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An Innovative Analytical Result of Two-Dimensional Heat Equation Using Joint Mechanism of Natural Transform and Adomian Decomposition Method

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Abstract. This research article has put forward an innovative analytical result of the 2D heat equation having a source term. The aforesaid non-homogenous PDE is solved through a new hybrid mechanism. whereas the technique supports the solution process, commencing with the parametric form of the 2D heat equation. Furthermore, this hybrid mechanism which consists of Natural transform (NT) and a series solution method of Adomian decomposition method (ADM) is applied properly. Next, the solution that is obtained for the unknown function is presented in series form. During the computational process, it was observed that the proposed mechanism is less time-consuming and efficient with accurate results. Which shows that the mechanism (proposed mechanism) makes a very useful contribution to the analytical solution of the 2D heat equation. The paper is properly supported with appropriate examples to verify the claim.

2020 Mathematics Subject Classifications: 26A33, 35Exx, 35R13

Key Words and Phrases: 2D Heat Equation; Natural Transform; Adomian Decomposition Method

1. Introduction

It is obvious that the subject of fractional calculus is (as old as) the differential calculus and is the generalization of the ordinary (differentiation and integration) to arbitrary (non-integer) order [1]. The popularity of fractional calculus not only attracts the attention of mathematicians but also attracts the attention of engineers and physicists. Due to which fractional calculus appears in many fields like electrochemistry, viscoelasticity,

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rheology, electromagnetism, etc., [2, 3]. Moreover, some new investigation/contribution to the theory of Fractional Differential Equation (FDE) like the results of existence for a three-point third-order nonlocal boundary value problem, structural stability of solutions, and uniqueness of nonlinear ordinary differential equations [4, 5].

Additionally, a number of fields of both pure and applied mathematics can be expanded using fractional calculus. In this regard, our interest is to extend the mechanism to such-like problems where uncertainty arises, and for such-like problems, Zadeh introduced fuzzy concepts in 1965 [6]. This approach of fuzziness has also been used in many fields like topological fuzziness, fuzzy fixed point theory, fuzzy system control, and so forth. Later on, fuzzy mapping and control were introduced together with the fuzzy set notion by Chang and Zadeh [7]. Moreover, many researchers introduce elementary calculus in a fuzzy sense (fuzzy calculus) on the basis of mapping (fuzzy mapping) and control (i.e., equations involving derivatives of fuzzy mappings) [8]. These mappings can be thought of as fuzzy relations as well. The integral of such fuzzy mappings over a crisp interval is defined using Zadeh's extension approach as well [9]. In addition, it is observed from the literature review that the problem of integrating mapping over fuzzy domains has been received. As there is not a unique approach, many definitions have been presented, each having its own approach [10].

Further, it may be noted that in recent studies, fuzzy fractional differential integral equations (FFDIEs) have an important role in the field of sciences (physical science(s)). Whereas, the basic concept for fuzzy integral equation(s) was presented by Dobius and Prade in [8].

Due to the fact that there are so many real-world issues in computer science, artificial intelligence, physics, and industrial engineering, fuzzy fractional integral equations have a lot more applications. which can be expressed by means of/through such equations. Additionally, operation research issues may be transformed into uncertain processes with fractional order derivative(s).

Furthermore, notice that partial differential equations (PDEs) have convincing properties in modeling various real-world physical problems, such as propagation of sound in wave(s) form, heat transfer phenomena, and water waves, etc. [11]. Additionally, fractional calculus is receiving more attention because numerical modeling of partial integro-differential equations of fractional order reveals intriguing characteristics in a variety of scientific fields [12]. In this regard, O.A. Arqub [13] considers a powerful tool/algorithm to investigate the solutions of singular Fredholm time-fractional partial integro-differential equations having Dirichlet functions. And hence, provide an appropriate solution in infinite series with accurate structures. Moreover, the aforesaid area has been enlarged to ordinary-order (fractional) calculus, and numerous successful outcomes are well-documented in the available literature [11]. Furthermore, fuzzy ordinary (fractional) order partial differential equation(s) have various properties in the field of dynamical systems, damped nonlinear string, nonlinear propagation of traveling wave, telecommunications, and in the field of electronics etc.,[14]. In addition, Non-linear Partial Differential Equations(NLPDEs) also have various properties in the field of physic and engineering. In more recent times, partial

differential equations have been approached using nonlinear fractional order differential equations. In this connection, different type of fractional-operators (Riemann-Liouville fractional-order, Caputo-Fabrizio, etc.,) which is based on the exponential kernel have been developed by different researchers. These fractional operators which is used in analyzing many non-linear real-word problems [15–18].
Being a Fuzzy Fractional Partial Differential Equation (FFPDEs), Heat Equation (HE) is one of the most important and widely studied topics in the world of mathematics, and the study of the aforesaid PDE is considered as a vital to the area of partial differential equations. Now to investigate the aforesaid FFPDEs, several valuable and important mechanism were used in the past, like the mechanism of Fourier integral transform, Sumudu transform, and laplace transform etc., On the other hand, some analytical method such as, Homotopy Perturbation Methods (HPMs), SAMD, ADM, LADM, taylors series method etc., were also utilized accordingly [19, 20]. To the best of our study, the aforesaid technique (joint mechanism of NT and ADM) have not been used to handled such like problems. Furthermore, the said equation(heat equation in 2-dimension) have been solved by double Laplace transform with no source term and are solved by HPM with a source term. Here, we aim to investigate the analytical solution by a joint mechanism of NT and ADM.

$$\begin{cases} \frac{\partial^\beta}{\partial t^\beta}\varphi(x,\breve{\lambda},t,y) = \frac{\partial^2}{\partial x^2}\varphi(x,\breve{\lambda},t,y) + \frac{\partial^2}{\partial y^2}\varphi(x,\breve{\lambda},t,y) + \phi(x,\breve{\lambda},t,y), & 0<\beta<1, 0\le\breve{\lambda}\le 1 \\ \varphi(x,0,y) = \zeta(x,y). \end{cases} \tag{1}$$

Moreover, the parametric form of the above fuzzy Eq: (1) will be;

$$\begin{cases} \frac{\partial^\beta}{\partial t^\beta}\breve{\varphi}(x,\breve{\lambda},t,y) = \frac{\partial^2}{\partial x^2}\breve{\varphi}(x,\breve{\lambda},t,y) + \frac{\partial^2}{\partial y^2}\breve{u}(x,\breve{\lambda},t,y) + \breve{\phi}(x,\breve{\lambda},t,y), & 0<\beta<1, 0\le\lambda\le 1 \\ \breve{\varphi}(x,0,y) = \zeta(x,y). \end{cases} \tag{2}$$

Where, $\breve{\varphi}(x,\breve{\lambda},t,y) = \{\underline{\varphi}(x,\underline{\lambda},t,y), \overline{\varphi}(x,\overline{\lambda},t,y)\}, \underline{\lambda} = \lambda - 1, \overline{\lambda} = 1-\lambda. \quad \lambda\in[0,1]$

2. Preliminaries

Here, we'll talk about several fundamental findings that were employed in this research.

Definition 1. *[21] A fuzzy set (which is a class of objects) "$M = \{(x,\mu_m(x)), x\in R\}$", where $\mu(x)$ is the grade of membership function whose range is between 0 and 1, i.e., ($\mu(x): R\to[0,1]$) is said to be a (fuzzy number) if, this fuzzy set satisfy some conditions such as;*
M is normal, fuzzy convex, continuous (upper and lower) and the closure of the set is compact set.

Definition 2. *[21] Consider the fuzzy valued function $\varphi(x)$, then the Riemann-Liouville (RL) fuzzy fractional order derivative is defined as below;*

$$D^{\beta}_{RL}\varphi(x) = \begin{cases} \frac{1}{\Gamma(m-\beta)} \odot (\frac{d}{dx})^m \int_{t_0}^{x} (x-t)^{m-\beta-1} \odot \varphi(t)dt, & m-1 < \beta < m \\ (\frac{d}{dx})^{m-1}\varphi(t), & \beta = m-1 \end{cases} \tag{3}$$

In addition, for $x \in [t_0, T]$. The above derivatives (fractional derivative) is defined in terms of a fractional integral (instead) of an integer integral.

Definition 3. *[21] In the above Riemann-Liouville (RL) fuzzy fractional order derivative, if the integer (classical) order of the derivative is an operator(s) inside of the integral and operating on operand(s) function $\varphi(x) \in \mathcal{F}_{\mathcal{R}}, t \in [t_0, T]$*

$$D^{\beta}_{C_{gH}}\varphi(x) = \begin{cases} \frac{1}{\Gamma(m-\beta)} \odot \int_{t_0}^{x} (x-t)^{m-\beta-1} \odot \varphi^m(t)dt, & m-1 < \beta < m \\ (\frac{d}{dx})^{m-1}\varphi(t), & \beta = m-1 \end{cases} \tag{4}$$

Definition 4. *[21] Mittage-Leffler function $E_\beta(\varphi)$ is defined as follows,*

$$E_\beta(\varphi) = \sum_{n=0}^{\infty} \frac{\varphi^n}{\Gamma(n\beta+1)}.$$

Definition 5. *[22] Natural Transform for a fuzzy function(s) "$\varphi(x,t)$" is defined as,*

$$\mathcal{R}(v,s) = \mathcal{N}[\varphi(x,t)] = \int_0^{\infty} (e)^{-st} \odot \varphi(x, vt)dt, t > 0.$$

Definition 6. *[23] In the Adomian Decomposition Method the fuzzy function(s) "$\varphi(x,t)$" can be shown in infinite series form as,*

$$\varphi(x,t) = \sum_{i=0}^{\infty} \varphi_i(x,t).$$

3. Mathematical Formulation of the Model

In this section we will show a general algorithm for the analytical solution(s) of parametric form of 2D heat equation as shown in Eq.(2). now, utilizing our suggested strategy, i-e, applying NT to both sides of Eq. (2) we have,

$$N\big(\frac{\partial^\beta}{\partial t^\beta}\breve{\varphi}(x,\breve{\lambda},t,y)\big) = N\bigg(\frac{\partial^2}{\partial x^2}\breve{\varphi}(x,\breve{\lambda},t,y) + \frac{\partial^2}{\partial y^2}\breve{\varphi}(x,\breve{\lambda},t,y) + \breve{\phi}(x,\breve{\lambda},t,y)\bigg). \tag{5}$$

$$\frac{s^\beta}{\varphi^\beta}N\bigg(\breve{\varphi}(x,\breve{\lambda},t,y)\bigg) - \sum_{i=0}^{\beta-1}\frac{s^{\beta-i-1}}{\varphi^{\beta-i}}\breve{\varphi}_i(0,x,y) = N\bigg(\frac{\partial^2}{\partial x^2}\breve{\varphi}(x,\breve{\lambda},t,y) + \frac{\partial^2}{\partial y^2}\breve{\varphi}(x,\breve{\lambda},t,y) + \breve{\phi}(x,\breve{\lambda},t,y)\bigg). \tag{6}$$

$$N\left(\breve{\varphi}(x,\breve{\lambda},t,y)\right) = \frac{1}{s}\breve{\zeta}(x,y)+\frac{\varphi^{\beta}}{s^{\beta}}N\left(\frac{\partial^2}{\partial x^2}\breve{\varphi}(x,\breve{\lambda},t,y)+\frac{\partial^2}{\partial y^2}\breve{\varphi}(x,\breve{\lambda},t,y)+\breve{\phi}(x,\breve{\lambda},t,y)\right). \quad (7)$$

Now applying inverse Natural transform, we have,

$$\breve{\varphi}(x,\breve{\lambda},t,y) = \breve{\zeta}(x,y)+N^{-1}\left(\frac{\varphi^{\beta}}{s^{\beta}}N\left(\frac{\partial^2}{\partial x^2}\breve{\varphi}(x,\breve{\lambda},t,y)+\frac{\partial^2}{\partial y^2}\breve{\varphi}(x,\breve{\lambda},t,y)+\breve{\phi}(x,\breve{\lambda},t,y)\right)\right) \quad (8)$$

In Adomian decomposition sense the unknown function can be shown in infinite series form as below;

$$\breve{\varphi}(x,\breve{\lambda},t,y) = \sum_{j=0}^{\infty}\breve{\varphi}_j(x,\breve{\lambda},t,y). \quad (9)$$

Plugging Eq.(9) in Eq.(8), we have,

$$\sum_{j=0}^{\infty}\breve{\varphi}_j(x,\breve{\lambda},t,y) = \breve{\zeta}(x,y)+N^{-1}\left(\frac{\varphi^{\beta}}{s^{\beta}}N\left(\frac{\partial^2}{\partial x^2}\sum_{j=0}^{\infty}\breve{\varphi}_j(x,\breve{\lambda},t,y)+\frac{\partial^2}{\partial y^2}\sum_{j=0}^{\infty}\breve{\varphi}_j(x,\breve{\lambda},t,y)+\breve{\phi}(x,\breve{\lambda},t,y)\right)\right). \quad (10)$$

$$\begin{cases} \breve{\varphi}_0(x,\breve{\lambda},t,y)+\breve{\varphi}_1(x,\breve{\lambda},t,y)+\breve{\varphi}_2(x,\breve{\lambda},t,y)+... = \breve{\zeta}(x,y)+N^{-1}\left(\frac{\varphi^{\beta}}{s^{\beta}}N\left(\frac{\partial^2}{\partial x^2}(\breve{\varphi}_0(x,\breve{\lambda},t,y)+\breve{\varphi}_1(x,\breve{\lambda},t,y)+\right.\right. \\ \left.\left.\breve{\varphi}_2(x,\breve{\lambda},t,y)+...)+\frac{\partial^2}{\partial y^2}(\breve{\varphi}_0(x,\breve{\lambda},t,y)+\breve{\varphi}_1(x,\breve{\lambda},t,y)+\breve{\varphi}_2(x,\breve{\lambda},t,y)+...)+\breve{\phi}(x,\breve{\lambda},t,y)\right)\right). \end{cases} \quad (11)$$

Comparing both the sides we have,

$$\breve{\varphi}_0(x,\breve{\lambda},t,y) = \breve{\zeta}(x,y). \quad (12)$$

$$\breve{\varphi}_1(x,\breve{\lambda},t,y) = N^{-1}\left(\frac{\varphi^{\beta}}{s^{\beta}}N\left(\frac{\partial^2}{\partial x^2}\breve{\varphi}_0(x,\breve{\lambda},t,y)+\frac{\partial^2}{\partial y^2}\breve{\varphi}_0(x,\breve{\lambda},t,y)+\breve{\phi}(x,\breve{\lambda},t,y)\right)\right). \quad (13)$$

$$\breve{\varphi}_2(x,\breve{\lambda},t,y) = N^{-1}\left(\frac{\varphi^{\beta}}{s^{\beta}}N\left(\frac{\partial^2}{\partial x^2}\breve{\varphi}_1(x,\breve{\lambda},t,y)+\frac{\partial^2}{\partial y^2}\breve{\varphi}_1(x,\breve{\lambda},t,y)\right)\right). \quad (14)$$

$$\vdots$$

$$\breve{\varphi}_n(x,\breve{\lambda},t,y) = N^{-1}\left(\frac{\varphi^{\beta}}{s^{\beta}}N\left(\frac{\partial^2}{\partial x^2}\breve{\varphi}_{n-1}(x,\breve{\lambda},t,y)+\frac{\partial^2}{\partial y^2}\breve{\varphi}_{n-1}(x,\breve{\lambda},t,y)\right)\right). \quad (15)$$

Where, $n \geq 1$.

4. Examples and Discussion

Example 1.

Let us assume the 2D fuzzy heat equation(s) as shown in [11].

$$\begin{cases} \frac{\partial^{\beta}}{\partial t^{\beta}}\breve{\varphi}(x,\breve{\lambda},t,y) = \frac{\partial^2}{\partial x^2}\breve{\varphi}(x,\breve{\lambda},t,y)+\frac{\partial^2}{\partial y^2}\breve{\varphi}(x,\breve{\lambda},t,y)+x+y+1, \quad 0<\breve{\lambda}\leq 1, 0<t. \\ \breve{\varphi}(x,0,y) = \breve{\lambda}e^{(-x-y)}, x<1, y>0. \end{cases} \quad (16)$$

Where $\breve{\lambda} = [\underline{\lambda}, \overline{\lambda}]$, such that $\underline{\lambda} = \lambda - 1, \overline{\lambda} = 1 - \lambda$ Now using equation(12) and (15), we can write as follows:

$$\breve{\varphi}_0(x, \breve{\lambda}, t, y) = \breve{\lambda} e^{-(x+y)},$$

$$\breve{\varphi}_1(x, \breve{\lambda}, t, y) = \breve{\lambda} 2e^{-(x+y)} + (x + y + 1)]\frac{t^{\beta}}{\Gamma(\beta + 1)},$$

$$\breve{\varphi}_2(x, \breve{\lambda}, t, y) = \breve{\lambda} 4e^{-(x+y)}\frac{t^{2\beta}}{\Gamma(2\beta + 1)},$$

$$\breve{\varphi}_3(x, \breve{\lambda}, t, y) = \breve{\lambda} 8e^{-(x+y)}\frac{t^{3\beta}}{\Gamma(3\beta + 1)},$$

$$\vdots\vdots\vdots$$

Proceeding in the same fashion the aforesaid analytical solution can be shown as,

$$\breve{\varphi}(x, \breve{\lambda}, t, y) = \breve{\varphi}_0(x, \breve{\lambda}, t, y) + \breve{\varphi}_1(x, \breve{\lambda}, t, y) + \breve{\varphi}_2(x, \breve{\lambda}, t, y) + \breve{\varphi}_3(x, \breve{\lambda}, t, y) \dots,$$

In the absence of the source term the solution is,

$$\breve{\varphi}_n(x, \breve{\lambda}, t, y) = \sum_{n=2}^{\infty} \frac{t^{n\beta}}{\Gamma(n\beta + 1)} \breve{\lambda} 2^n e^{(-x-y)}.$$

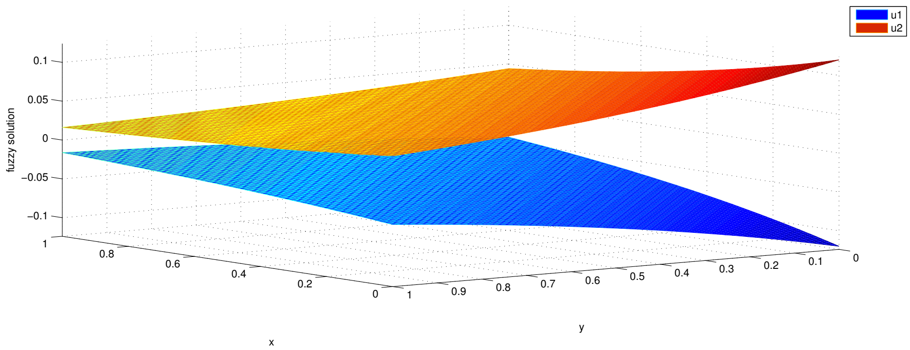

Figure 1: Fuzzy solution of Example 1.

Example 2.

Let's take another two-dimension fuzzy heat equation as in [11].

$$\begin{cases} \frac{\partial^{\beta}}{\partial t^{\beta}} \breve{\varphi}(x, \breve{\lambda}, t, y) = \frac{\partial^2}{\partial x^2} \breve{\varphi}(x, \breve{\lambda}, t, y) + \frac{\partial^2}{\partial y^2} \breve{\varphi}(x, \breve{\lambda}, t, y) + x + y + t^2, \quad \breve{\lambda} \in [0, 1], 0 < \beta < 1, 0 < t, \\ \breve{\varphi}(x, 0, y) = \breve{\lambda} sin(\pi(x + y)), 1 > (x, y) > 0. \end{cases} \tag{17}$$

Furthermore, $\breve{\lambda} = [\underline{\lambda}, \overline{\lambda}]$, such that $\underline{\lambda} = \lambda - 1, \overline{\lambda} = 1 - \lambda$. Now using equation(12) and (15), we can write as follows:

$$\breve{\varphi}_0(x, \breve{\lambda}, t, y) \breve{\lambda} \sin \pi(x + y),$$

$$\breve{\varphi}_1(x,\breve{\lambda},t,y)\breve{\lambda}[-2\pi^2\sin\pi(x+y)+(x+y)]\frac{t^{\beta}}{\Gamma(\beta+1)}+\frac{t^{\beta+2}}{\Gamma(\beta+3)},$$
$$\breve{\varphi}_2(x,\breve{\lambda},t,y)\breve{\lambda}[4\pi^4\sin\pi(x+y)]\frac{t^{2\beta}}{\Gamma(2\beta+1)},$$
$$\breve{\varphi}_3(x,\breve{\lambda},t,y)\breve{\lambda}[8\pi^6\sin\pi(x+y)]\frac{t^{3\beta}}{\Gamma(3\beta+1)},$$

and so on.
Proceeding in the same fashion the final solution(s) can be written as

$$\breve{\varphi}(x,\breve{\lambda},t,y)\breve{\varphi}_0(x,\breve{\lambda},t,y)+\breve{\varphi}_1(x,\breve{\lambda},t,y) + \breve{\varphi}_2(x,\breve{\lambda},t,y)+\breve{\varphi}_3(x,\breve{\lambda},t,y)+\ldots.$$

Now if the source term is zero then the solution will take the form

$$\breve{\varphi}_n(x,\breve{\lambda},t,y)=\sum_{n=2}^{\infty}\frac{(-1)^n2^nt^{n\beta}\pi^{2n}}{\Gamma(n\beta+1)}\breve{\lambda}\sin(\pi(x+y)).$$

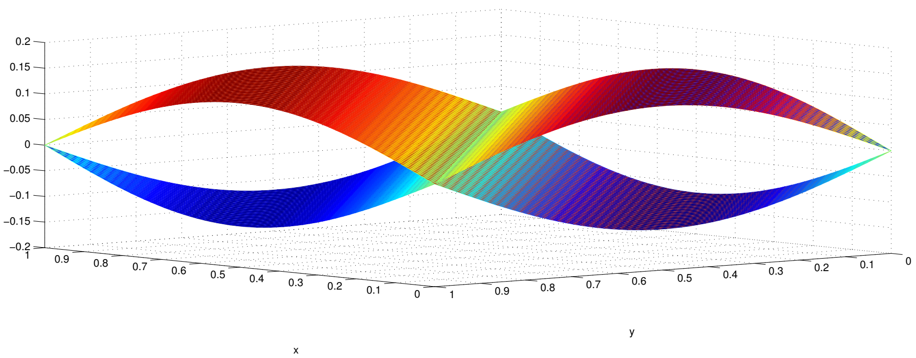

Figure 2: Fuzzy solution of Example 2.

Example 3.
Let's imagine another two-dimension fuzzy heat equation as in [11].

$$\begin{cases} \frac{\partial^{\beta}}{\partial t^{\beta}}\breve{\varphi}(x,\breve{\lambda},t,y)=\frac{1}{2}(x+y)\left[\frac{\partial^2}{\partial x^2}\breve{\varphi}(x,\breve{\lambda},t,y)+\frac{\partial^2}{\partial y^2}\breve{\varphi}(x,\breve{\lambda},t,y)\right]+t^4+x+y, \quad 1\geq\breve{\lambda}\geq 0; 0<\beta<1; t>0, \\ \breve{\varphi}(x,y,0)=\breve{\lambda}(x+y)^2, x<1, y>0, \end{cases} \tag{18}$$

where

$$\breve{\lambda}=[\underline{\lambda},\overline{\lambda}],$$

such that

$$\underline{\lambda}=\lambda-1, \overline{\lambda}=1-\lambda.$$

Now using equation(12) and (15), we can write as follows:

$$\breve{\varphi}_0(x,\breve{\lambda},t,y) = \breve{k}(x+y)^2,$$
$$\breve{\varphi}_1(x,\breve{\lambda},t,y) = \breve{\lambda}[(2).(x+y)^{(2)} + (x+y)]\frac{t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{\beta+4}}{\Gamma(\beta+5)},$$

$$\breve{\varphi}_2(x,\breve{\lambda},t,y) = \breve{\lambda}[(4).(x+y)^{(2)}]\frac{t^{2\beta}}{\Gamma(2\beta+1)},$$

$$\breve{\varphi}_3(x,\breve{\lambda},t,y) = \breve{\lambda}[(8).(x+y)^{(2)}]\frac{t^{3\beta}}{\Gamma(3\beta+1)}.$$

Hence the solution (final solution) can be written as follows,

$$\breve{\varphi}(x,\breve{\lambda},t,y) = \breve{\varphi}_0(x,\breve{\lambda},t,y) + \breve{\varphi}_1(x,\breve{\lambda},t,y) + \breve{\varphi}_2(x,\breve{\lambda},t,y) + \breve{\varphi}_3(x,\breve{\lambda},t,y)\ldots.$$

Now if the source term is zero, then the analytical solution of the aforesaid problem can be written as;

$$\breve{\varphi}(x,\breve{\lambda},t,y) = \sum_{n=2}^{\infty}\frac{2^n t^{n\beta}}{\Gamma(n\beta+1)}\lambda(x+y)^2.$$

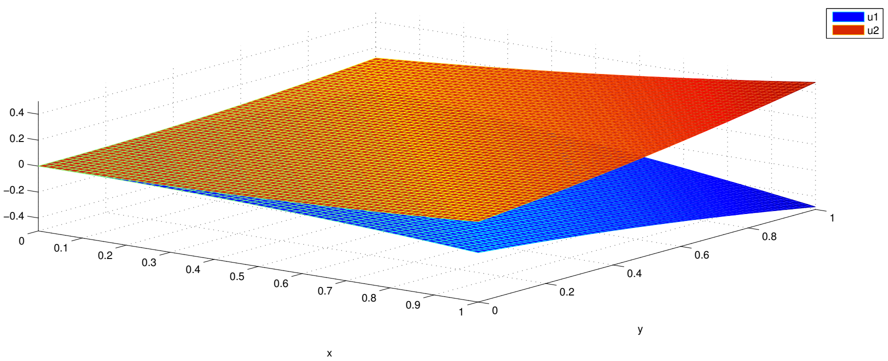

Figure 3: Fuzzy solution of Example 3.

5. Conclusion

In this work a new and a hybrid mechanism that consisting of Natural Transform (NT) and a series solution method (Adomian decomposition method) (ADM) have been successfully handled and a general algorithm is developed for the solution of 2-D fuzzy fractional order heat equations having a source term. The process of the solution started by the parametric form of the 2-D heat equation. Later on, the aforesaid mechanism is applied to the problem, the solution which is obtained for the unknown function is written is infinite series form. During analysis, it was observed that as the uncertainty($\breve{\lambda}$) increases the distance between the upper and lower solution increases. Moreover, the joint mechanism which is developed for the problem were found rapid and accurate. Hence the mechanism is suggested for the solution of linear and non-linear problems.

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