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On Pseudo BN-algebras

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Abstract. In this paper, we introduce the notion of pseudo BN-algebras and investigate some of its basic properties. Moreover, its relationship with the already established pseudo BF-algebra is investigated. Furthermore, subalgebras, ideals and pseudo-normality of pseudo BN-algebras are studied. Illustrations are provided for new concepts. Finally, we establish the equivalency of pseudo-normal subalgebras and pseudo-normal ideals.

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1. Introduction

Logic algebras form the mathematical foundation of reasoning in artificial intelligence, cybernetics, and computer science, offering a formal structure for representing and manipulating logical statements. Evolving from set theory and non-classical logic, they extend beyond traditional Boolean frameworks to support more advanced reasoning systems.

The development of logic algebras has led to various generalizations. In 1966, Imai and Iséki introduced BCK- and BCI-algebras [1, 2], expanding the study of non-classical logic by generalizing set-theoretic differences and propositional calculus. Many-valued logics further contributed to this field, with Chang introducing MV-algebras [3] and Hájek extending them to BL-algebras [4]. Another significant contribution is the BN-algebra, introduced by C.B. Kim and H.S. Kim [5], which is a subclass of BF-algebras introduced by Walendziak [6]. These structures provide valuable insights into the algebraic foundations of logic.

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A key advancement in algebraic logic is the introduction of pseudo-algebras, which extend classical structures by relaxing commutativity constraints. These include pseudo-BCK algebras [7], pseudo-BCI algebras [8], pseudo-MV algebras [9], and pseudo-BL algebras [10–13].

An essential aspect of algebraic structures is the study of subalgebras, normality, and ideals, which play a crucial role in understanding the internal organization of algebraic systems. Subalgebras help classify algebraic structures by identifying subsets that inherit operations from the parent algebra, while normality and ideals are fundamental in defining quotient structures and homomorphic properties. Investigating these aspects in pseudo-algebras provides deeper insight into their structural behavior and their connections to existing logical frameworks.

Despite extensive research on logic algebras and pseudo-algebras, pseudo BN-algebras have not yet been formally studied. This paper introduces the pseudo-variant of BN-algebras, developed in light of the established structural connections between BN-algebras and BF-algebras. This study aims to define pseudo BN-algebras, establish their fundamental properties, explore their subalgebra structures, normality conditions, and ideal theory and examine their relationships with existing pseudo-algebraic frameworks. By doing so, it contributes to the ongoing development of algebraic logic and noncommutative reasoning frameworks.

2. Preliminaries

Some preliminary concepts are given below, as well as results that are needed in this study.

Definition 1. [5] A *BN-algebra* is an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following axioms: for any $x, y, z \in X$,

$(BN1)$ $x * x = 0$,

$(BN2)$ $x * 0 = x$, and

$(BN3)$ $(x * y) * z = (0 * z) * (y * x)$.

Example 1. ([5], Example 2.26) Let $X = \{0, 1, 2, 3\}$ be a set and $*$ be a binary operation defined by the following Cayley table.

$*$	0	1	2	3
0	0	1	2	3
1	1	0	1	1
2	2	1	0	1
3	3	1	1	0

Then $(X, *, 0)$ is a BN-algebra.

Example 2. ([14], Example 5) Let $\mathbb{R}$ be the set of real numbers. Define the operation "$*$" on $\mathbb{R}$ as follows: for $x, y \in \mathbb{R}$,

$$x * y = \begin{cases} x, & \text{if } y = 0, \\ y, & \text{if } x = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Then $(\mathbb{R}, *, 0)$ is a BN-algebra.

Example 3. Let $\mathbb{R}$ be the set of real numbers. Define the operation "$*$" on $\mathbb{R}$ as follows: for $x, y \in \mathbb{R}$,

$$x * y = \begin{cases} x + y, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases} \tag{2}$$

Then $(\mathbb{R}, *, 0)$ is a BN-algebra.

Theorem 1. ([5], Theorem 2.3) *If $(X, *, 0)$ is a BN-algebra, then $(X, *, 0)$ is a BF-algebra.*

Remark 1. The converse of Theorem 1 does not hold in general as it was shown in [5], Example 2.4.

Proposition 1. [5] *If $(X, *, 0)$ is a BN-algebra, then for any $x, y, z \in X$,*

(i) $0 * (0 * x) = x$;

(ii) $y * x = (0 * x) * (0 * y)$;

(iii) $(0 * x) * y = (0 * y) * x$;

(iv) $x * y = 0 \implies y * x = 0$;

(v) $0 * x = 0 * y \implies x = y$;

(vi) $(x * z) * (y * z) = (z * y) * (z * x)$.

Definition 2. ([15], Definition 4) An algebra $(E, \bullet, \star, 0)$ of type $(2, 2, 0)$ is said to be a *pseudo-BF-algebra*, if the following axioms are satisfied for all $a, b \in E$:

(pBF1): $a \bullet a = 0$ and $a \star a = 0$,

(pBF2): $a \bullet 0 = a$ and $a \star 0 = a$,

(pBF3): $0 \bullet (a \star b) = b \star a$ and $0 \star (a \bullet b) = b \bullet a$.

Example 4. [15] Consider the additive group $(G, +, 0)$. Define the operations "$\bullet$" and "$\star$" on G by:

$$a \bullet b = (-b) + a \quad \text{and} \quad a \star b = (-b) + a \quad \text{for all} \quad a, b \in G.$$

Then $(G, \bullet, \star, 0)$ is a pseudo-BF-algebra.

Example 5. ([15], Example 2) Define the operations "$\bullet$" and "$\star$" on $E = \{0, 1, 2, 3\}$ by the following Cayley tables.

$\bullet$	0	1	2	3
0	0	1	2	3
1	1	0	3	0
2	2	3	0	2
3	3	0	2	0

$\star$	0	1	2	3
0	0	1	2	3
1	1	0	1	1
2	2	1	0	1
3	3	1	1	0

Then $(E, \bullet, 0)$ and $(E, \star, 0)$ are BF-algebras (shown in [6]). It can also be verified that $(E, \bullet, \star, 0)$ is a pseudo-BF-algebra.

3. Results

This section explores key properties of pseudo BN-algebras, distinguishing them from classical BN-algebras and establishing their structural foundations.

3.1. Some Properties of Pseudo BN-algebras

The results in this section provide insights into the algebraic nature of pseudo BN-algebras, setting the stage for further exploration of their operations and applications.

Definition 3. A *pseudo BN-algebra* is a structure $\mathcal{X} = (X, *, \circ, 0)$, where X is a set, "$*$" and "$\circ$" are binary operations on X, and "0" is a distinguished element called the *zero element*, satisfying: for all $x, y, z \in X$,

$(PBN1)$: $x * x = 0$ and $x \circ x = 0$,

$(PBN2)$: $x * 0 = x$ and $x \circ 0 = x$,

$(PBN3)$: $(x * y) \circ z = (0 * z) \circ (y * x)$ and $(x \circ y) * z = (0 \circ z) * (y \circ x)$.

Example 6. Define the binary operations "$*$" and "$\circ$" on $X = \{0, 1, 2, 3\}$ by the following Cayley tables:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	1	1
2	2	1	0	1
3	3	1	1	0

$\circ$	0	1	2	3
0	0	1	2	3
1	1	0	3	0
2	2	3	0	2
3	3	0	2	0

It can be verified that both $(X, *, 0)$ and $(X, \circ, 0)$ satisfy the axioms of a BN-algebra. It can also be verified that $(X, *, \circ, 0)$ is a pseudo BN-algebra.

Observe that if $(X, *, 0)$ is a BN-algebra, defining $x * y = x \circ y$ yields a pseudo BN-algebra $(X, *, \circ, 0)$. Thus, every BN-algebra is a special case of a pseudo BN-algebra where $\circ$ coincides with $*$, inherently satisfying its structure.

Example 7. Consider the set $\mathbb{R}$ of real numbers, and define the binary operations " $*$ " and " $\circ$ " on $\mathbb{R}$ as follows:

$$x * y = \begin{cases} x, & \text{if } y = 0 \\ y, & \text{if } x = 0 \\ 0, & \text{otherwise} \end{cases} \qquad x \circ y = \begin{cases} x + y, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$$

Then it can be shown that $(\mathbb{R}, *, \circ, 0)$ is a pseudo BN-algebra. The same holds for when $\mathbb{R}$ is replaced by $\mathbb{Z}$ and $\mathbb{Q}$.

Remark 2. The zero element "0" in Definition 3 is unique such that it is the only element that satisfies properties PBN1, PBN2 and PBN3.

Remark 3. If $(X, *, 0)$ and $(X, \circ, 0)$ are BN-algebras, then $(X, *, \circ, 0)$ is not necessarily a pseudo BN-algebra as shown in the following example:

Example 8. Define the operations "$*$" and "$\circ$" on $X = \{0, 1, 2\}$, by the following Cayley tables:

$*$	0	1	2
0	0	1	2
1	1	0	1
2	2	1	0

$\circ$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Both $(X, *, 0)$ and $(X, \circ, 0)$ are BN-algebras. But $(X, *, \circ, 0)$ is not a pseudo BN-algebra since $(0 * 1) \circ 2 = 2 \neq 1 = (0 * 2) \circ (1 * 0)$.

Theorem 2. *If $(X, *, \circ, 0)$ is a pseudo BN-algebra, then $(X, *, \circ, 0)$ is a pseudo BF-algebra.*

Proof. If we let $z = 0$ in $(PBN3)$, we obtain

$$x * y = (x * y) \circ 0 = (0 * 0) \circ (y * x) = 0 \circ (y * x)$$

and

$$x \circ y = (x \circ y) * 0 = (0 \circ 0) * (y \circ x) = 0 * (y \circ x)$$

Hence, $(X, *, \circ, 0)$ is a pseudo BF-algebra. □

Remark 4. The converse of Theorem 2 does not hold in general as shown in the following example:

Example 9. Consider the set $\mathbb{R}$ of real numbers, and define the binary operations " $*$ " and " $\circ$ " on $\mathbb{R}$ as follows:

$$x * y = \begin{cases} x, & \text{if } y = 0 \\ y, & \text{if } x = 0 \\ 0, & \text{otherwise} \end{cases} \qquad x \circ y = \begin{cases} x, & \text{if } y = 0 \\ 0, & \text{if } x = 0 \text{ or } x = y \\ y \circ x & \text{otherwise} \end{cases}$$

Then $(\mathbb{R}, *, \circ, 0)$ is a pseudo BF-algebra. But it is not a pseudo BN-algebra since if we let $x = y = z$ we have $(x * y) \circ z = 0 \circ z = 0 \neq z = z \circ 0 = (0 * z) \circ (y * x)$.

The following proposition gives the routinary properties of pseudo BN-algebras.

Proposition 2. *If* $(X, *, \circ, 0)$ *is a pseudo* BN*-algebra for all* $x, y, z \in X$ *then*

(i) $0 * (0 * x) = x$ *and* $0 \circ (0 \circ x) = x$,

(ii) $0 * (0 \circ x) = x$ *and* $0 \circ (0 * x) = x$,

(iii) $y * x = (0 \circ x) * (0 \circ y)$ *and* $y \circ x = (0 * x) \circ (0 * y)$,

(iv) $(0 * x) \circ y = (0 * y) \circ x$ *and* $(0 \circ x) * y = (0 \circ y) * x$,

(v) $0 * x = 0 \circ y \implies x = y$,

(vi) $x * y = 0$ *implies* $y * x = 0$ *and* $x \circ y = 0$ *implies* $y \circ x = 0$,

(vii) $(x * z) \circ (y * z) = (z * y) \circ (z * x)$ *and* $(x \circ z) * (y \circ z) = (z \circ y) * (z \circ x)$.

Proof.

(i) For any $x \in X$, applying $(PBN2)$ and Theorem 2, we get

$$0 * (0 * x) = 0 * [0 \circ (x * 0)] = 0 * (0 \circ x) = x \circ 0 = x$$

and similarly,

$$0 \circ (0 \circ x) = 0 \circ [0 * (x \circ 0)] = 0 \circ (0 * x) = x * 0 = x.$$

(ii) Setting $y = 0$ and $z = 0$ in $(PBN3)$, we obtain

$$(x * 0) \circ 0 = (0 * 0) \circ (x * 0).$$

Using $(PBN1)$ and $(PBN2)$, we conclude $x = 0 \circ (0 * x)$. By a similar argument,

$$x = x * 0 = (x \circ 0) * 0 = 0 * (0 \circ x).$$

(iii) Applying $(PBN3)$ and $(PBN2)$, we derive

$$y * x = (0 \circ x) * (0 \circ y), \quad y \circ x = (0 * x) \circ (0 * y).$$

(iv) From $(PBN3)$ and $(PBN2)$, we get

$$(0 * x) \circ y = (0 * y) \circ x.$$

(v) If $x * 0 = y \circ 0$, then by Proposition 2(i),

$$x = 0 * (0 * x) = 0 * (0 \circ y) = y.$$

(vi) Suppose $x * y = 0$. Then by (PBN_2) and Theorem 2, $0 = 0 \circ 0 = 0 \circ (x * y) = y * x$. Hence, $y * x = 0$. Similarly, if $x \circ y = 0$ then $y \circ x = 0$.

(vii) Using $(PBN3)$ and Theorem 2, we obtain

$$(x * z) \circ (y * z) = (z * y) \circ (z * x), \quad (x \circ z) * (y \circ z) = (z \circ y) * (z \circ x). \quad \square$$

Definition 4. An algebra $(X, *, \circ, 0)$ is said to be 0-*commutative* if for all $x, y \in X$,

$$x * (0 * y) = y * (0 * x) \text{ and } x \circ (0 \circ y) = y \circ (0 \circ x).$$

Example 10. Consider the Cayley table in Example 6. Since all conditions are satisfied for both operations $*$ and $\circ$, the algebra is a 0-commutative.

Definition 5. *An algebra $(X, *, \circ, 0)$ is said to be pseudo 0-commutative if*

$$x * (0 \circ y) = y * (0 \circ x) \textit{ and } x \circ (0 * y) = y \circ (0 * x)$$

for all $x, y \in X$.

Example 11. Using the Cayley table in Example 6, all conditions are satisfied for both operations $*$ and $\circ$. Hence, the algebra is pseudo 0-commutative pseudo BN-algebra.

Proposition 3. *If $(X, *, \circ, 0)$ is a pseudo BN-algebra, then it is 0-commutative and pseudo 0-commutative.*

Proof. Let $x, y \in X$. Using Theorem 2(ii), $(PBN3)$, and $(PBN2)$, we derive:

$$x * (0 * y) = [0 \circ (0 * y)] * [(0 * x) \circ 0] = y * (0 * x).$$

Similarly,

$$x \circ (0 \circ y) = [0 * (0 \circ y)] \circ [(0 \circ x) * 0] = y \circ (0 \circ x).$$

Applying Proposition 2(i), $(PBN3)$, and $(PBN2)$, we also obtain:

$$x * (0 \circ y) = [0 \circ (0 \circ y)] * [(0 \circ x) \circ 0] = y * (0 \circ x),$$

and

$$x \circ (0 * y) = [0 * (0 * y)] \circ [(0 * x) * 0] = y \circ (0 * x).$$

Thus, the result follows. $\square$

Theorem 3. *Let $(X, *, \circ, 0)$ be a pseudo BN-algebra. Then $(X, *, \circ, 0)$ is* 0-commutative *if and only if it is* pseudo 0-commutative.

Proof. Suppose $(X, *, \circ, 0)$ satisfies 0-commutativity:

$$x * (0 * y) = y * (0 * x), \quad x \circ (0 \circ y) = y \circ (0 \circ x).$$

We show it satisfies pseudo-0-commutativity:

$$x * (0 \circ y) = y * (0 \circ x), \quad x \circ (0 * y) = y \circ (0 * x).$$

For the first identity, using 0-commutativity:

$$\begin{aligned} x * (0 \circ y) &= x * [(0 * y) \circ 0] = x * (0 * y) = y * (0 * x) \\ &= y * [(0 * x) \circ 0] = y * (0 \circ x). \end{aligned}$$

For the second identity, again by 0-commutativity:

$$\begin{aligned} x \circ (0 * y) &= x \circ [(0 \circ y) * 0] = x \circ (0 \circ y) = y \circ (0 \circ x) \\ &= y \circ [(0 \circ x) * 0] = y \circ (0 * x). \end{aligned}$$

Conversely, assume pseudo-0-commutativity holds. We show that 0-commutativity follows:

$$\begin{aligned} x * (0 * y) &= x * [(0 \circ y) * 0] = x * (0 \circ y) = y * (0 \circ x) \\ &= y * [(0 \circ x) * 0] = y * (0 * x). \end{aligned}$$

Similarly,

$$\begin{aligned} x \circ (0 \circ y) &= x \circ [(0 * y) \circ 0] = x \circ (0 * y) = y \circ (0 * x) \\ &= y \circ [(0 * x) \circ 0] = y \circ (0 \circ x). \quad \square \end{aligned}$$

Example 12. Define the operations "$*$" and "$\circ$" on $E = \{0, 1, 2, 3\}$, by the following Cayley tables:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	0
2	2	3	0	2
3	3	0	2	0

$\circ$	0	1	2	3
0	0	1	2	3
1	1	0	1	1
2	2	1	0	1
3	3	1	1	0

Then $(E, *, \circ, 0)$ is a 0-commutative pseudo BF-algebra. Futhermore, it can be shown that $(E, *, \circ, 0)$ pseudo BN-algebra.

For 0-commutative pseudo BF-algebras, the converse of Theorem 2 holds.

Proposition 4. *If $(X, *, \circ, 0)$ is a 0-commutative pseudo BF-algebra, then it is a pseudo BN-algebra.*

Proof. Let $x, y, z \in X$. Using $(pBF3)$, 0-commutativity, and $(PBN2)$, we derive:

$$(0 \circ z) * (y \circ x) = (0 \circ z) * [0 * (x \circ y)] = (x \circ y) * (z \circ 0) = (x \circ y) * z.$$

Similarly,

$$(0 * z) \circ (y * x) = (0 * z) \circ [0 \circ (x * y)] = (x * y) \circ (z * 0) = (x * y) \circ z.$$

Hence, $(X, *, \circ, 0)$ is a pseudo BN-algebra. □

By Theorem 2 and Proposition 4, we obtain the following result.

Corollary 1. *$(X, *, \circ, 0)$ is a 0 -commutative pseudo BF-algebra if and only if it is a pseudo BN-algebra.*

3.2. Subalgebras and Ideals of Pseudo BN-algebras

This subsection discusses subalgebras and ideals, exploring their key properties and the relationships between them. We examine how subalgebras and ideals interact within algebraic structures, highlighting fundamental characteristics and significant results.

Definition 6. Let $(X, *, \circ, 0)$ be a pseudo BN-algebra, and let I be a subset of X containing 0. If S is a pseudo BN-algebra with respect to the operations $*$ and $\circ$ on X, we say that S is a *subalgebra* of X.

Theorem 4. *Let I be a nonempty subset of a pseudo BN-algebra X. Then I is a subalgebra of X if and only if $x * y,\ x \circ y \in I,$ for all $\ x, y \in I.$*

Proof. Suppose I is a subalgebra of X. By definition, I must be closed under the operations of X, meaning: $x * y \in I,\ x \circ y \in I,$ for all $x, y \in I$.

Conversely, suppose I is a nonempty subset of X satisfying these closure conditions. Then, since I inherits the algebraic structure of X restricted to I, it follows that I satisfies the definition of a subalgebra. □

Example 13. Given the pseudo BN-algebra with the Cayley table found in Example 6. The subalgebras, which are closed under both operations $*$ and $\circ$, are:

$$\{0\}, \quad \{0,1\}, \quad \{0,2\}, \quad \{0,3\}, \quad \{0,1,3\}, \quad \{0,1,2,3\}.$$

However, $\{0,1,2\}$ is not a subalgebra since $1 \circ 2 = 3$. Also, $\{0,2,3\}$ is not a subalgebra since $2 * 3 = 1$.

Proposition 5. *Let $S(X)$ denote the family of all sub-algebras of a pseudo BN-algebra X. $S(X)$ forms a complete lattice.*

Proof. Let $\{S_i\}_{i\in I}$ be a family of sub-algebras of pseudo BN-algebra X. Then $0 \in \bigcap\limits_{i\in I} S_i$ since $0 \in S_i$ for every $i \in I$. Let us take $x, y \in X$ such that $x \in \bigcap\limits_{i\in I} S_i$ and $y \in \bigcap\limits_{i\in I} S_i$. This means $x \in S_i$ and $y \in S_i$ for each $i \in I$. Thus $x * y \in S_i$ for each $i \in I$ because S_i is a sub-algebra in X. Hence $x * y \in \bigcap\limits_{i\in I} S_i$.

Let Y be the family of all sub-algebras in a pseudo BN-algebra X that contain $\bigcup\limits_{i\in I} S_i$. Then $\bigcap Y$ is a sub-algebra in X according to the first step of this proof. If we put $\bigvee\limits_{i\in I} S_i \wedge S = \bigcap Y$ and $\bigwedge\limits_{i\in I} S_i \vee S = \bigcap\limits_{i\in I} S_i$, then $(S(X),\ ,\wedge)$ is a complete lattice. □

Lemma 1. *Let S be a subalgebra of a pseudo BN-algebra $(X, *, \circ, 0)$. If $x * y \in S$ and $x \circ y \in S$, then $y * x \in S$ and $y \circ x \in S$.*

Proof. Suppose $x * y \in S$ and $x \circ y \in S$. By the definition of a subalgebra, we have $0 \in S$. By Theorem 2 and Theorem 4, $y * x = 0 \circ (x * y) \in S$ and $y \circ x = 0 * (x \circ y) \in S$. Thus, $y * x \in S$ and $y \circ x \in S$. □

We will now explore ideals of pseudo BN-algebras. In what follows, we consider X as the pseudo BN-algebra $(X, *, \circ, 0)$.

Definition 7. The set I is called an *ideal* of X if it satisfies the following:

$(pI1)$**:** $0 \in I$;

$(pI2)$**:** $x * y \in I$, $x \circ y \in I$ and $y \in I$ imply $x \in I$ for any $x, y \in X$.

In Example 6, the only ideals are $\{0\}$ and $\{0, 1, 2, 3\}$. So, let us consider another example.

Example 14. Define the operations "$*$" and "$\circ$" on $X = \{0, 1, 2, 3, 4\}$, by the following Cayley tables:

$*$	0	1	2	3	4
0	0	1	2	3	4
1	1	0	4	2	3
2	2	4	0	3	0
3	3	2	3	0	2
4	4	3	0	2	0

$\circ$	0	1	2	3	4
0	0	1	2	3	4
1	1	0	3	4	2
2	2	3	0	4	1
3	3	4	4	0	2
4	4	2	1	2	0

Subset $J_2 = \{0, 2\}$ is not an ideal in X because, for example, we have $4 * 2 = 0 \in J_2$ and $2 \in J_2$ but $4 \notin J_2$.

Subset $J_3 = \{0, 3\}$ is not an ideal in X because, for example, we have $2 * 3 = 3 \in J_3$ and $3 \in J_3$ but $2 \notin J_3$.

Subset $J_4 = \{0, 4\}$ is not an ideal in X because, for example, we have $2 * 4 = 0 \in J_4$ and $4 \in J_4$ but $2 \notin J_4$.

Subset $J_5 = \{0, 1, 2\}$ is not an ideal in X because, for example, we have $3 * 2 = 1 \in J_5$ and $2 \in J_5$ but $3 \notin J_5$.

Subset $J_6 = \{0, 1, 3\}$ is not an ideal in X because, for example, we have $4 * 1 = 3 \in J_6$ and $1 \in J_6$ but $4 \notin J_6$.

The subsets $\{0, 1, 4\}$, $\{0, 2, 3\}$, $\{0, 2, 4\}$, and $\{0, 3, 4\}$ are not ideals in X either.

Then $(X, *, \circ, 0)$ is a pseudo BN-algebra. Now, the ideals are

$$\{0\}, \{0, 1\}, \{0, 2\}, \{0, 3\}, \{0, 4\}, \{0, 1, 3\}, \text{ and } X.$$

Remark 5. $\{0\}$ and X are always ideals of X.

Proposition 6. *Let $J(X)$ denote the family of all ideals of a pseudo BN-algebra X. $J(X)$ forms a complete lattice.*

Proof. Analogous to the proof of Proposition 5. □

Let X be a pseudo BN-algebra as in Example 14. The subsets $\{0\}$, $\{0, 1\}$, $\{0, 2\}$, $\{0, 3\}$, and $\{0, 4\}$ are sub-algebras in X, while the subsets $\{0, 1, 2\}$, $\{0, 1, 3\}$, $\{0, 1, 4\}$, $\{0, 2, 3\}$, $\{0, 2, 4\}$, and $\{0, 3, 4\}$ are not.

Remark 6. Sub-algebra of X need not be an ideal in X, as shown in Example 14. For example, the sub-algebra $\{0, 2\}$ in X is not an ideal in X, and vice versa.

Definition 8. A nonempty subset N of X is said to be *pseudo-normal* if it satisfies the following conditions:

(i) If $x * y$, $a * b \in N$, then $(x * a) \circ (y * b) \in N$.

(ii) If $x \circ y$, $a \circ b \in N$, then $(x \circ a) * (y \circ b) \in N$.

Since N is pseudo-normal, any element $x \in N$ satisfies $x * 0 = x \in N$ by PBN2. Furthermore, using PBN1, we obtain $0 = 0 \circ 0 = (x * x) \circ (0 * 0)$. But $(x * x) \circ (0 * 0) \in B$ by pseudo-normality of N which implies that $0 \in N$. This leads us to the following observation.

Remark 7. *If N is pseudo-normal, then $0 \in N$.*

Example 15. Given the pseudo BN-algebra with the following Cayley tables:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	1
2	2	3	0	2
3	3	1	2	0

$\circ$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

The following are the pseudo-normal subsets of X: $\{0\}, \{0, 3\}$, and X.

Example 16. In Example 14, the set $I = \{0\}$ is not pseudo-normal because $(1*1) = 0 \in I$ and $(2*4) = 0 \in I$ but $(1*2) \circ (1*4) = 4 \circ 3 = 2 \notin I$.

Remark 8. In general, the set $\{0\}$ is not pseudo-normal.

Proposition 7. *Let $N(X)$ denote the family of all pseudo-normal subsets of a pseudo BN-algebra X. $N(X)$ forms a complete lattice.*

Proof. Analogous to the proof of Proposition 5 . □

Proposition 8. *Every pseudo-normal subset N of a pseudo BN-algebra $(X, *, \circ, 0)$ is a subalgebra of X.*

Proof. Suppose N is a subset of X and $x, y \in N$. Then by (PBN2), $x * 0,\ y * 0,\ x \circ 0,\ y \circ 0 \in N$. By PBN2 and pseudo-normalily of N, we obtain

$$x * y = (x * y) \circ 0 = (x * y) \circ (0 * 0) \in N, \quad x \circ y = (x \circ y) * 0 = (x \circ y) * (0 \circ 0) \in N.$$

By Theorem 4, N is a subalgebra of X. □

The converse of Proposition 8 is not true in general. The set $I = \{0\}$ in Example 14 is a subalgebra but not pseudo-normal as shown in Example 16.

A pseudo-normal subalgebra is a special type of subalgebra in a pseudo BN-algebra. It is formed by a subset that follows the pseudo-normality conditions. This naming helps distinguish it from other subalgebras that do not necessarily follow the pseudo-normality conditions, making its role in the structure clear.

Definition 9. A *pseudo-normal ideal* is an ideal which satisfies the pseudo-normality conditions.

Example 17. In Example 15, it can be shown that the set $I = \{0, 3\}$ is a pseudo-normal ideal. However, in Example 14, the ideal $J = \{0, 3\}$ is not pseudo-normal since $(4*1) = 3 \in J$ and $(2*3) = 3 \in J$ but $(4*2) \circ (1*3) = 0 \circ 2 = 2 \notin J$. Hence, J is not a pseudo-normal ideal.

The next result follows from Propositions 6 and 7.

Corollary 2. *Let X be a pseudo BN-algebra and $\{N_i \mid i \in I\}$ be a nonempty family of pseudo-normal ideals, then $\bigcap_{i \in I} N_i$ is also a pseudo-normal ideal in X.*

The next result established the equivalency of pseudo-normal subalgebra and pseudo-normal ideal of a pseudo BN-algebra.

Theorem 5. *Let X be a pseudo BN-algebra and let $N \subseteq X$. Then N is a pseudo-normal subalgebra of X if and only if N is a pseudo-normal ideal of X.*

Proof. Suppose X is a pseudo BN-algebra and $N \subseteq X$. Assume that N is a pseudo-normal subalgebra of X. By Remark 7, we have $0 \in N$, satisfying (pI1) of Definition 7.

To verify (pI2), suppose $x * y \in N$, $x \circ y \in N$, and $y \in N$. Since $0 \in N$, by Theorem 4, $0 * y,\ 0 \circ y \in N$. By PBN2 [applied twice] and pseudo-normality in N,

$$x = x * 0 = (x * 0) \circ 0 = (x * 0) \circ (y * y) \in N.$$

Hence, N is a pseudo-normal ideal of X. The converse follows directly from Proposition 8. □

Notice in the proof of Theorem 5, we did not use $x \circ y \in N$ and $0 \circ y \in N$. But we can also show that $x \in N$ using these:

$$x = x \circ 0 = (x \circ 0) * 0 = (x \circ 0) * (y \circ y) \in N.$$

4. Conclusion

In this paper, we introduced the concept of pseudo BN-algebras as a natural extension of BN-algebras, relaxing commutativity constraints to explore broader algebraic structures. We established fundamental properties of pseudo BN-algebras, analyzed their relationships with pseudo BF-algebras, and examined their structural components, including subalgebras and ideals. Finally, we established the equivalency of subalgebras and ideals in terms of pseudo-normality. These investigations provide a deeper understanding of their internal organization and potential applications in non-classical logic and abstract algebra.

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