcal{B}}{}^{c}) \cap (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}) &= \left[\aleph^{c}_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^{c}_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph^{c}_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^{c}_{\mathcal{B}}(\mathfrak{s})}\right] * \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] \\
&= \left[\aleph^{c}_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^{c}_{\mathcal{A}}(\mathfrak{s})} * \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] \\
&= \left[\aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] \qquad (4.8)
\end{aligned}$$

From equation 4.7 and 4.8

$$(\mathcal{E}_{\mathcal{A}}{}^{c} \cap \mathcal{E}_{\mathcal{B}}) \cup (\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}{}^{c}) = (\mathcal{E}_{\mathcal{A}}{}^{c} \cup \mathcal{E}_{\mathcal{B}}{}^{c}) \cap (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}).$$

Case 5 . $\aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}), \aleph^c_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}), \aleph^c_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s})$ and $\aleph^c_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) \leq \aleph^c_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s})$.

$$\begin{aligned}
(\mathcal{E}_{\mathcal{A}}{}^c \cap \mathcal{E}_{\mathcal{B}}) \cup (\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}{}^c) &= \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{A}}(\mathfrak{s})} * \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] \oplus \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} * \aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] \\
&= \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] \\
&= \left[\aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] \qquad (4.9) \\
(\mathcal{E}_{\mathcal{A}}{}^c \cup \mathcal{E}_{\mathcal{B}}{}^c) \cap (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}) &= \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] * \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] \\
&= \left[\aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})} * \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})}\right] \\
&= \left[\aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] \qquad (4.10)
\end{aligned}$$

From equation 4.9 and 4.10

$$(\mathcal{E}_{\mathcal{A}}{}^c \cap \mathcal{E}_{\mathcal{B}}) \cup (\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}{}^c) = (\mathcal{E}_{\mathcal{A}}{}^c \cup \mathcal{E}_{\mathcal{B}}{}^c) \cap (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}).$$

Case 6 . $\aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}), \aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \leq \aleph^c_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}), \aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) \leq \aleph^c_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s})$ and $\aleph^c_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) \leq \aleph^c_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s})$.

$$\begin{aligned}
(\mathcal{E}_{\mathcal{A}}{}^c \cap \mathcal{E}_{\mathcal{B}}) \cup (\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}{}^c) &= \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{A}}(\mathfrak{s})} * \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] \oplus \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} * \aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] \\
&= \left[\aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})} \oplus \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})}\right] \\
&= \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})}\right] \qquad (4.11) \\
(\mathcal{E}_{\mathcal{A}}{}^c \cup \mathcal{E}_{\mathcal{B}}{}^c) \cap (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}) &= \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] * \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] \\
&= \left[\aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})} * \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})}\right] \\
&= \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})}\right] \qquad (4.12)
\end{aligned}$$

From equation 4.11 and 4.12

$$(\mathcal{E}_{\mathcal{A}}{}^c \cap \mathcal{E}_{\mathcal{B}}) \cup (\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}{}^c) = (\mathcal{E}_{\mathcal{A}}{}^c \cup \mathcal{E}_{\mathcal{B}}{}^c) \cap (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}).$$

Case 7 . $\aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}), \aleph^c_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}), \aleph^c_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s})$ and $\aleph^c_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) \leq \aleph^c_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s})$.

$$\begin{aligned}
(\mathcal{E}_{\mathcal{A}}{}^c \cap \mathcal{E}_{\mathcal{B}}) \cup (\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}{}^c) &= \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{A}}(\mathfrak{s})} * \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] \oplus \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} * \aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] \\
&= \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})} \oplus \aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] \\
&= \left[\aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] \qquad (4.13) \\
(\mathcal{E}_{\mathcal{A}}{}^c \cup \mathcal{E}_{\mathcal{B}}{}^c) \cap (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}) &= \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] * \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] \\
&= \left[\aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})} * \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})}\right]
\end{aligned}$$

$$= \left[\aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] \tag{4.14}$$

From equation 4.13 and 4.14

$$\left({\mathcal{E}_{\mathcal{A}}}^c \cap \mathcal{E}_{\mathcal{B}}\right) \cup \left(\mathcal{E}_{\mathcal{A}} \cap {\mathcal{E}_{\mathcal{B}}}^c\right) = \left({\mathcal{E}_{\mathcal{A}}}^c \cup {\mathcal{E}_{\mathcal{B}}}^c\right) \cap \left(\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}\right).$$

Case 8 . $\aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}), \aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \leq \aleph^c_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}), \aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) \leq \aleph^c_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s})$ and $\aleph^c_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) \leq \aleph^c_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s})$.

$$\begin{aligned}
\left({\mathcal{E}_{\mathcal{A}}}^c \cap \mathcal{E}_{\mathcal{B}}\right) \cup \left(\mathcal{E}_{\mathcal{A}} \cap {\mathcal{E}_{\mathcal{B}}}^c\right) &= \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{A}}(\mathfrak{s})} * \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] \oplus \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} * \aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] \\
&= \left[\aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})} \oplus \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})}\right] \\
&= \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})}\right] \qquad (4.15) \\
\left({\mathcal{E}_{\mathcal{A}}}^c \cup {\mathcal{E}_{\mathcal{B}}}^c\right) \cap \left(\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}\right) &= \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] * \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] \\
&= \left[\aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})} * \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})}\right] \\
&= \left[\aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] \qquad (4.16)
\end{aligned}$$

From equation 4.15 and 4.16

$$\left({\mathcal{E}_{\mathcal{A}}}^c \cap \mathcal{E}_{\mathcal{B}}\right) \cup \left(\mathcal{E}_{\mathcal{A}} \cap {\mathcal{E}_{\mathcal{B}}}^c\right) = \left({\mathcal{E}_{\mathcal{A}}}^c \cup {\mathcal{E}_{\mathcal{B}}}^c\right) \cap \left(\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}\right).$$

Therefore, the formula for symmetrical difference is valid for all cases.

Theorem 2. *The union, intersection and complement function of exponential $\mathcal{EFS}$s $\mathcal{E}_{\mathcal{A}}$ and $\mathcal{E}_{\mathcal{B}}$ is an equivalence relation.*

Proof. $\mathcal{E}_{\mathcal{A}}$ and $\mathcal{E}_{\mathcal{B}}$ be two $\mathcal{EFS}$s. To demonstrate the equivalence relation.

$$\left({\mathcal{E}_{\mathcal{A}}}^c \cup \mathcal{E}_{\mathcal{B}}\right) \cap \left(\mathcal{E}_{\mathcal{A}} \cup {\mathcal{E}_{\mathcal{B}}}^c\right) = \left({\mathcal{E}_{\mathcal{A}}}^c \cap {\mathcal{E}_{\mathcal{B}}}^c\right) \cup \left(\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}\right).$$

Case 1 . $\aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}), \aleph^c_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}), \aleph^c_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s})$ and $\aleph^c_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \leq \aleph^c_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s})$.

$$\begin{aligned}
\left({\mathcal{E}_{\mathcal{A}}}^c \cup \mathcal{E}_{\mathcal{B}}\right) \cap \left(\mathcal{E}_{\mathcal{A}} \cup {\mathcal{E}_{\mathcal{B}}}^c\right) &= \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] * \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] \\
&= \left[\aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})} \oplus \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})}\right] \\
&= \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})}\right] \qquad (4.17) \\
\left({\mathcal{E}_{\mathcal{A}}}^c \cap {\mathcal{E}_{\mathcal{B}}}^c\right) \cup \left(\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}\right) &= \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{A}}(\mathfrak{s})} * \aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] \oplus \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} * \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] \\
&= \left[\aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})} \oplus \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})}\right] \\
&= \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})}\right] \qquad (4.18)
\end{aligned}$$

From equation 4.17 and 4.18

$$\left({\mathcal{E}_{\mathcal{A}}}^c \cup \mathcal{E}_{\mathcal{B}}\right) \cap \left(\mathcal{E}_{\mathcal{A}} \cup {\mathcal{E}_{\mathcal{B}}}^c\right) = \left({\mathcal{E}_{\mathcal{A}}}^c \cap {\mathcal{E}_{\mathcal{B}}}^c\right) \cup \left(\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}\right).$$

Case 2 . $\aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}), \aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \leq \aleph^c_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}), \aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) \leq \aleph^c_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s})$ and $\aleph^c_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \leq \aleph^c_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s})$.

$$\begin{aligned}
(\mathcal{E_A}^c \cup \mathcal{E_B}) \cap (\mathcal{E_A} \cup \mathcal{E_B}^c) &= \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] * \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] \\
&= \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{A}}(\mathfrak{s})} * \aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] \\
&= \left[\aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] \qquad (4.19) \\
(\mathcal{E_A}^c \cap \mathcal{E_B}^c) \cup (\mathcal{E_A} \cap \mathcal{E_B}) &= \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{A}}(\mathfrak{s})} * \aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] \oplus \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} * \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] \\
&= \left[\aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})} \oplus \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})}\right] \\
&= \left[\aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] \qquad (4.20)
\end{aligned}$$

From equation 4.19 and 4.20

$$(\mathcal{E_A}^c \cup \mathcal{E_B}) \cap (\mathcal{E_A} \cup \mathcal{E_B}^c) = (\mathcal{E_A}^c \cap \mathcal{E_B}^c) \cup (\mathcal{E_A} \cap \mathcal{E_B}).$$

Case 3 . $\aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}), \aleph^c_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}), \aleph^c_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s})$ and $\aleph^c_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \leq \aleph^c_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s})$.

$$\begin{aligned}
(\mathcal{E_A}^c \cup \mathcal{E_B}) \cap (\mathcal{E_A} \cup \mathcal{E_B}^c) &= \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] * \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] \\
&= \left[\aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{R}(\mathfrak{s})} * \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})}\right] \\
&= \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})}\right] \qquad (4.21) \\
(\mathcal{E_A}^c \cap \mathcal{E_B}^c) \cup (\mathcal{E_A} \cap \mathcal{E_B}) &= \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{A}}(\mathfrak{s})} * \aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] \oplus \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} * \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] \\
&= \left[\aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})} \oplus \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})}\right] \\
&= \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})}\right] \qquad (4.22)
\end{aligned}$$

From equation 4.21 and 4.22

$$(\mathcal{E_A}^c \cup \mathcal{E_B}) \cap (\mathcal{E_A} \cup \mathcal{E_B}^c) = (\mathcal{E_A}^c \cap \mathcal{E_B}^c) \cup (\mathcal{E_A} \cap \mathcal{E_B}).$$

Case 4 . $\aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}), \aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \leq \aleph^c_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}), \aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) \leq \aleph^c_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s})$ and $\aleph^c_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \leq \aleph^c_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s})$.

$$\begin{aligned}
(\mathcal{E_A}^c \cup \mathcal{E_B}) \cap (\mathcal{E_A} \cup \mathcal{E_B}^c) &= \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] * \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] \\
&= \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{A}}(\mathfrak{s})} * \aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] \\
&= \left[\aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] \qquad (4.23) \\
(\mathcal{E_A}^c \cap \mathcal{E_B}^c) \cup (\mathcal{E_A} \cap \mathcal{E_B}) &= \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{A}}(\mathfrak{s})} * \aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] \oplus \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} * \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] \\
&= \left[\aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph^c_{\mathcal{B}}(\mathfrak{s})} \oplus \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})}\right]
\end{aligned}$$

$$= \left[\aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] \tag{4.24}$$

From equation 4.23 and 4.24

$$({\mathcal{E}_{\mathcal{A}}}^c \cup \mathcal{E}_{\mathcal{B}}) \cap (\mathcal{E}_{\mathcal{A}} \cup {\mathcal{E}_{\mathcal{B}}}^c) = ({\mathcal{E}_{\mathcal{A}}}^c \cap {\mathcal{E}_{\mathcal{B}}}^c) \cup (\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}).$$

Case 5 . $\aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}), \aleph^c_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}), \aleph^c_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s})$ and $\aleph^c_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) \leq \aleph^c_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s})$.

$$\begin{aligned}
({\mathcal{E}_{\mathcal{A}}}^c \cup \mathcal{E}_{\mathcal{B}}) \cap (\mathcal{E}_{\mathcal{A}} \cup {\mathcal{E}_{\mathcal{B}}}^c) &= \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph^c_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph_{\mathcal{B}}(\mathfrak{s})}\right] * \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] \\
&= \left[\aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph_{\mathcal{B}}(\mathfrak{s})} * \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph_{\mathcal{A}}(\mathfrak{s})}\right] \\
&= \left[\aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph_{\mathcal{B}}(\mathfrak{s})}\right] &(4.25) \\
({\mathcal{E}_{\mathcal{A}}}^c \cap {\mathcal{E}_{\mathcal{B}}}^c) \cup (\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}) &= \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph^c_{\mathcal{A}}(\mathfrak{s})} * \aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] \oplus \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph_{\mathcal{A}}(\mathfrak{s})} * \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph_{\mathcal{B}}(\mathfrak{s})}\right] \\
&= \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph^c_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph_{\mathcal{B}}(\mathfrak{s})}\right] \\
&= \left[\aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph_{\mathcal{B}}(\mathfrak{s})}\right] &(4.26)
\end{aligned}$$

From equation 4.25 and 4.26

$$({\mathcal{E}_{\mathcal{A}}}^c \cup \mathcal{E}_{\mathcal{B}}) \cap (\mathcal{E}_{\mathcal{A}} \cup {\mathcal{E}_{\mathcal{B}}}^c) = ({\mathcal{E}_{\mathcal{A}}}^c \cap {\mathcal{E}_{\mathcal{B}}}^c) \cup (\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}).$$

Case 6 . $\aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}), \aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \leq \aleph^c_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}), \aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) \leq \aleph^c_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s})$ and $\aleph^c_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) \leq \aleph^c_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s})$.

$$\begin{aligned}
({\mathcal{E}_{\mathcal{A}}}^c \cup \mathcal{E}_{\mathcal{B}}) \cap (\mathcal{E}_{\mathcal{A}} \cup {\mathcal{E}_{\mathcal{B}}}^c) &= \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph^c_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph_{\mathcal{B}}(\mathfrak{s})}\right] * \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] \\
&= \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph^c_{\mathcal{A}}(\mathfrak{s})} * \aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] \\
&= \aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph^c_{\mathcal{A}}(\mathfrak{s})} &(4.27) \\
({\mathcal{E}_{\mathcal{A}}}^c \cap {\mathcal{E}_{\mathcal{B}}}^c) \cup (\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}) &= \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph^c_{\mathcal{A}}(\mathfrak{s})} * \aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] \oplus \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph_{\mathcal{A}}(\mathfrak{s})} * \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph_{\mathcal{B}}(\mathfrak{s})}\right] \\
&= \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph^c_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph_{\mathcal{B}}(\mathfrak{s})}\right] \\
&= \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph^c_{\mathcal{A}}(\mathfrak{s})}\right] &(4.28)
\end{aligned}$$

From equation 4.27 and 4.28

$$({\mathcal{E}_{\mathcal{A}}}^c \cup \mathcal{E}_{\mathcal{B}}) \cap (\mathcal{E}_{\mathcal{A}} \cup {\mathcal{E}_{\mathcal{B}}}^c) = ({\mathcal{E}_{\mathcal{A}}}^c \cap {\mathcal{E}_{\mathcal{B}}}^c) \cup (\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}).$$

Case 7 . $\aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}), \aleph^c_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}), \aleph^c_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s})$ and $\aleph^c_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) \leq \aleph^c_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s})$.

$$\begin{aligned}
({\mathcal{E}_{\mathcal{A}}}^c \cup \mathcal{E}_{\mathcal{B}}) \cap (\mathcal{E}_{\mathcal{A}} \cup {\mathcal{E}_{\mathcal{B}}}^c) &= \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph^c_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph_{\mathcal{B}}(\mathfrak{s})}\right] * \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph^c_{\mathcal{B}}(\mathfrak{s})}\right] \\
&= \left[\aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph_{\mathcal{B}}(\mathfrak{s})} * \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal \aleph_{\mathcal{A}}(\mathfrak{s})}\right]
\end{aligned}$$

$$= \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal^{\aleph_{\mathcal{B}}(\mathfrak{s})}} \quad (4.29)$$

$$\begin{aligned}
(\mathcal{E_A}^c \cap \mathcal{E_B}^c) \cup (\mathcal{E_A} \cap \mathcal{E_B}) &= \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal^{\aleph^c_{\mathcal{A}}(\mathfrak{s})}} * \aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal^{\aleph^c_{\mathcal{B}}(\mathfrak{s})}}\right] \oplus \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal^{\aleph_{\mathcal{A}}(\mathfrak{s})}} * \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal^{\aleph_{\mathcal{B}}(\mathfrak{s})}}\right] \\
&= \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal^{\aleph^c_{\mathcal{A}}(\mathfrak{s})}} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal^{\aleph_{\mathcal{B}}(\mathfrak{s})}}\right] \\
&= \left[\aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal^{\aleph_{\mathcal{B}}(\mathfrak{s})}}\right] \quad (4.30)
\end{aligned}$$

From equation 4.29 and 4.30

$$(\mathcal{E_A}^c \cup \mathcal{E_B}) \cap (\mathcal{E_A} \cup \mathcal{E_B}^c) = (\mathcal{E_A}^c \cap \mathcal{E_B}^c) \cup (\mathcal{E_A} \cap \mathcal{E_B}).$$

Case 8 . $\aleph_{\mathcal{E_B}}(\mathfrak{s}) \leq \aleph_{\mathcal{E_B}}(\mathfrak{s}), \aleph_{\mathcal{E_B}}(\mathfrak{s}) \leq \aleph^c_{\mathcal{E_A}}(\mathfrak{s}), \aleph_{\mathcal{E_A}}(\mathfrak{s}) \leq \aleph^c_{\mathcal{E_B}}(\mathfrak{s})$ and $\aleph^c_{\mathcal{E_A}}(\mathfrak{s}) \leq \aleph^c_{\mathcal{E_B}}(\mathfrak{s})$.

$$\begin{aligned}
(\mathcal{E_A}^c \cup \mathcal{E_B}) \cap (\mathcal{E_A} \cup \mathcal{E_B}^c) &= \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal^{\aleph^c_{\mathcal{A}}(\mathfrak{s})}} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal^{\aleph_{\mathcal{B}}(\mathfrak{s})}}\right] * \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal^{\aleph_{\mathcal{A}}(\mathfrak{s})}} \oplus \aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal^{\aleph^c_{\mathcal{B}}(\mathfrak{s})}}\right] \\
&= \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal^{\aleph^c_{\mathcal{A}}(\mathfrak{s})}} * \aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal^{\aleph^c_{\mathcal{B}}(\mathfrak{s})}}\right] \\
&= \aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal^{\aleph^c_{\mathcal{A}}(\mathfrak{s})}} \quad (4.31) \\
(\mathcal{E_A}^c \cap \mathcal{E_B}^c) \cup (\mathcal{E_A} \cap \mathcal{E_B}) &= \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal^{\aleph^c_{\mathcal{A}}(\mathfrak{s})}} * \aleph^c_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal^{\aleph^c_{\mathcal{B}}(\mathfrak{s})}}\right] \oplus \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal^{\aleph_{\mathcal{A}}(\mathfrak{s})}} * \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal^{\aleph_{\mathcal{B}}(\mathfrak{s})}}\right] \\
&= \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal^{\aleph^c_{\mathcal{A}}(\mathfrak{s})}} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal^{\aleph_{\mathcal{B}}(\mathfrak{s})}}\right] \\
&= \left[\aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal^{\aleph^c_{\mathcal{A}}(\mathfrak{s})}}\right] \quad (4.32)
\end{aligned}$$

From equation 4.31 and 4.32

$$(\mathcal{E_A}^c \cup \mathcal{E_B}) \cap (\mathcal{E_A} \cup \mathcal{E_B}^c) = (\mathcal{E_A}^c \cap \mathcal{E_B}^c) \cup (\mathcal{E_A} \cap \mathcal{E_B}).$$

From the all cases $\mathcal{E_A}$ and $\mathcal{E_B}$ is equivalence relation.

Theorem 3. *Any finite collection of $\mathcal{EFS}$s is always an $\mathcal{EFS}$s for union and intersection.*

Proof.

case i . Let $\mathcal{E}_{\mathcal{A}1}, \mathcal{E}_{\mathcal{A}2}, \mathcal{E}_{\mathcal{A}3}, ...\mathcal{E}_{\mathcal{A}m}$ be $\mathcal{EFS}$s and its membership functions is $\aleph_{\mathcal{E}_{\mathcal{A}1}}, \aleph_{\mathcal{E}_{\mathcal{A}2}}, \aleph_{\mathcal{E}_{\mathcal{A}3}}, ..., \aleph_{\mathcal{E}_{\mathcal{A}m}}$. $\aleph'_{\mathcal{E_A}}(\mathfrak{s}) = max\left[\aleph_{E_{A_1}}, \aleph_{E_{A_2}}, \aleph_{E_{A_3}}, ..., \aleph_{E_{A_m}}\right]$. Now,

$$\begin{aligned}
\mathcal{E}_{\mathcal{A}1} \cup \mathcal{E}_{\mathcal{A}2} \cup ...... \cup \mathcal{E}_{\mathcal{A}m} &= \left[\aleph_{A_1}(\mathfrak{s})\mathfrak{e}^{-\intercal^{\aleph_{A_1}(\mathfrak{s})}} \oplus \aleph_{A_2}(\mathfrak{s})\mathfrak{e}^{-\intercal^{\aleph_{A_2}(\mathfrak{s})}} \oplus ..... \oplus \aleph_{A_m}(\mathfrak{s})\mathfrak{e}^{-\intercal^{\aleph_{A_m}(\mathfrak{s})}}\right] \\
&= \aleph'_{A_1}(\mathfrak{s})\mathfrak{e}^{-\intercal^{\aleph'_{A_1}(\mathfrak{s})}} \\
&= E'_{A_1}.
\end{aligned}$$

case ii . $\mathcal{E}_{\mathcal{A}1}, \mathcal{E}_{\mathcal{A}2}, \mathcal{E}_{\mathcal{A}3}, ...\mathcal{E}_{\mathcal{A}m}$ be any m $\mathcal{EFS}$s and $\aleph_{A_1}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{A_1}(\mathfrak{s})}, \aleph_{A_2}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{A_2}(\mathfrak{s})}, ...\aleph_{A_m}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{A_m}(\mathfrak{s})}$ denotes the membership functions of these $\mathcal{EFS}$s.
$\aleph'_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) = min\left[\aleph_{E_{A_1}}, \aleph_{E_{A_2}}, \aleph_{E_{A_3}}, ..., \aleph_{E_{A_m}}\right]$. Now,

$$\begin{aligned}
\mathcal{E}_{\mathcal{A}1} \cap \mathcal{E}_{\mathcal{A}2} \cap ...... \cap \mathcal{E}_{\mathcal{A}m} &= \left[\aleph_{A_1}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{A_1}(\mathfrak{s})} * \aleph_{A_2}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{A_2}(\mathfrak{s})} * ..... * \aleph_{A_m}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{A_m}(\mathfrak{s})}\right] \\
&= \aleph'_{A_1}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph'_{A_1}(\mathfrak{s})} \\
&= E'_{A_1}.
\end{aligned}$$

Which is also a $\mathcal{EFS}$.

Theorem 4. *For any two $\mathcal{EFS}$ $\mathcal{E}_{\mathcal{A}}$ and $\mathcal{E}_{\mathcal{B}}$, the union and intersection function with the same function for determining the phase term satisfy:*

$$\sum_{j=1, \mathfrak{s}_j \in \mathfrak{Z}}^{\mathfrak{M}} \left|\aleph_{\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}}(\mathfrak{s}_i)\right| \leq \sum_{j=1, \mathfrak{s}_j \in \mathfrak{Z}}^{\mathfrak{M}} \left|\aleph_{\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}}(\mathfrak{s}_i)\right|.$$

Proof. The expression function of union and intersection are define by

$$\aleph_{\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) = max\left[\aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}), \aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s})\right]$$

and

$$\aleph_{\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) = min\left[\aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}), \aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s})\right].$$

As

$$\begin{aligned}
\left|\aleph_{\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}}(u)\right| &\leq \left|\aleph_{\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}}(u)\right| \\
\left|\aleph_{\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}}(v)\right| &\leq \left|\aleph_{\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}}(v)\right| \\
\vdots \quad &\vdots \quad \vdots \\
\left|\aleph_{\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}}(\mathfrak{s}_m)\right| &\leq \left|\aleph_{\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}}(\mathfrak{s}_m)\right|.
\end{aligned}$$

Sum of all above inequalities we obtained

$$\sum_{i=1, \mathfrak{s}_i \in \mathfrak{Z}}^{\mathfrak{M}} \left|\aleph_{\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}}(\mathfrak{s}_i)\right| \leq \sum_{i=1, \mathfrak{s}_i \in \mathfrak{Z}}^{\mathfrak{M}} \left|\aleph_{\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}}(\mathfrak{s}_i)\right|.$$

Theorem 5. *For any $\mathcal{EFS}$s $\mathcal{E}_{\mathcal{A}}, \mathcal{E}_{\mathcal{B}}$ and $\mathcal{E}_{\mathcal{C}}$, the intersection union functions with the same function for determining the phase term stratify the distributive law.*

Proof. First, we prove the distributive law for any $\mathcal{EFS}$s $\mathcal{E}_{\mathcal{A}}, \mathcal{E}_{\mathcal{B}}$ and $\mathcal{E}_{\mathcal{C}}$, six cases arise here. We prove distributive law of union over intersection.

Case 1 $\aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{C}}}(\mathfrak{s})$

$$\mathcal{E}_{\mathcal{A}} \cup (\mathcal{E}_{\mathcal{B}} \cap \mathcal{E}_{\mathcal{C}}) = \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \left(\aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})} * \aleph_{C}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{C}(\mathfrak{s})}\right)\right]$$
$$= \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right]$$
$$\mathcal{E}_{\mathcal{A}} \cup (\mathcal{E}_{\mathcal{B}} \cap \mathcal{E}_{\mathcal{C}}) = \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}. \tag{4.33}$$
$$(\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}) \cap (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{C}}) = \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] * \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{C}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{C}(\mathfrak{s})}\right]$$
$$= \left[\aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})} * \aleph_{C}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{C}(\mathfrak{s})}\right]$$
$$(\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}) \cap (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{C}}) = \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})} \tag{4.34}$$

From equation (4.33) and (4.34), we have

$$\mathcal{E}_{\mathcal{A}} \cup (\mathcal{E}_{\mathcal{B}} \cap \mathcal{E}_{\mathcal{C}}) = (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}) \cap (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{C}})$$

Case 2 $\aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{C}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s})$

$$\mathcal{E}_{\mathcal{A}} \cup (\mathcal{E}_{\mathcal{B}} \cap \mathcal{E}_{\mathcal{C}}) = \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \left(\aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})} * \aleph_{C}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{C}(\mathfrak{s})}\right)\right]$$
$$= \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right]$$
$$\mathcal{E}_{\mathcal{A}} \cup (\mathcal{E}_{\mathcal{B}} \cap \mathcal{E}_{\mathcal{C}}) = \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})}. \tag{4.35}$$
$$(\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}) \cap (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{C}}) = \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] * \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{C}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{C}(\mathfrak{s})}\right]$$
$$= \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} * \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right]$$
$$(\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}) \cap (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{C}}) = \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} \tag{4.36}$$

From equation (4.35) and (4.36)

$$\mathcal{E}_{\mathcal{A}} \cup (\mathcal{E}_{\mathcal{B}} \cap \mathcal{E}_{\mathcal{C}}) = (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}) \cap (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{C}})$$

Case 3 $\aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{C}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s})$

$$\mathcal{E}_{\mathcal{A}} \cup (\mathcal{E}_{\mathcal{B}} \cap \mathcal{E}_{\mathcal{C}}) = \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \left(\aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})} * \aleph_{C}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{C}(\mathfrak{s})}\right)\right]$$
$$= \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{C}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{C}(\mathfrak{s})}\right]$$
$$\mathcal{E}_{\mathcal{A}} \cup (\mathcal{E}_{\mathcal{B}} \cap \mathcal{E}_{\mathcal{C}}) = \aleph_{C}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{C}(\mathfrak{s})}. \tag{4.37}$$
$$(\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}) \cap (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{C}}) = \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] * \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{C}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{C}(\mathfrak{s})}\right]$$
$$= \left[\aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})} * \aleph_{C}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{C}(\mathfrak{s})}\right]$$
$$(\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}) \cap (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{C}}) = \aleph_{C}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{C}(\mathfrak{s})} \tag{4.38}$$

From equation (4.37) and (4.38)

$$\mathcal{E}_{\mathcal{A}} \cup (\mathcal{E}_{\mathcal{B}} \cap \mathcal{E}_{\mathcal{C}}) = (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}) \cap (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{C}}).$$

Case 4 $\aleph_{\mathcal{E}_{\mathcal{C}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s})$

$$\begin{aligned} \mathcal{E}_{\mathcal{A}} \cup (\mathcal{E}_{\mathcal{B}} \cap \mathcal{E}_{\mathcal{C}}) &= \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{\mathcal{A}}(\mathfrak{s})}} \oplus \left(\aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{\mathcal{B}}(\mathfrak{s})}} * \aleph_{C}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{C}(\mathfrak{s})}}\right)\right] \\ &= \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{\mathcal{A}}(\mathfrak{s})}} \oplus \aleph_{C}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{C}(\mathfrak{s})}}\right] \\ \mathcal{E}_{\mathcal{A}} \cup (\mathcal{E}_{\mathcal{B}} \cap \mathcal{E}_{\mathcal{C}}) &= \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{\mathcal{A}}(\mathfrak{s})}}. \qquad (4.39) \\ (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}) \cap (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{C}}) &= \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{\mathcal{A}}(\mathfrak{s})}} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{\mathcal{B}}(\mathfrak{s})}}\right] * \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{\mathcal{A}}(\mathfrak{s})}} \oplus \aleph_{C}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{C}(\mathfrak{s})}}\right] \\ &= \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{\mathcal{A}}(\mathfrak{s})}} * \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{\mathcal{A}}(\mathfrak{s})}}\right] \\ (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}) \cap (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{C}}) &= \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{\mathcal{A}}(\mathfrak{s})}} \qquad (4.40) \end{aligned}$$

From equation (4.39) and (4.40),

$$\mathcal{E}_{\mathcal{A}} \cup (\mathcal{E}_{\mathcal{B}} \cap \mathcal{E}_{\mathcal{C}}) = (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}) \cap (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{C}})$$

Case 5 $\aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{C}}}(\mathfrak{s})$

$$\begin{aligned} \mathcal{E}_{\mathcal{A}} \cup (\mathcal{E}_{\mathcal{B}} \cap \mathcal{E}_{\mathcal{C}}) &= \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{\mathcal{A}}(\mathfrak{s})}} \oplus \left(\aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{\mathcal{B}}(\mathfrak{s})}} * \aleph_{C}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{C}(\mathfrak{s})}}\right)\right] \\ &= \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{\mathcal{A}}(\mathfrak{s})}} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{\mathcal{B}}(\mathfrak{s})}}\right] \\ \mathcal{E}_{\mathcal{A}} \cup (\mathcal{E}_{\mathcal{B}} \cap \mathcal{E}_{\mathcal{C}}) &= \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{\mathcal{A}}(\mathfrak{s})}}. \qquad (4.41) \\ (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}) \cap (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{C}}) &= \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{\mathcal{A}}(\mathfrak{s})}} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{\mathcal{B}}(\mathfrak{s})}}\right] * \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{\mathcal{A}}(\mathfrak{s})}} \oplus \aleph_{C}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{C}(\mathfrak{s})}}\right] \\ &= \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{\mathcal{A}}(\mathfrak{s})}} * \aleph_{C}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{C}(\mathfrak{s})}}\right] \\ (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}) \cap (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{C}}) &= \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{\mathcal{A}}(\mathfrak{s})}} \qquad (4.42) \end{aligned}$$

From equation (4.41) and (4.42)

$$\mathcal{E}_{\mathcal{A}} \cup (\mathcal{E}_{\mathcal{B}} \cap \mathcal{E}_{\mathcal{C}}) = (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}) \cap (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{C}})$$

Case 6 $\aleph_{\mathcal{E}_{\mathcal{C}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s})$

$$\begin{aligned} \mathcal{E}_{\mathcal{A}} \cup (\mathcal{E}_{\mathcal{B}} \cap \mathcal{E}_{\mathcal{C}}) &= \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{\mathcal{A}}(\mathfrak{s})}} \oplus \left(\aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{\mathcal{B}}(\mathfrak{s})}} * \aleph_{C}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{C}(\mathfrak{s})}}\right)\right] \\ &= \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{\mathcal{A}}(\mathfrak{s})}} \oplus \aleph_{C}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{C}(\mathfrak{s})}}\right] \\ \mathcal{E}_{\mathcal{A}} \cup (\mathcal{E}_{\mathcal{B}} \cap \mathcal{E}_{\mathcal{C}}) &= \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{\mathcal{A}}(\mathfrak{s})}}. \qquad (4.43) \\ (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}) \cap (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{C}}) &= \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{\mathcal{A}}(\mathfrak{s})}} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{\mathcal{B}}(\mathfrak{s})}}\right] * \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{\mathcal{A}}(\mathfrak{s})}} \oplus \aleph_{C}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{C}(\mathfrak{s})}}\right] \\ &= \left[\aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{\mathcal{B}}(\mathfrak{s})}} * \aleph_{R}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{R}(\mathfrak{s})}}\right] \\ (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}) \cap (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{C}}) &= \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}^{\aleph_{\mathcal{A}}(\mathfrak{s})}} \qquad (4.44) \end{aligned}$$

From equation (4.43) and (4.44), we have

$$\mathcal{E}_{\mathcal{A}} \cup (\mathcal{E}_{\mathcal{B}} \cap \mathcal{E}_{\mathcal{C}}) = (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}) \cap (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{C}})$$

The Law is valid for all above cases.
Similar way the distributive law of intersection over union is prove.

$$\mathcal{E}_{\mathcal{A}} \cap (\mathcal{E}_{\mathcal{B}} \cup \mathcal{E}_{\mathcal{C}}) = (\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}) \cup (\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{C}})$$

Theorem 6. *For any $\mathcal{EFS}$ $\mathcal{E}_{\mathcal{A}}$, the union, intersection, complement function with the same function for determining the phase term satisfy the following:*

i . $\mathcal{E}_{\mathcal{A}} \cup {\mathcal{E}_{\mathcal{A}}}^c = \mathcal{E}_{\mathcal{A}}$ *or* $\mathcal{E}_{\mathcal{A}} \cup {\mathcal{E}_{\mathcal{A}}}^c = {\mathcal{E}_{\mathcal{A}}}^c$.

ii . $\mathcal{E}_{\mathcal{A}} \cap {\mathcal{E}_{\mathcal{A}}}^c = \mathcal{E}_{\mathcal{A}}$ *or* $\mathcal{E}_{\mathcal{A}} \cap {\mathcal{E}_{\mathcal{A}}}^c = {\mathcal{E}_{\mathcal{A}}}^c$.

Proof. To prove (i) and (ii), two cases arise here.

i . $\mathcal{E}_{\mathcal{A}} \cup {\mathcal{E}_{\mathcal{A}}}^c = \mathcal{E}_{\mathcal{A}}$ or $\mathcal{E}_{\mathcal{A}} \cup {\mathcal{E}_{\mathcal{A}}}^c = {\mathcal{E}_{\mathcal{A}}}^c$.

Case 1 .

$$\begin{aligned}
\aleph^c_{\mathcal{A}}(\mathfrak{s}) &\leq \aleph_{\mathcal{A}}(\mathfrak{s}) \\
\mathcal{E}_{\mathcal{A}} \cup {\mathcal{E}_{\mathcal{A}}}^c &= \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph^c_{\mathcal{A}}(\mathfrak{s})}\right] \\
&= \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{\mathcal{A}}(\mathfrak{s})} \\
&= \mathcal{E}_{\mathcal{A}}.
\end{aligned}$$

Case 2 .

$$\begin{aligned}
\aleph_{\mathcal{A}}(\mathfrak{s}) &\leq \aleph^c_{\mathcal{A}}(\mathfrak{s}) \\
\mathcal{E}_{\mathcal{A}} \cup {\mathcal{E}_{\mathcal{A}}}^c &= \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph^c_{\mathcal{A}}(\mathfrak{s})}\right] \\
&= \aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph^c_{\mathcal{A}}(\mathfrak{s})} \\
&= {\mathcal{E}_{\mathcal{A}}}^c.
\end{aligned}$$

ii .$\mathcal{E}_{\mathcal{A}} \cap {\mathcal{E}_{\mathcal{A}}}^c = \mathcal{E}_{\mathcal{A}}$ or $\mathcal{E}_{\mathcal{A}} \cap {\mathcal{E}_{\mathcal{A}}}^c = {\mathcal{E}_{\mathcal{A}}}^c$.

Case 1 .

$$\begin{aligned}
\aleph^c_{\mathcal{A}}(\mathfrak{s}) &\leq \aleph_{\mathcal{A}}(\mathfrak{s}) \\
\mathcal{E}_{\mathcal{A}} \cap {\mathcal{E}_{\mathcal{A}}}^c &= \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{\mathcal{A}}(\mathfrak{s})} * \aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph^c_{\mathcal{A}}(\mathfrak{s})}\right] \\
&= \aleph^c_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph^c_{\mathcal{A}}(\mathfrak{s})} \\
&= {\mathcal{E}_{\mathcal{A}}}^c.
\end{aligned}$$

Case 2 .

$$\begin{aligned}\aleph_{\mathcal{A}}(\mathfrak{s}) &\leq \aleph^{c}_{\mathcal{A}}(\mathfrak{s})\\ \mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{A}}{}^{c} &= \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{\mathcal{A}}(\mathfrak{s})} * \aleph^{c}_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph^{c}_{\mathcal{A}}(\mathfrak{s})}\right]\\ &= \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{\mathcal{A}}(\mathfrak{s})}\\ &= \mathcal{E}_{\mathcal{A}}.\end{aligned}$$

Theorem 7. *For any $\mathcal{EFS}$s $\mathcal{E}_{\mathcal{A}}$ and $\mathcal{E}_{\mathcal{B}}$ over a crisp set $\mathfrak{Z}$, union and intersection function with the max function for determining the phase term does not satisfy the absorption law.*

Proof. The absorption laws for crisp set are $\mathcal{EFS}$s $\mathcal{E}_{\mathcal{A}}$ and $\mathcal{E}_{\mathcal{B}}$, the absorption laws do not hold. If $\aleph_{\mathcal{A}}(\mathfrak{s}) \leq \aleph_{\mathcal{B}}(\mathfrak{s})$.

$$\begin{aligned}\mathcal{E}_{\mathcal{A}} \cap (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}) &= \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{\mathcal{A}}(\mathfrak{s})} * \left(\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{\mathcal{B}}(\mathfrak{s})}\right)\right]\\ &= \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{\mathcal{A}}(\mathfrak{s})} * \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{\mathcal{B}}(\mathfrak{s})}\\ &= \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{\mathcal{B}}(\mathfrak{s})} \neq \mathcal{E}_{\mathcal{A}}.\\ \mathcal{E}_{\mathcal{A}} \cup (\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}) &= \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \left(\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{\mathcal{A}}(\mathfrak{s})} * \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{\mathcal{B}}(\mathfrak{s})}\right)\right]\\ &= \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{\mathcal{A}}(\mathfrak{s})}\\ &= \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{\mathcal{B}}(\mathfrak{s})} \neq \mathcal{E}_{\mathcal{A}}.\end{aligned}$$

Also if $\aleph_{\mathcal{B}}(\mathfrak{s}) \leq \aleph_{\mathcal{A}}(\mathfrak{s})$.

$$\begin{aligned}\mathcal{E}_{\mathcal{A}} \cup (\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}) &= \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{\mathcal{A}}(\mathfrak{s})} * \left(\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{\mathcal{B}}(\mathfrak{s})}\right)\right]\\ &= \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{\mathcal{A}}(\mathfrak{s})} * \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{\mathcal{A}}(\mathfrak{s})}\\ &= \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{\mathcal{B}}(\mathfrak{s})} \neq \mathcal{E}_{\mathcal{A}}.\\ \mathcal{E}_{\mathcal{A}} \cup (\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}) &= \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \left(\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{\mathcal{A}}(\mathfrak{s})} * \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{\mathcal{B}}(\mathfrak{s})}\right)\right]\\ &= \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{\mathcal{B}}(\mathfrak{s})}\\ &= \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{\mathcal{B}}(\mathfrak{s})} \neq \mathcal{E}_{\mathcal{A}}.\end{aligned}$$

Hence, the absorption law does not hold for any $\mathcal{EFS}$s.

Theorem 8. *For any $\mathcal{EFS}$s $\mathcal{E}_{\mathcal{A}}, \mathcal{E}_{\mathcal{B}}$ and E_C, the complement, intersection, union function for determining the phase term does not satisfy the distributive laws.*

Proof. The distributive law of union over intersection is $\mathcal{E}_{\mathcal{A}} \cup (\mathcal{E}_{\mathcal{B}} \cap E_C) = (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}) \cap (\mathcal{E}_{\mathcal{A}} \cup E_C)$. If $\aleph_{\mathcal{E}_{\mathcal{A}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{\mathcal{B}}}(\mathfrak{s}) \leq \aleph_{\mathcal{E}_{C}}(\mathfrak{s})$.

$$\mathcal{E}_{\mathcal{A}} \cup (\mathcal{E}_{\mathcal{B}} \cap E_C) = \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \left(\aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{\mathcal{B}}(\mathfrak{s})} * \aleph_{C}(\mathfrak{s})\mathfrak{e}^{-\intercal\aleph_{C}(\mathfrak{s})}\right)\right]$$

$$= \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{C}(\mathfrak{s})}$$

$$= \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{C}(\mathfrak{s})} \tag{4.45}$$

$$(\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}) \cap (\mathcal{E}_{\mathcal{A}} \cup E_C) = \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] * \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{C}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{C}(\mathfrak{s})}\right]$$

$$= \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})} * \aleph_{C}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{C}(\mathfrak{s})}$$

$$= \aleph_{B}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{A}(\mathfrak{s})}. \tag{4.46}$$

From equation 4.45 and 4.46, we have

$$\mathcal{E}_{\mathcal{A}} \cup (\mathcal{E}_{\mathcal{B}} \cap E_C) \neq (\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}) \cap (\mathcal{E}_{\mathcal{A}} \cup E_C)$$

Now distributive law of intersection over union

$$\mathcal{E}_{\mathcal{A}} \cap (\mathcal{E}_{\mathcal{B}} \cup E_C) = \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} * \left(\aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})} \oplus \aleph_{C}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{C}(\mathfrak{s})}\right)\right]$$

$$= \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} * \aleph_{C}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{C}(\mathfrak{s})}$$

$$= \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{C}(\mathfrak{s})} \tag{4.47}$$

$$(\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}) \cup (\mathcal{E}_{\mathcal{A}} \cap E_C) = \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} * \aleph_{\mathcal{B}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{B}}(\mathfrak{s})}\right] \oplus \left[\aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} * \aleph_{C}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{C}(\mathfrak{s})}\right]$$

$$= \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} \oplus \aleph_{A}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{A}(\mathfrak{s})}$$

$$= \aleph_{\mathcal{A}}(\mathfrak{s})\mathfrak{e}^{-\mathsf{T}\aleph_{\mathcal{A}}(\mathfrak{s})} \tag{4.48}$$

From equation 4.47 and 4.48, we have

$$\mathcal{E}_{\mathcal{A}} \cap (\mathcal{E}_{\mathcal{B}} \cup E_C) \neq (\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}) \cup (\mathcal{E}_{\mathcal{A}} \cap E_C).$$

Hence, the absorption law does not hold for any $\mathcal{EFS}$s.

5. Application: AI-Powered Investment Decision-Making Using the Weighted Mean Method

Investments are essential for people because they help grow wealth, provide financial security, and ensure a stable future. By investing, individuals can increase their money over time through interest, dividends, or asset appreciation, rather than relying solely on savings. Investments also protect against inflation, which reduces the value of money, ensuring that purchasing power remains strong. They also act as a source of financial relief in cases of emergencies and enable individuals to fulfill long-term objectives like purchasing a house, covering education expenses, or saving for retirement. Investments are also capable of creating passive income from dividends, rental properties, or bonds, giving financial security without hard labor.

Investment diversification among various assets minimizes risk even more, providing financial security and a balanced future. Finally, achieving long-term wealth and financial freedom requires investment. Nowadays, investing money poses a variety of difficulties for individuals, making it difficult to increase wealth and safeguard one?s financial future. A significant obstacle is a lack of financial literacy, since most individuals do not know where or how to start or how investments work. Limited capital is another problem; some individuals believe that only rich people can invest since they lack sufficient funds. Making the right investment decision can be complex due to uncertainty, market fluctuations, and multiple investment options.

Our exponential fuzzy investment decision system is designed to assist investors in choosing the most suitable investment scheme based on a exponential fuzzy set approach. This application evaluates investment opportunities by considering multiple factors, such as risk level, expected return, investment horizon, and financial goals. Unlike fuzzy logic decision-making models, exponential fuzzy logic enables a more flexible and human-like reasoning process, allowing for better handling of imprecise and uncertain data this application helps you make data-driven investment decisions with confidence.

In this section, we use the weighted mean method for AI-powered investment decision-making, where financial experts (with different importance weights) assess stocks, and their opinions are aggregated using $\mathcal{EFS}$s.

Let $\mathcal{S} = \{\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4, \mathfrak{s}_5, \mathfrak{s}_6, \mathfrak{s}_7, \mathfrak{s}_8, \mathfrak{s}_9, \mathfrak{s}_{10}\}$ be the set of Investments where $\mathfrak{s}_1$ (Bonds), $\mathfrak{s}_2$ (Public provident fund), $\mathfrak{s}_3$ (Stocks), $\mathfrak{s}_4$ (Real estate), $\mathfrak{s}_5$ (Treasurys), $\mathfrak{s}_6$ (Cryptocurrencies), $\mathfrak{s}_7$ (Mutual funds), $\mathfrak{s}_8$ (Fixed deposits), $\mathfrak{s}_9$ (Gold), $\mathfrak{s}_{10}$ (National Pension Scheme) . The decision-makers are:

- $\mathcal{E}_1$: Junior Analyst (Weight $= w_1 = 0.2$)
- $\mathcal{E}_2$: Senior Analyst (Weight $= w_2 = 0.3$)
- $\mathcal{E}_3$: AI Model (Weight $= w_3 = 0.5$)

where $\sum_{j=1}^{3} w_j = 1$ and $\intercal = 2$.

Experts assign fuzzy membership values $\aleph_{A_j}(\mathfrak{s})$ to each stock:

Investments	$\aleph_{A_1}(\mathfrak{s})$	$\aleph_{A_2}(\mathfrak{s})$	$\aleph_{A_3}(\mathfrak{s})$
$\mathfrak{s}_1$	0.3	0.4	0.5
$\mathfrak{s}_2$	0.6	0.7	0.8
$\mathfrak{s}_3$	0.8	0.9	1.0
$\mathfrak{s}_4$	0.5	0.6	0.7
$\mathfrak{s}_5$	0.9	1.0	0.9
$\mathfrak{s}_6$	0.7	0.4	0.8
$\mathfrak{s}_7$	0.5	1.0	0.3
$\mathfrak{s}_8$	1.0	0.3	0.2
$\mathfrak{s}_9$	0.1	0.5	0.9
$\mathfrak{s}_{10}$	0.7	1.0	0.4

Table 1: Fuzzy Membership Values for Each Stock

The exponential fuzzy membership formula is

$$\aleph_{\mathcal{E}_{\mathcal{A}_j}}(\mathfrak{s}) = \aleph_{\mathcal{A}_j}(\mathfrak{s})\mathfrak{e}^{-2\aleph_{\mathcal{A}_j}(\mathfrak{s})} \tag{5.1}$$

Investment	$\aleph_{E_{A_1}}(\mathfrak{s})$	$\aleph_{E_{A_2}}(\mathfrak{s})$	$\aleph_{E_{A_3}}(\mathfrak{s})$
$\mathfrak{s}_1$	0.1646	0.1797	0.1839
$\mathfrak{s}_2$	0.1807	0.1726	0.1615
$\mathfrak{s}_3$	0.1615	0.1487	0.1353
$\mathfrak{s}_4$	0.1839	0.1807	0.1726
$\mathfrak{s}_5$	0.1487	0.1353	0.1487
$\mathfrak{s}_6$	0.1726	0.1797	0.1615
$\mathfrak{s}_7$	0.1839	0.1353	0.1646
$\mathfrak{s}_8$	0.1353	0.1646	0.1340
$\mathfrak{s}_9$	0.0818	0.1839	0.1487
$\mathfrak{s}_{10}$	0.1726	0.1353	0.1737

Table 2: exponential fuzzy membership values for each stock

Weighted Mean Aggregation Using the formula:

$$\aleph_{\mathcal{E}_G}(\mathfrak{s}) = \sum_{j=1}^{3} w_j \aleph_{\mathcal{E}_{A_j}}(\mathfrak{s}) \tag{5.2}$$

We compute the aggregated fuzzy values:

Investment	$\aleph_{\mathcal{E}}(\mathfrak{s})$
$\mathfrak{s}_1$	0.1787
$\mathfrak{s}_2$	0.1686
$\mathfrak{s}_3$	0.1445
$\mathfrak{s}_4$	0.1772
$\mathfrak{s}_5$	0.1446
$\mathfrak{s}_6$	0.1691
$\mathfrak{s}_7$	0.1596
$\mathfrak{s}_8$	0.1434
$\mathfrak{s}_9$	0.1458
$\mathfrak{s}_{10}$	0.1619

Table 3: Aggregated Membership Values of $\mathcal{EFS}$

The stock with the highest aggregated value in Table 3:

$$\max_{\mathfrak{s}\in\mathfrak{Z}} \aleph_{\mathcal{E}}(\mathfrak{s}) = \aleph_{\mathcal{E}}(\mathfrak{s}_1) = 0.1784 \tag{5.3}$$

Thus, the best investment decision is Bonds.

5.1. Sensitivity Analysis

Sensitivity Analysis Sensitivity analysis assists in assessing the effect of changes in input parameters on the investment decision. By varying expert-specified weights or membership values, we can find the stability of the exponential fuzzy investment decision system. Sensitivity analysis investigates the effect of membership value changes on the investment ranking.

5.1.1. Initial Aggregation Analysis

With the fuzzy membership values for each stock from Table 2, the fuzzy aggregated value for investment decision is:

$$\max_{\mathfrak{s}\in\mathfrak{Z}} \aleph_{\mathcal{E}}(\mathfrak{s}) = \aleph_{\mathcal{E}}(\mathfrak{s}_1) = 0.1784$$

Therefore, the optimal investment choice is Bonds ($\mathfrak{s}_1$).

5.1.2. Scenario 1: Membership Value Changes

Scenario 1: Membership Value Changes To measure stability, we adjust the membership values by raising and lowering each entry by 5% and recompute the aggregated values.

Case 1.1: Raise Membership Values by 5% If membership values are raised by 5%, new fuzzy values are recomputed, and the new maximum aggregated value is:

$$\max_{\mathfrak{s}\in\mathfrak{Z}} \aleph_{\mathcal{E}}(\mathfrak{s}) = \aleph_{\mathcal{E}}(\mathfrak{s}_1{}') = 0.1832$$

The best investment choice does not change Bonds ($\mathfrak{s}_1$).

Case 1.2:Lower Membership Values by 5% If all membership values are reduced by 5%, the new maximum aggregated value is:

$$\max_{\mathfrak{s}\in\mathfrak{Z}} \aleph_{\mathcal{E}}(\mathfrak{s}) = \aleph_{\mathcal{E}}(\mathfrak{s}_1{}'') = 0.1741$$

Once again, Bonds ($\mathfrak{s}_1$) is the best choice.

Scenario 2: Varying Expert Weights Final ranking is determined by weights allocated to investment criteria. Let's assume two varied weight scenarios

Case 2.1: Same Weights for All Factors Allocating same weights to all three fuzzy membership values:

$$w_1 = w_2 = w_3 = 0.33$$

Recalculating aggregated values, we observe $\mathfrak{s}_1$ continues to have the maximum value, verifying the consistency of the decision.

Case 2.2:Increased Weight on Risk Factor If the expert gives greater weight (0.5) to the risk factor but leaves the others unchanged at 0.25, the ranking is slightly different. But Bonds ($\mathfrak{s}_1$) is still among the best investment options.

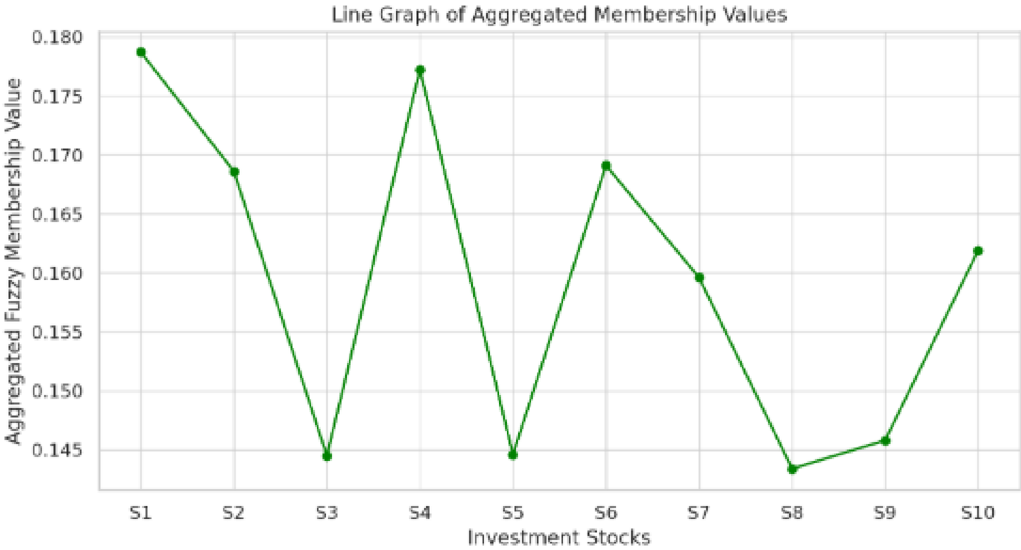

Figure 2: Investment: Stocks

5.2. Comparison Analysis of Exponential Fuzzy Sets $\mathcal{EFS}$ with Traditional Fuzzy Models

One of the key advantages of $\mathcal{EFS}$ over traditional fuzzy models is their ability to handle uncertainty more effectively. Traditional fuzzy models assign a direct membership degree to an investment option, which may not fully capture the gradual decline in confidence as uncertainty increases. In contrast, EFS introduces an exponential decay function, which provides a more refined approach to uncertainty modeling. This is particularly beneficial in investment decision-making, where risk levels vary significantly, and a more nuanced representation of uncertainty leads to better investment choices.

Flexibility in decision-making is another critical factor where EFS outperforms traditional fuzzy models. Traditional fuzzy logic uses a linear membership assignment, making it less adaptable in differentiating between investments with similar membership values. $\mathcal{EFS}$, however, applies an exponential transformation, ensuring a smoother transition between choices. This allows for a more sensitive response to small variations in investment attributes, improving the accuracy of financial assessments.

Numerical stability and sensitivity analysis further highlight the advantages of EFS. Traditional fuzzy models often experience abrupt shifts in decision outcomes when input data changes. In contrast, $\mathcal{EFS}$ incorporates a sensitivity mechanism that stabilizes rankings even with small fluctuations in the membership values. As demonstrated in the sensitivity analysis, adjusting membership values by 5% does not significantly alter the ranking of the best investment choice in the $\mathcal{EFS}$ model, confirming its robustness in real-world applications.

A graphical comparison further strengthens the argument for EFS. In a traditional fuzzy model, membership values are assigned directly, leading to a more rigid classification of investment options. The $\mathcal{EFS}$ model, however, applies an exponential transformation that results in a sharper differentiation of choices. This means that $\mathcal{EFS}$ can more effectively distinguish between investments with slight variations in risk and return, leading to more precise decision-making.

Finally, in practical applications, $\mathcal{EFS}$ proves to be a more effective tool for investment decision-making. Traditional fuzzy models rely on fixed weight allocations and may struggle in highly uncertain investment environments. $\mathcal{EFS}$, by incorporating an exponential factor, ensures a more dynamic and realistic risk assessment. This feature makes it particularly useful for financial forecasting, portfolio optimization, and strategic investment planning. By improving uncertainty management and decision flexibility, $\mathcal{EFS}$ provides investors with a more reliable approach to selecting optimal investment schemes.

6. Conclusion

The $\mathcal{EFS}$ offers a strong tool for managing uncertainty in mathematical modeling and decision-making. With the use of an exponential membership function, $\mathcal{EFS}$ successfully describes systems with high uncertainty rates of change. This paper has discussed the basic properties and operations of $\mathcal{EFS}$, as well as its uses in AI-based investment decisions. Our exponential fuzzy investment decision system showcases the strengths of $\mathcal{EFS}$ in making optimal investment decisions. Compared to conventional fuzzy models, $\mathcal{EFS}$ supports more accurate and dynamic decision-making, especially in unstable financial markets. The integration of the weighted mean method within the $\mathcal{EFS}$ system maximizes investment analysis by combining expert opinions efficiently.

6.1. Future Research Directions

The $\mathcal{EFS}$ concept we can extend in Intuitionistic fuzzy set, Neutrosophic fuzzy set all areas like graph theory, BCI Algebra and different type of algebras. Then decision making problems.

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