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Modeling Virus Mutation Dynamics Using Piecewise Fractional Derivatives

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Abstract. A virus mutation model under piecewise fractional order derivatives involving Mittag-Leffler type kernel has been studied in this manuscript. As mutation is an important phenomenon for the survival of virus. The concerned study aims to detect the crossover behavior of the dynamics of virus mutation. The considered problem explains a compartmental model with pre and post mutation of virus. Fundamental results related to local and global stability of equilibrium points have been studied by using tools of nonlinear functional analysis. We have deduced positivity and feasibility of for the mentioned model using the fractional order derivatives. In addition, both trivial and non-trivial equilibrium points are computed and reproductive number is also derived. With the help of the considered numerical scheme based on Adam Bashforth method, we have simulated our results graphically for using different values of fractional order.

2020 Mathematics Subject Classifications: 6A33, 34A08, 03Cxx

Key Words and Phrases: Virus mutation model, Voltera- Lyapunov functions, piecewise derivative, Global stability, Numerical results

1. Introduction

A virus must undergo mutation in order to exist. According to the study of virology, virus mutation happens naturally and frequently. It evolved to fit its surroundings more effectively and spread from host to host. A mutation may improve a virus's capacity to quickly multiply or attach more effectively to the surface of human cells [1]. A virus may occasionally be prevented from gaining characteristics that aid in transmission due to the rate of its mutation. This explains why viruses can change and arise, then vanish. Viral alterations may be dangerous, especially if they make it easier for them to evade the defences of our immune system. They mutate often due to viral mutation. These changes

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and adjustments encourage the virus's enhanced spread [2]. Clinics can illustrates the importance of viral mutation rates with anti-HIV therapy histories. Azidothymidine (AZT), a nucleoside analogue, was the first anti-HIV therapy to be approved; unfortunately, drug-resistant variants soon made AZT ineffective. Every base-on-single substitution, including resistance-causing mutations The HIV-1 virus, which is continually developing inside of a patient, produces AZT every day. The subsequent success of highly active antiretroviral therapy rested more on carefully combining other drugs, such as AZT, to lower the risk of resistance mutations arising, than it did on simply increasing the potency of the medicine. The hepatitis C virus (HCV) and other rapidly developing viruses are qualitatively similar to one another. Protease inhibitor and non-nucleoside polymerase inhibitor resistance pre-exists naturally in individuals who have never had treatment, or in the absence of selection favoring these mutations, according to a population sequencing study. Some clinical studies, we refer to [3–5]. A similar scenario applies to antiviral immunity. High mutation rates increase the likelihood that a virus will effectively evade immunity. Hepatitis B virus (HBV), HCV, and HIV-1 are three rapidly evolving viruses that can result in chronic infections. These viruses frequently evade antibodies as well as cytotoxic T lymphocytes (CTL). Point mutations in HBV, the most common cause of hepatitis worldwide with about 350 million chronic carriers, have been associated with immunological escape and vaccine failure [6]. The most famous example is the influenza virus which mutates every year to produce new strains that require yearly adjustments to the vaccine. It is possible to get vaccinated against the flu. It is well known that the influenza virus RNA-polymerase lacks a proofreading component. Consequently, during viral replication, faulty nucleotides are often integrated at a rate of 103 to 104, resulting in notable mutation rates [7]. In order to prevent the chance of future mutation and to prevent falling ill in the first place, patients are continuously advised to receive their flu shots. Thus, the design of antiviral techniques is significantly influenced by viral genetic diversity, which is ultimately dictated by mutation rates [8].

Actually, the phrase "virus mutation" describes genetic changes that occur in the genetic material of a virus, which is usually RNA or DNA [9]. These changes result in a new strain of the virus or variations in its genetic composition and characteristics which raise questions regarding the virus's possible impact on the severity of the disease, the efficacy of vaccinations, available treatments, and its ability to weaken the host's immune system. Since RNA replication enzymes are not as capable of proofreading as DNA polymerase, RNA viruses such as HIV and influenza have a tendency to mutate more frequently than DNA viruses, which include smallpox. The hosts' immune system, ultra violet radiation, and chemicals are the main causes of viral modifications [10, 11]. Precise estimates of virus mutation rates are crucial to our understanding of how viruses evolve and their defence. Scientists and researchers closely monitor changes to viruses, especially those like SARS-CoV-2. This mutation tracking makes it feasible to comprehend the behavior of the virus, anticipate potential outbreaks, and alter diagnostic techniques, drugs, and vaccines. Since March 2020, Ohio State researchers have been sequencing the genomes of the SARS-CoV-2 virus, despite the fact that this happens worldwide. It allows us to monitor the evolution of COVID-19 and assess if newly discovered variations are more contagious or transmissible than more well-known variants (we refer to [12, 13]). Here in the given figure 1, we give a rough sketch of SARS-CoV-2 virus in sub figure (a) and influenza virus mutation in sub figure (b).

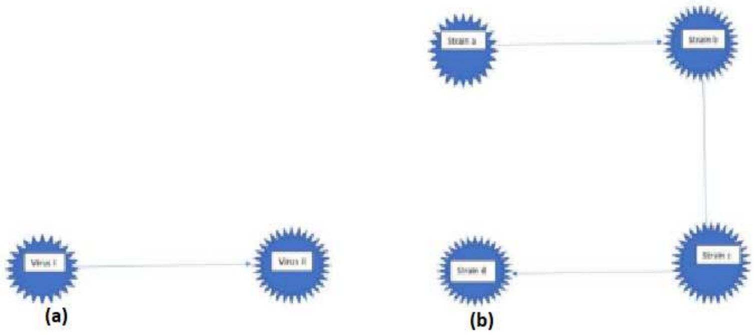

Figure 1: (a). Mutation in SARS-CoV-2 virus. (b). Mutation in influenza virus.

It's hard to predict when or for how long new strains of SARS-CoV-2 may emerge. Getting the required immunizations and donning masks in crowded areas, where COVID-19 is more likely to spread among people are essential actions we can all take to prevent the disease from spreading. Viral infections have the potential to become more contagious in populations with low rates of immunization. Getting immunized and wearing a mask need to be requirements. Most hospital patients with COVID-19 are not immunized. Immunization is one way to stop the spread and replication of COVID-19. Mutations are more likely to occur when COVID-19 is allowed to multiply. To assess the ongoing risk to the public's health, it is imperative to monitor these mutations and the novel variants they produce. Certain mutations may result in modifications that do not immediately jeopardies an organism's enhanced pathogenicity. Certain adjustments might even exacerbate the illness. Nonetheless, physicians continue to monitor both of these viruses and others because a new virus mutation could alter the current scenario. It's important to keep in mind that, even in cases where virus mutations lead to modifications in public health practises, not all virus mutations have a deep effect on virus behavior or public health. Many mutations either have no effect on the virus or could even be harmful to it (we refer to [14, 15]). However, understanding infectious diseases (including how to manage medicine resistance, immunological escape, vaccination, pathogenesis, and the emergence of novel diseases) and developing effective prophylactic measures depend on tracking and studying changes in viruses. We can gain a better understanding of infectious diseases through mathematical modeling.

The typical methods for studying epidemiological models have been difference equations or traditional order derivatives. Nevertheless, the global dynamical behavior of events cannot be fully captured by these traditional local operators. This has led to a growing trend of academics modeling many types of systems using fractional order derivatives in disciplines such as dynamical analysis and mathematical biology. Because fractional calculus can be used to a wide range of real-world situations, it has been widely adopted in these domains and has generated significant interest and breakthroughs in the field. Fractional calculus has a wide range of real-world applications, as demonstrated by the works referenced as [16, 17]. As references [18, 19] demonstrate, fractional calculus has been found useful in solving a wide range of issues, including those pertaining to mathematical biology, epidemiological modeling, and disease dynamics. Furthermore, a wide range of scientific and technological domains, including those listed in references [20, 21] have adopted fractional calculus. Because fractional calculus is globally applicable and can take memory effects in dynamic systems into account, it is becoming widely used in the moderation of many diseases. The scientific literature and research publications devoted to the fractional calculus and its use in disease modeling as cited in [22], provide a wealth of information and detailed specifics on this subject. Researchers have modified and enhanced conventional analytical and numerical techniques to address the difficulties presented by issues involving fractional order derivatives. As mentioned in references [23, 24], these improvements have made it possible for them to address a wider variety of scientific and engineering problems that profit from fractional calculus. As an example, scholars have expanded on standard techniques like as perturbation, transform, and decomposition to investigate a wide range of dynamic phenomena using fractional order derivatives. Furthermore, as mentioned in [25–27], conventional numerical methods have been gradually improved upon and modified to solve fractional order derivative problems. A new idea has surfaced recently to explain the crossover behavior seen in dynamic systems. Abrupt shifts in the stages of different evolutionary processes give birth to this behavior. These effects show that both classical and fractional derivatives, in their conventional forms, are not sufficient to describe the multi-phase behaviors that are being studied. Thus, for the examined model, a concerted effort is undertaken to conduct numerical simulations in reference [28], with a particular emphasis on examining multi-phase behaviors across various fractional orders. The researchers use a numerical technique based on Newton interpolation polynomials to accomplish this goal. The research described in reference [29] demonstrates the strength of the suggested numerical method and its capacity to manage dynamic behaviors that are complex and involve fractional order derivatives. It contributes to the increasing amount of studies that recognize fractional calculus as an important instrument for comprehending and simulating real-world events.

Recently, authors [30] have studied the following model of virus mutation involving time delay as

$$\begin{cases} \dot{\mathcal{S}}(t) = \Lambda - \rho\mathcal{S}(t) - \epsilon_1\mathcal{S}(t)\mathcal{I}_{pre}(t) - \epsilon_2\mathcal{S}(t)\mathcal{I}_{post}(t) \\ \dot{\mathcal{L}}_{\mathcal{E}}(t) = \epsilon_1\mathcal{S}(t)\mathcal{I}_{pre}(t) + \epsilon_2\mathcal{S}(t)\mathcal{I}_{post}(t) - (\varepsilon+\rho)\mathcal{L}_{\mathcal{E}} \\ \dot{\mathcal{I}}_{pre}(t) = \varepsilon\mathcal{L}_{\mathcal{E}} - (\rho+\kappa_1)\mathcal{I}_{pre}(t) - \alpha\mathcal{I}_{pre}(t-\theta) \\ \dot{\mathcal{I}}_{post}(t) = \alpha\mathcal{I}_{pre}(t\theta) - (\rho+\kappa_2)\mathcal{I}_{post}(t) \\ \dot{\mathcal{R}}(t) = \kappa_1\mathcal{I}_{pre}(t) + \kappa_2\mathcal{I}_{post}(t) - \rho\mathcal{R}(t), \end{cases} \tag{1}$$

where initial values are given by

$$\begin{aligned} &\mathcal{S}(0) = \mathcal{S}_0 > 0,\ \mathcal{L}_E(0) = \mathcal{L}_{\mathcal{E}0} \geq 0, \\ &\mathcal{I}_{\text{pre}}(0) = \mathcal{I}_{\text{pre}_0} \geq 0,\ \mathcal{I}_{\text{post}}(0) = \mathcal{I}_{\text{post}_0} \geq 0, \\ &\mathcal{R}(0) = \mathcal{R}_0 \geq 0, \end{aligned} \tag{2}$$

additionally classes and nomenclatures are defined as:
$\mathcal{S}$: Represents the susceptible individuals in the population. $\mathcal{L}_{\mathcal{E}}$: Represents the latently Individuals who have been exposed (those who are infected but not yet contagious). $\mathcal{I}_{pre}$: Represents the pre-mutation patients (infectious individuals with the original strain of the virus). $\mathcal{I}_{post}$: Represents the post-mutation patients (infectious individuals with a mutated strain of the virus). $\mathcal{R}$: Represents the recovered individuals. The parameters used in the model are described in Table 1.

Parameter	Physical description
Λ	Represents the birth rate of new individuals into the population
ρ	Represents the death rate in the population
ϵ_1 and ϵ_2	represent the rates at which contact with pre-mutation and post-mutation patients causes susceptible people to become latently exposed
ε	Represents the rate at which latently exposed people get disease
κ_1 and κ_2	represent the rates of recovery for patients before and after a mutation, respectively
α	Represents the rate of mutation from the pre-mutation to post-mutation state
θ	denotes delay term

Table 1: Physical description of parameters involve in model (1).

Motivated by the importance of piecewise derivatives of fractional order, we update the model (1) by omitting the delays terms and involving fractional order derivative as

$$\begin{cases} {}^{\mathbb{PABC}}\mathrm{D}^{\xi}_{+0}(\mathcal{S})(t) = \Lambda^{\xi} - \rho^{\xi}\mathcal{S}(t) - \epsilon_1^{\xi}\mathcal{S}(t)\mathcal{I}_{pre}(t) - \epsilon_2^{\xi}\mathcal{S}(t)\mathcal{I}_{post}(t) \\ {}^{\mathbb{PABC}}\mathrm{D}^{\xi}_{+0}(\mathcal{L}_{\mathcal{E}})(t) = \epsilon_1^{\xi}\mathcal{S}(t)\mathcal{I}_{pre}(t) + \epsilon_2^{\xi}\mathcal{S}(t)\mathcal{I}_{post}(t) - (\varepsilon^{\xi} + \rho^{\xi})\mathcal{L}_{\mathcal{E}} \\ {}^{\mathbb{PABC}}\mathrm{D}^{\xi}_{+0}(\mathcal{I}_{pre})(t) = \varepsilon^{\xi}\mathcal{L}_{\mathcal{E}} - (\rho^{\xi} + \kappa_1^{\xi})\mathcal{I}_{pre}(t) - \alpha^{\xi}\mathcal{I}_{pre}(t) \\ {}^{\mathbb{PABC}}\mathrm{D}^{\xi}_{+0}(\mathcal{I}_{post})(t) = \alpha^{\xi}\mathcal{I}_{pre}(t) - (\rho^{\xi} + \kappa_2^{\xi})\mathcal{I}_{post}(t) \\ {}^{\mathbb{PABC}}\mathrm{D}^{\xi}_{+0}(\mathcal{R})(t) = \kappa_1^{\xi}\mathcal{I}_{pre}(t) + \kappa_2^{\xi}\mathcal{I}_{post}(t) - \rho^{\xi}\mathcal{R}(t). \end{cases} \tag{3}$$

The model (3) explains a compartmental model with pre and post mutation of virus. To include genetic traits and historical background in the model (3), a new approach called the piecewise differential and integral operators are used. Extending the standard analysis used in [31, 32], we compute fundamental ratio, equilibrium points and predict the local asymptotical stability of trivial equilibrium point. Recently, researchers have applied various tools to investigate different fractional order models. We refer some published works as [33–36]. Moreover, a Voltera- Lyapunov function is constructed to show the global stability for endemic equilibrium. Here we use the adopted derivative to deduce the required results. Additionally, we will use fixed point theory, and numerical tools based on to investigate the existence theory and simulate the model (3). Various graphical presentations with different fractional orders are displayed for the proposed model.

2. Some Preliminaries

We define here the Banach space by $\mathbb{B} = \mathbb{C}(\mathbb{J})$, where $\mathbb{J} = [0, T]$, $\mathbb{J}_1 = [0, t_1]$, $\mathbb{J}_2 = (t_1, T]$ with norm defined by $\|\Omega\| = \sup_{t\in\mathbb{J}} \{|\Omega(t)|,\ \Omega \in \mathbb{B}\}$. Here, we recollect some basic tools needed throughout this paper.

Definition 1. *[28] The $\mathbb{ABC}$ derivative of a function $\psi \in \mathbb{H}^1(\mathbb{J})$ is defined as follows:*

$$ {}^{\mathbb{ABC}}D^{\xi}_{+0}(\psi(t)) = \frac{\mathbb{ABC}(\xi)}{1-\xi}\int_0^t \frac{d}{d\zeta}\psi(\ell)\mathbb{E}_{\xi}\left[\frac{-\xi}{1-\xi}\Big(t-\ell\Big)^{\xi}\right]d\ell. \tag{4}$$

where $\mathbb{ABC}(0) = \mathbb{ABC}(1) = 1$. Also $\mathbb{E}_{\xi}$ stands for Mittag-Leffler function.

Definition 2. *[28] If $\psi \in L(\mathbb{J})$, then $\mathbb{AB}$ fractional integral is given by*

$$ {}^{\mathbb{ABC}}I^{\xi}_{+0}\psi(t) = \frac{1-\xi}{\mathbb{ABC}(\xi)}\psi(t) + \frac{\xi}{\mathbb{ABC}(\xi)\Gamma(\xi)}\int_0^t (t-\ell)^{\xi-1}\psi(\ell)d\ell. \tag{5}$$

Definition 3. *[28] For differentiable function ψ, the piecewise fractional derivative in $\mathbb{ABC}$ sense is given as*

$$ {}^{\mathbb{PABC}}D^{\xi}_{+0}\psi(t) = \begin{cases} {}^{C}_{0}D^{\xi}_{t}\psi(t), & \mathbb{J}_1, \\ {}^{\mathbb{ABC}}D^{\xi}_{+0}\psi(t) & \mathbb{J}_2, \end{cases},$$

here ${}^{PABC}D^{\xi}_{+0}\psi(t)$ is Caputo derivative for $\mathbb{J}_1$ and fractional $\mathbb{ABC}$ derivative for $\mathbb{J}_2$.

Definition 4. *[28] The piecewise integral in $\mathbb{AB}$ sense is defined as for continuous function $\psi \in L(\mathbb{J})$*

$$ {}^{\mathbb{PAB}}I_{+0}\psi(t) = \begin{cases} \dfrac{1}{\Gamma\xi}\displaystyle\int_{t_1}^t (t-\ell)^{\xi-1}\psi(\ell)d(\ell), & \mathbb{J}_1, \\ \dfrac{1-\xi}{\mathbb{ABC}\xi}\psi(t) + \dfrac{\xi}{\mathbb{ABC}\xi\Gamma\xi}\displaystyle\int_{t_1}^t (t-\ell)^{\xi-1}\psi(\ell)d(\ell), & \mathbb{J}_2, \end{cases},$$

here ${}^{\mathbb{PAB}}I_{+0}\psi(t)$ is Riemann-Liouville integration for $\mathbb{J}_1$ and $\mathbb{AB}$ integration for $\mathbb{J}_2$.

Lemma 1. *[28] The equation*

$$ {}^{\mathbb{PABC}}D^{\xi}_{+0}\psi(t) = \phi(t), \quad 0 < \xi \le 1$$

has a unique solution described by

$$\psi(t) = \begin{cases} \psi_0 + \dfrac{1}{\Gamma(\xi)}\displaystyle\int_0^t \phi(\ell))(t-\ell)^{\xi-1}d\ell, & \mathbb{J}_1, \\ \psi(t_1) + \dfrac{1-\xi}{\mathbb{ABC}(\xi)}\phi(t)) + \dfrac{\xi}{\mathbb{ABC}\xi\Gamma(\xi)}\displaystyle\int_{t_1}^t (t-\ell)^{\xi-1}\phi(\ell))d(\ell), & \mathbb{J}_2. \end{cases}$$

Lemma 2. *[32] If $\mathcal{Z} \in \mathbb{R}^+$ is continues function and for $t \geq t_0$*

$$^{\mathbb{C}}D^{\xi}_{+0}\left[\mathcal{Z}(t) - \widehat{\mathcal{Z}} - \widehat{\mathcal{Z}}\log\left(\frac{\mathcal{Z}}{\widehat{\mathcal{Z}}}\right)\right], \widehat{\mathcal{Z}} \in \mathbb{R}^+,$$

for $\xi \in (0,1)$.

3. Feasibility, Positivity and Existence of Solution

Here, we prove a few results concerning equilibrium points, positivity and fundamental reproduction number, invariant region, etc. Here, we note that the $\mathbb{ABC}$ derivative and Laplace transform are used to establish the aforementioned results.

Theorem 1. *The sector defined by $\Theta = \{(\mathcal{S}, \mathcal{L}_{\mathcal{E}}, \mathcal{I}_{pre}, \mathcal{I}_{post}, \mathcal{R}) \in \mathbb{R}^5_+ : \mathbb{N} \leq \frac{\Lambda^\xi}{\rho^\xi}\}$ is positive invariant. Additionally, every solution is drawn to $\mathbb{R}^5_+$.*

Proof. Adding all equations of model (3), and assuming that the population as a whole is $\mathbb{N}$

$$\begin{aligned} ^{\mathbb{PABC}}D^{\xi}_{+0}\mathbb{N}(t) &= \Lambda^\xi - \rho^\xi[\mathcal{S} + \mathcal{L}_{\mathcal{E}} + \mathcal{I}_{pre} + \mathcal{I}_{post} + \mathcal{R}] \\ &= \Lambda - \rho^\xi \mathbb{N}(t) \end{aligned} \tag{6}$$

which further can be written as

$$^{\mathbb{PABC}}D^{\xi}_{+0}\mathbb{N}(t) = \begin{cases} ^{C}D^{\xi}_{+0}\mathbb{N}(t) = \Lambda^\xi - \rho^\xi \mathbb{N}(t), \ t \in \mathbb{J}_1, \\ ^{\mathbb{ABC}}D^{\xi}_{+0}\mathbb{N}(t) = \Lambda^\xi - \rho^\xi \mathbb{N}(t), \ t \in \mathbb{J}_2. \end{cases} \tag{7}$$

Using Laplace transform in (7) with $\mathbb{N}_0$ is initial population, for $t \in \mathbb{J}_1$, one has

$$\mathbb{N}(t) = \mathbb{E}_\xi(-\rho^\xi t^\xi) + \frac{\Lambda^\xi}{\rho^\xi}\left[1 - \mathbb{E}_\xi(-\rho^\xi t^\xi)\right]. \tag{8}$$

If $t \to \infty$ in (8), we see that

$$\mathbb{N}(t) \to \frac{\Lambda^\xi}{\rho^\xi}. \tag{9}$$

In the same way for case when $t \in \mathbb{J}_2$, using $\varrho = \frac{1}{1-\xi}$, $\varrho + \rho^\xi = \kappa$, $\frac{\rho^\xi \xi \rho^\xi}{\rho^\xi + \rho^\xi} = \lambda$, $\frac{1}{\kappa} = \kappa_1$, one has

$$\mathscr{L}[\mathbb{N}\exp(\rho^\xi t)] \leq \frac{\Lambda^\xi \kappa_1 (s^\xi + \xi\varrho)}{\nu s(s^\xi + \lambda)} + \frac{\mathbb{N}_0 \kappa_1}{s^\xi + \lambda}. \tag{10}$$

From (10) on using inverse Laplace transform, one has

$$N(t) = \frac{\Lambda^\xi}{\rho^\xi}\left[1 - \mathbb{E}_\xi(-\lambda t^\xi)\right] + \mathbb{N}_0 \kappa_1 t^{\xi-1}\mathbb{E}_{\xi,\xi}(-\lambda t^\xi). \tag{11}$$

At $t \to \infty$, $\mathbb{E}_\xi(-\lambda t^\xi) \to 0$ and utilizing L.Hoptital's procedure of limit repeatedly, we have

$$\lim_{t\to\infty}[t^{\xi-1}\mathbb{E}_{\xi,\xi}(-\lambda t^\xi)] = 0.$$

Hence (11) implies that

$$\mathbb{N}(t) \to \frac{\Lambda^\xi}{\rho^\xi}. \tag{12}$$

Hence, from (9) and (12), we have

$$\mathbb{N}(t) \to \frac{\Lambda^\xi}{\rho^\xi}$$

the set Θ is feasible region for bounded solution, where all solutions lie in the region Θ.

Theorem 2. *If* $\{(\mathcal{S}(0), \mathcal{L}_\mathcal{E}(0), \mathcal{I}_{pre}(0), \mathcal{I}_{post}(0), \mathcal{R}(0)) \geq 0\} \in \mathbb{R}^5_+$, *then the solution* $\{(\mathcal{S}, \mathcal{L}_\mathcal{E}, \mathcal{I}_{pre}, \mathcal{I}_{post}, \mathcal{R})\}$ *of model* (3) *subject to the positive initial data is positive for all* $t > 0$.

Proof. One has by using first equation of model (3)

$$\begin{aligned} {}^{\mathbb{PABC}}\mathrm{D}^\xi_{+0}(\mathcal{S})(t) &= \Lambda^\xi - \rho^\xi\mathcal{S} - \epsilon_1^\xi\mathcal{S}(\mathcal{I}_{pre} - \epsilon_2^\xi\mathcal{S}\mathcal{I}_{post} \\ &\geq -\left(\rho^\xi + \epsilon_1^\xi\mathcal{I}_{pre} + \epsilon_2^\xi\mathcal{I}_{post}\right)\mathcal{S}(t). \end{aligned} \tag{13}$$

Use $\Delta = \rho^\xi + \epsilon_1^\xi\mathcal{I}_{pre} + \epsilon_2^\xi\mathcal{I}_{post}$ in (13) implies that

$${}^{\mathbb{PABC}}\mathrm{D}^\xi_{+0}[\mathcal{S}(t)] \geq -\Delta\mathcal{S}. \tag{14}$$

Keeping in mind the definition of ${}^{\mathbb{PABC}}\mathrm{D}^\xi_{+0}$, we discuss two cases here as:
Case I: When $t \in \mathbb{J}_1$, then

$${}^{C}\mathrm{D}^\xi_{+0}(\mathcal{S})(t) \geq -\Delta\mathcal{S} \tag{15}$$

which yields on applying Laplace transform

$$\mathcal{S}(t) \geq \mathcal{S}_0\mathbb{E}_\xi(-\Delta t^\xi) > 0 \text{ at } t > 0. \tag{16}$$

Case II: In addition if $t \in \mathbb{J}_2$, then

$${}^{\mathbb{ABC}}\mathrm{D}^\xi_{+0}[\mathcal{S}(t)] \geq -\Delta\mathcal{S}. \tag{17}$$

Applying Laplace transform to (17) on using $\sigma_1 = \varrho + \Delta$ with $\varrho = \frac{1}{1-\xi}, \sigma_2 = \frac{\Delta\xi\varrho}{\sigma_1}$, we get

$$\mathscr{L}[\mathcal{S}(t)] \geq \frac{\mathcal{S}_0\sigma_1}{s^\xi + \sigma_2},$$

which further can be expressed as

$$\mathscr{L}[\mathcal{S}(t)] \geq \frac{\mathcal{S}_0\sigma_1}{\sigma_2}\sum_{n=0}^{\infty}\left(\frac{-1}{\sigma_2}\right)^n\frac{1}{s^{n\xi}}. \tag{18}$$

In view of inverse transform of Laplace, (18) implies that

$$\mathcal{S}(t) \geq \frac{\mathcal{S}_0\sigma_1}{\sigma_2}\sum_{n=0}^{\infty}\left(\frac{-1}{\sigma_2}\right)^n\frac{t^{-n\xi-1}}{\Gamma(-n\xi)}. \tag{19}$$

For simplicity using Wright function notation, one has $\mathsf{W}(u,v;z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n+1)\Gamma(u-vn)}$. Therefore, (21) is also expressed as

$$\mathcal{S}(t) \geq \frac{\mathcal{S}_0\sigma_1}{\sigma_2}\mathsf{W}\left(0,-\xi;\frac{-1}{\sigma_2 t^{\xi}}\right). \tag{20}$$

Finally from (20), we have $\frac{\mathcal{S}_0\sigma_1}{\sigma_2} > 0$, at $t > 0$, $\mathsf{W} > 0$. Thus we conclude from (16) and (20) that $\mathcal{S} > 0$, for all $t > 0$. Repeating the same procedure for other compartment of model (3), we can easily prove that

$$\mathcal{L}_{\mathcal{E}} > 0,\ \mathcal{I}_{pre} > 0,\ \mathcal{I}_{post} > 0,\ \mathcal{R} > 0,\ \text{for all } t > 0.$$

Further, from (3) by putting system equal to zero and for disease free equilibrium sitting $\mathcal{L}_{\mathcal{E}} = 0, \mathcal{I}_{pre} = 0,\ \mathcal{I}_{post} = 0,\ \mathcal{R} = 0$, we have from $\mathcal{S}^0 = \frac{\Lambda^{\xi}}{\rho^{\xi}}$. Thus trivial equilibrium is given by

$$\mathbf{E}^0 = \left(\mathcal{S}^0, \mathcal{L}^0_{\mathcal{E}}, \mathcal{I}^0_{pre},\ \mathcal{I}^0_{post},\ \mathcal{R}^0\right) = \left(\frac{\Lambda^{\xi}}{\rho^{\xi}}, 0, 0, 0, 0\right).$$

If $\overbrace{\mathbf{E}} = \left(\overbrace{\mathcal{S}}, \overbrace{\mathcal{L}}_{\mathcal{E}}, \overbrace{\mathcal{I}}_{pre}, \overbrace{\mathcal{I}}_{post}, \overbrace{\mathcal{R}}\right)$ be endemic equilibrium point of proposed model (3), then we compute its value by equating (3) left sides equal to zero after using $\overbrace{\mathcal{S}}, \overbrace{\mathcal{L}}_{\mathcal{E}}, \overbrace{\mathcal{I}}_{pre}, \overbrace{\mathcal{I}}_{post}, \overbrace{\mathcal{R}}$, one has

$$\begin{aligned}
\overbrace{\mathcal{S}} &= \frac{(\varepsilon^{\xi}+\rho^{\xi})(\rho^{\xi}+\kappa_1^{\xi}-\alpha^{\xi})(\rho^{\xi}+\kappa_2^{\xi})}{\varepsilon^{\xi}[\varepsilon^{\xi}(\rho^{\xi}+\kappa_2^{\xi})+(\alpha\epsilon_2)^{\xi}]}\\
\overbrace{\mathcal{L}}_{\mathcal{E}} &= \frac{\Lambda^{\xi}(\varepsilon^{\xi})^2[\varepsilon^{\xi}(\rho^{\xi}+\kappa_2^{\xi})+(\alpha\epsilon_2)^{\xi}]-\rho^{\xi}(\varepsilon^{\xi}+\rho^{\xi})(\rho^{\xi}+\kappa_1^{\xi}-\alpha^{\xi})(\rho^{\xi}+\kappa_2^{\xi})}{\varepsilon^{\xi}\alpha^{\xi}(\varepsilon^{\xi}+\rho^{\xi})[\epsilon_1^{\xi}(\rho^{\xi}+\kappa_2^{\xi})+\epsilon_2^{\xi}]}\\
\overbrace{\mathcal{I}}_{pre} &= \frac{\Lambda^{\xi}(\varepsilon^{\xi})^2[\varepsilon^{\xi}(\rho^{\xi}+\kappa_2^{\xi})+(\alpha\epsilon_2)^{\xi}]-\rho^{\xi}(\varepsilon^{\xi}+\rho^{\xi})(\rho^{\xi}+\kappa_1^{\xi}-\alpha^{\xi})(\rho^{\xi}+\kappa_2^{\xi})}{\alpha^{\xi}(\varepsilon^{\xi}+\rho^{\xi})(\rho^{\xi}+\kappa_1^{\xi}-\alpha^{\xi})[\epsilon_1^{\xi}(\rho^{\xi}+\kappa_2^{\xi})+\epsilon_2^{\xi}]}\\
\overbrace{\mathcal{I}}_{post} &= \frac{\Lambda^{\xi}(\varepsilon^{\xi})^2[\varepsilon^{\xi}(\rho^{\xi}+\kappa_2^{\xi})+(\alpha\epsilon_2)^{\xi}]-\rho^{\xi}(\varepsilon^{\xi}+\rho^{\xi})(\rho^{\xi}+\kappa_1^{\xi}-\alpha^{\xi})(\rho^{\xi}+\kappa_2^{\xi})}{(\varepsilon^{\xi}+\rho^{\xi})(\rho^{\xi}+\kappa_1^{\xi}-\alpha^{\xi})(\rho^{\xi}+\kappa_2^{\xi})[\epsilon_1^{\xi}(\rho^{\xi}+\kappa_2^{\xi})+\epsilon_2^{\xi}]}
\end{aligned}$$

$$\begin{aligned}\widehat{\mathcal{R}} &= \frac{\kappa_1^\xi}{\rho^\xi}\left[\frac{\Lambda^\xi(\varepsilon^\xi)^2[\varepsilon^\xi(\rho^\xi+\kappa_2^\xi)+(\alpha\epsilon_2)^\xi]-\rho^\xi(\varepsilon^\xi+\rho^\xi)(\rho^\xi+\kappa_1^\xi-\alpha^\xi)(\rho^\xi+\kappa_2^\xi)}{\alpha^\xi(\varepsilon^\xi+\rho^\xi)(\rho^\xi+\kappa_1^\xi-\alpha^\xi)[\epsilon_1^\xi(\rho^\xi+\kappa_2^\xi)+\epsilon_2^\xi]}\right] \\ &+ \frac{\kappa_2^\xi}{\rho^\xi}\left[\frac{\Lambda^\xi(\varepsilon^\xi)^2[\varepsilon^\xi(\rho^\xi+\kappa_2^\xi)+(\alpha\epsilon_2)^\xi]-\rho^\xi(\varepsilon^\xi+\rho^\xi)(\rho^\xi+\kappa_1^\xi-\alpha^\xi)(\rho^\xi+\kappa_2^\xi)}{(\varepsilon^\xi+\rho^\xi)(\rho^\xi+\kappa_1^\xi-\alpha^\xi)(\rho^\xi+\kappa_2^\xi)[\epsilon_1^\xi(\rho^\xi+\kappa_2^\xi)+\epsilon_2^\xi]}\right]. \quad (21)\end{aligned}$$

Further, considered the second equation of (3) as

$$^{\mathbb{PABC}}\mathrm{D}_{+0}^{\xi}(\mathcal{L}_{\mathcal{E}})(t) = \epsilon_1^\xi \mathcal{S}(t)\mathcal{I}_{pre}(t) + \epsilon_2^\xi \mathcal{S}(t)\mathcal{I}_{post}(t) - (\varepsilon^\xi + \rho^\xi)\mathcal{L}_{\mathcal{E}}. \quad (22)$$

Using next generation matrix approach [31], one can compute the basic reproductive number given by

$$\mathbf{R} = \mathbf{R}_1 + \mathbf{R}_2,$$

where

$$\mathbf{R}_1 = \frac{\varepsilon^\xi\epsilon_1^\xi\Lambda^\xi}{\rho^\xi(\varepsilon^\xi+\rho^\xi)(\rho^\xi+\kappa_1^\xi+\alpha^\xi)}, \mathbf{R}_2 = \frac{\varepsilon^\xi\epsilon_2^\xi\Lambda^\xi}{\rho^\xi(\varepsilon^\xi+\rho^\xi)(\rho^\xi+\kappa_1^\xi+\alpha^\xi)(\rho^\xi+\kappa_2^\xi)}.$$

Remark 1. *Moreover, if* $\mathbf{R} < 1$*, the unique trivial equilibrium* $\left(\frac{\Lambda^\xi}{\rho^\xi}, 0, 0, 0, 0\right)$ *is stable asymptotically and globally. In addition if* $\mathbf{R} > 1$*, then the trivial equilibrium is unstable and we have a unique non-trivial equilibrium been computed in* (21) *which stable globally and asymptotically.*

Theorem 3. *The nontrivial equilibrium point* $\widehat{\mathbf{E}}$ *is stable globally and asymptotically if* $\mathbf{R} > 1$*.*

Proof. We construct a Voltera- Lyapunov function

$$\mathcal{Y} : \{(\mathcal{S}, \mathcal{L}_{\mathcal{E}}, \mathcal{I}_{pre}, \mathcal{I}_{post}, \mathcal{R}) \in \Theta : (\mathcal{S} > 0, \mathcal{L}_{\mathcal{E}} > 0, \mathcal{I}_{pre} > 0, \mathcal{I}_{post} > 0, \mathcal{R} > 0\} \to \mathbb{R}$$

by

$$\begin{aligned}\mathcal{Y}(t) &= a_1\left(\mathcal{S} - \widehat{\mathcal{S}} - \widehat{\mathcal{S}}\log\left[\frac{\mathcal{S}}{\widehat{\mathcal{S}}}\right]\right) + a_2\left(\mathcal{L}_{\mathcal{E}} - \widehat{\mathcal{L}_{\mathcal{E}}} - \widehat{\mathcal{L}_{\mathcal{E}}}\log\left[\frac{\mathcal{L}_{\mathcal{E}}}{\widehat{\mathcal{L}_{\mathcal{E}}}}\right]\right) \\ &+ a_3\left(\mathcal{I}_{pre} - \widehat{\mathcal{I}_{pre}} - \widehat{\mathcal{I}_{pre}}\log\left[\frac{\mathcal{I}_{pre}}{\widehat{\mathcal{I}_{pre}}}\right]\right) + a_4\left(\mathcal{I}_{post} - \widehat{\mathcal{I}_{post}} - \widehat{\mathcal{I}_{post}}\log\left[\frac{\mathcal{I}_{post}}{\widehat{\mathcal{I}_{post}}}\right]\right) \\ &+ a_5\left(\mathcal{R} - \widehat{\mathcal{R}} - \widehat{\mathcal{R}}\log\left[\frac{\mathcal{R}}{\widehat{\mathcal{R}}}\right]\right), \quad (23)\end{aligned}$$

the function $\mathcal{Y}$ is continuous and positive definite for all $\mathcal{S} > 0, \mathcal{L}_{\mathcal{E}} > 0, \mathcal{I}_{pre} > 0, \mathcal{I}_{post} > 0, \mathcal{R} > 0$, while $a_i > 0(i = 1, 2, .., 5$ are constants which can be fixed later. Now using Lemma 2, from (23), we have

$$
\begin{aligned}
{}^{\mathbb{PABC}}\mathrm{D}^{\xi}_{+0}[\mathcal{Y}(t)] &\leq a_1\left(\frac{\mathcal{S}-\overbrace{\mathcal{S}}}{\mathcal{S}}\right){}^{\mathbb{PABC}}\mathrm{D}^{\xi}_{+0}\mathcal{S}(t)+a_2\left(\frac{\mathcal{L}_{\mathcal{E}}-\overbrace{\mathcal{L}}_{\mathcal{E}}}{\mathcal{L}_{\mathcal{E}}}\right){}^{\mathbb{PABC}}\mathrm{D}^{\xi}_{+0}\mathcal{L}_{\mathcal{E}}(t)\\
&+ a_3\left(\frac{\mathcal{I}_{pre}-\overbrace{\mathcal{I}}_{pre}}{\overbrace{\mathcal{I}}_{pre}}\right){}^{\mathbb{PABC}}\mathrm{D}^{\xi}_{+0}\mathcal{I}_{pre}(t)+a_4\left(\frac{\mathcal{I}_{post}-\overbrace{\mathcal{I}}_{post}}{\overbrace{\mathcal{I}}_{post}}\right){}^{\mathbb{PABC}}\mathrm{D}^{\xi}_{+0}\mathcal{I}_{post}(t)\\
&+ a_5\left(\frac{\mathcal{R}-\overbrace{\mathcal{R}}}{\mathcal{R}}\right){}^{\mathbb{PABC}}\mathrm{D}^{\xi}_{+0}\mathcal{R}(t), \qquad (24)
\end{aligned}
$$

from model (3), (24) yields that

$$
\begin{aligned}
{}^{\mathbb{PABC}}\mathrm{D}^{\xi}_{+0}[\mathcal{Y}(t)] &\leq a_1\left(\frac{\mathcal{S}-\overbrace{\mathcal{S}}}{\mathcal{S}}\right)\Big[\Lambda^{\xi}-\rho^{\xi}\mathcal{S}(t)-\epsilon_1^{\xi}\mathcal{S}(t)\mathcal{I}_{pre}(t)-\epsilon_2^{\xi}\mathcal{S}(t)\mathcal{I}_{post}(t)\Big]\\
&+ a_2\left(\frac{\mathcal{L}_{\mathcal{E}}-\overbrace{\mathcal{L}}_{\mathcal{E}}}{\mathcal{L}_{\mathcal{E}}}\right)\Big[\epsilon_1^{\xi}\mathcal{S}(t)\mathcal{I}_{pre}(t)+\epsilon_2^{\xi}\mathcal{S}(t)\mathcal{I}_{post}(t)-(\varepsilon^{\xi}+\rho^{\xi})\mathcal{L}_{\mathcal{E}}\Big]\\
&+ a_3\left(\frac{\mathcal{I}_{pre}-\overbrace{\mathcal{I}}_{pre}}{\overbrace{\mathcal{I}}_{pre}}\right)\Big[\varepsilon^{\xi}\mathcal{L}_{\mathcal{E}}-(\rho^{\xi}+\kappa_1^{\xi})\mathcal{I}_{pre}(t)-\alpha^{\xi}\mathcal{I}_{pre}(t)\Big]\\
&+ a_4\left(\frac{\mathcal{I}_{post}-\overbrace{\mathcal{I}}_{post}}{\overbrace{\mathcal{I}}_{post}}\right)\Big[\alpha^{\xi}\mathcal{I}_{pre}(t)-(\rho^{\xi}+\kappa_2^{\xi})\mathcal{I}_{post}(t)\Big]\\
&+ a_5\left(\frac{\mathcal{R}-\overbrace{\mathcal{R}}}{\mathcal{R}}\right)\Big[\kappa_1^{\xi}\mathcal{I}_{pre}(t)+\kappa_2^{\xi}\mathcal{I}_{post}(t)-\rho^{\xi}\mathcal{R}(t)\Big]. \qquad (25)
\end{aligned}
$$

Plugging

$$
\mathcal{S}=\mathcal{S}-\overbrace{\mathcal{S}}, \mathcal{L}_{\mathcal{E}}=\mathcal{L}_{\mathcal{E}}-\overbrace{\mathcal{L}}_{\mathcal{E}}, \mathcal{I}_{pre}=\mathcal{I}_{pre}-\overbrace{\mathcal{I}}_{pre}, \mathcal{I}_{post}=\mathcal{I}_{post}-\overbrace{\mathcal{I}}_{post}, \mathcal{R}=\mathcal{R}-\overbrace{\mathcal{R}}
$$

in (25), we have

$$
{}^{\mathbb{PABC}}\mathrm{D}^{\xi}_{+0}[\mathcal{Y}(t)] \leq a_1\left(\frac{\mathcal{S}-\overbrace{\mathcal{S}}}{\mathcal{S}}\right)\Big[\Lambda^{\xi}-\rho^{\xi}(\mathcal{S}-\overbrace{\mathcal{S}})
$$

$$
\begin{aligned}
&- \quad \epsilon_1^{\xi}(\mathcal{S}-\overbrace{\mathcal{S}})(\mathcal{I}_{pre}-\overbrace{\mathcal{I}}_{pre}) - \epsilon_2^{\xi}(\mathcal{S}-\overbrace{\mathcal{S}})(\mathcal{I}_{post}-\overbrace{\mathcal{I}}_{post})\Bigg] \\
&+ \quad a_2\left(\frac{\mathcal{L}_{\mathcal{E}}-\overbrace{\mathcal{L}}_{\mathcal{E}}}{\mathcal{L}_{\mathcal{E}}}\right)\Bigg[\epsilon_1^{\xi}(\mathcal{S}-\overbrace{\mathcal{S}})(\mathcal{I}_{pre}-\overbrace{\mathcal{I}}_{pre}) + \epsilon_2^{\xi}(\mathcal{S}-\overbrace{\mathcal{S}})(\mathcal{I}_{post}-\overbrace{\mathcal{I}}_{post}) \\
&- \quad (\varepsilon^{\xi}+\rho^{\xi})(\mathcal{L}_{\mathcal{E}}-\overbrace{\mathcal{L}}_{\mathcal{E}})\Bigg] \\
&+ \quad a_3\left(\frac{\mathcal{I}_{pre}-\overbrace{\mathcal{I}}_{pre}}{\overbrace{\mathcal{I}}_{pre}}\right)\Bigg[\varepsilon^{\xi}(\mathcal{L}_{\mathcal{E}}-\overbrace{\mathcal{L}}_{\mathcal{E}}) - (\rho^{\xi}+\kappa_1^{\xi})(\mathcal{I}_{pre}-\overbrace{\mathcal{I}}_{pre}) - \alpha^{\xi}(\mathcal{I}_{pre}-\overbrace{\mathcal{I}}_{pre})\Bigg] \\
&+ \quad a_4\left(\frac{\mathcal{I}_{post}-\overbrace{\mathcal{I}}_{post}}{\overbrace{\mathcal{I}}_{post}}\right)\Bigg[\alpha^{\xi}(\mathcal{I}_{pre}-\overbrace{\mathcal{I}}_{pre}) - (\rho^{\xi}+\kappa_2^{\xi})(\mathcal{I}_{post}-\overbrace{\mathcal{I}}_{post})\Bigg] \\
&+ \quad a_5\left(\frac{\mathcal{R}-\overbrace{\mathcal{R}}}{\mathcal{R}}\right)\Bigg[\kappa_1^{\xi}(\mathcal{I}_{pre}-\overbrace{\mathcal{I}}_{pre}) + \kappa_2^{\xi}(\mathcal{I}_{post}-\overbrace{\mathcal{I}}_{post}) - \rho^{\xi}(\mathcal{R}-\overbrace{\mathcal{R}})\Bigg]. \qquad (26)
\end{aligned}
$$

Re-arranging the terms, we can write (26) using $a_i = 1(i = 1, 2, ..., 5)$ as

$$
{}^{\mathbb{PABC}}\mathrm{D}^{\xi}_{+0}[\mathcal{Y}(t)] \quad \leq \quad \Omega_1 - \Omega_2, \qquad (27)
$$

where

$$
\begin{aligned}
\Omega_1 \quad = \quad & \left(\frac{\mathcal{S}-\overbrace{\mathcal{S}}}{\mathcal{S}}\right)\Lambda^{\xi} + \left(\frac{\mathcal{L}_{\mathcal{E}}-\overbrace{\mathcal{L}}_{\mathcal{E}}}{\mathcal{L}_{\mathcal{E}}}\right)\Bigg[\epsilon_1^{\xi}(\mathcal{S}-\overbrace{\mathcal{S}})(\mathcal{I}_{pre}-\overbrace{\mathcal{I}}_{pre}) + \epsilon_2^{\xi}(\mathcal{S}-\overbrace{\mathcal{S}})(\mathcal{I}_{post}-\overbrace{\mathcal{I}}_{post})\Bigg] \\
+ \quad & \left(\frac{\mathcal{I}_{pre}-\overbrace{\mathcal{I}}_{pre}}{\overbrace{\mathcal{I}}_{pre}}\right)\varepsilon^{\xi}(\mathcal{L}_{\mathcal{E}}-\overbrace{\mathcal{L}}_{\mathcal{E}}) + \left(\frac{\mathcal{I}_{post}-\overbrace{\mathcal{I}}_{post}}{\overbrace{\mathcal{I}}_{post}}\right)\alpha^{\xi}(\mathcal{I}_{pre}-\overbrace{\mathcal{I}}_{pre}) \\
+ \quad & \left(\frac{\mathcal{R}-\overbrace{\mathcal{R}}}{\mathcal{R}}\right)\Bigg[\kappa_1^{\xi}(\mathcal{I}_{pre}-\overbrace{\mathcal{I}}_{pre}) + \kappa_2^{\xi}(\mathcal{I}_{post}-\overbrace{\mathcal{I}}_{post})\Bigg]
\end{aligned}
$$

and

$$
\begin{aligned}
\Omega_2 \quad = \quad & \frac{\epsilon_2^{\xi}}{\mathcal{S}}\Big[\mathcal{S}-\overbrace{\mathcal{S}}\Big]^2(\mathcal{I}_{post}-\overbrace{\mathcal{I}}_{post}) + \frac{(\varepsilon^{\xi}+\rho^{\xi})}{\mathcal{L}_{\mathcal{E}}}\Big[\mathcal{L}_{\mathcal{E}}-\overbrace{\mathcal{L}}_{\mathcal{E}}\Big]^2 \\
+ \quad & \frac{(\rho^{\xi}+\kappa_1^{\xi}+\alpha^{\xi})}{\overbrace{\mathcal{I}}_{pre}}\Big[\mathcal{I}_{pre}-\overbrace{\mathcal{I}}_{pre}\Big]^2 + \frac{(\rho^{\xi}+\kappa_2^{\xi})}{\overbrace{\mathcal{I}}_{post}}\Big[\mathcal{I}_{post}-\overbrace{\mathcal{I}}_{post}\Big]^2 + \frac{\rho^{\xi}}{\mathcal{R}}\Big[\mathcal{R}-\overbrace{\mathcal{R}}\Big]^2.
\end{aligned}
$$

We observe from (27), if $\Omega_1 < \Omega_2$, then ${}^{\mathbb{PABC}}\mathrm{D}^{\xi}_{+0}[\mathcal{Y}(t)] \leq 0$. Also if

$$\mathcal{S} = \widehat{\mathcal{S}}, \mathcal{L}_{\mathcal{E}} = \widehat{\mathcal{L}}_{\mathcal{E}}, \mathcal{I}_{pre} = \widehat{\mathcal{I}}_{pre}, \mathcal{I}_{post} = \widehat{\mathcal{I}}_{post}, \mathcal{R} = \widehat{\mathcal{R}},$$

then one has

$${}^{\mathbb{PABC}}\mathrm{D}^{\xi}_{+0}[\mathcal{Y}(t)] = 0$$

, hence our proposed model will turned in the largest invariant set given by

$$\left\{(\widehat{\mathcal{S}}, \widehat{\mathcal{L}}_{\mathcal{E}}, \widehat{\mathcal{I}}_{pre}, \widehat{\mathcal{I}}_{post}, \widehat{\mathcal{R}}) \in \Theta :^{\mathbb{PABC}} \mathrm{D}^{\xi}_{+0}[\mathcal{Y}(t)] = 0\right\}.$$

Hence in view of Lasalle's invariant principle, the non-trivial equilibrium of our proposed model is globally asymptotically stable if $\Omega_1 < \Omega_2$.

Here, we derive some sufficient requirements for the existence of a solution to the suggested model by applying analysis results. (3) can be expressed as a general system using $\mathcal{X} = (\mathcal{S}, \mathcal{L}_{\mathcal{E}}, \mathcal{I}_{pre}, \mathcal{I}_{post}, \mathcal{R})$

$$\begin{cases} {}^{\mathbb{PABC}}\mathrm{D}^{\xi}_{+0}(\mathcal{X})(t) = \mathcal{F}(t, \mathcal{X}(t)), \\ \mathcal{X}(0) = \mathcal{X}_0. \end{cases} \tag{28}$$

The solution of (28) is given by inview of Lemma 1 as

$$\mathcal{X}(t) = \begin{cases} \mathcal{X}_0 + \dfrac{1}{\Gamma(\xi)} \displaystyle\int_0^t \mathcal{F}(\ell, \mathcal{X}(\ell))(t-\ell)^{\xi-1} d\ell, \ t \in \mathbb{J}_1, \\ \mathcal{X}(t_1) + \dfrac{1-\xi}{\mathbb{ABC}(\xi)} \mathcal{F}(t, \mathcal{X}(t)) + \dfrac{\xi}{\mathbb{ABC}\xi\Gamma(\xi)} \displaystyle\int_{t_1}^t (t-\ell)^{\xi-1} \mathcal{F}(\ell, \mathcal{X}(\ell)) d(\ell), \ t \in \mathbb{J}_2. \end{cases} \tag{29}$$

The hypothesis holds:

(H_1) For $\mathcal{X}, \bar{\mathcal{X}} \in \mathbb{B}$, and real value $\mathscr{K}_{\mathcal{F}} > 0$, one has

$$|\mathcal{F}(t, \mathcal{X}) - \mathcal{F}(t, \bar{\mathcal{X}})| \leq \mathscr{K}_{\mathcal{F}} |\mathcal{X} - \overline{\mathcal{X}}|.$$

Theorem 4. *In view of* (H_1), *if* $\max\left\{\frac{t_1^{\xi}}{\Gamma(\xi+1)}, \frac{\Gamma(\xi)+T^{\xi}}{\mathbb{ABC}(\xi)\Gamma(\xi)}\right\}\mathscr{K}_{\mathcal{F}} < 1$, *then* (28) *has a unique solution which justifies the unique solution for model* (3).

Proof. Let $\mathbb{T} : \mathbb{B} \to \mathbb{B}$ defined by

$$\mathbb{T}[\mathcal{X}(t)] = \begin{cases} \mathcal{X}_0 + \dfrac{1}{\Gamma(\xi)} \displaystyle\int_0^t \mathcal{F}(\ell, \mathcal{X}(\ell))(t-\ell)^{\xi-1} d\ell, \ t \in \mathbb{J}_1, \\ \mathcal{X}(t_1) + \dfrac{1-\xi}{\mathbb{ABC}(\xi)} \mathcal{F}(t, \mathcal{X}(t)) + \dfrac{\xi}{\mathbb{ABC}(\xi)\Gamma(\xi)} \displaystyle\int_{t_1}^t (t-\ell)^{\xi-1} \mathcal{F}(\ell, \mathcal{X}(\ell)) d(\ell), \ t \in \mathbb{J}_2. \end{cases} \tag{30}$$

For $t \in \mathbb{J}_1$ and $\mathcal{X}, \bar{\mathcal{X}} \in \mathbb{B}$, from (31), one has

$$\begin{aligned} \|\mathbb{T}(\mathcal{X}) - \mathbb{T}(\overline{\mathcal{X}})\| &\leq \max_{t\in\mathbb{J}_1} \frac{1}{\Gamma(\xi)} \int_0^t (t-\ell)^{\xi-1} |\mathcal{F}(\ell, \mathcal{X}(\ell)) - \mathcal{F}(\ell, \overline{\mathcal{X}}(\ell))| d\ell \\ &\leq \frac{\mathscr{K}_{\mathcal{F}} t_1^{\xi}}{\Gamma(\xi+1)} \|\mathcal{X} - \overline{\mathcal{X}}\|. \end{aligned} \tag{31}$$

In the same way, if $t \in \mathbb{J}_2$ and $\mathcal{X}, \bar{\mathcal{X}} \in \mathbb{B}$, from (31), one has

$$\begin{aligned} \|\mathbb{T}(\mathcal{X}) - \mathbb{T}(\overline{\mathcal{X}})\| &\leq \frac{1-\xi}{\mathbb{A}\mathbb{B}\mathbb{C}(\xi)} \max_{t\in\mathbb{J}_2} \left| \mathcal{F}(t, \mathcal{X}(t)) - \mathcal{F}(t, \overline{\mathcal{X}}(t)) \right| \\ &+ \max_{t\in\mathbb{J}_2} \frac{\xi}{\mathbb{A}\mathbb{B}\mathbb{C}(\xi)\Gamma(\xi)} \int_{t_1}^t (t-\ell)^{\xi-1} |\mathcal{F}(\ell, \mathcal{X}(\ell)) - \mathcal{F}(\ell, \overline{\mathcal{X}}(\ell))| d\ell \\ &\leq \frac{\mathscr{K}_{\mathcal{F}}}{\mathbb{A}\mathbb{B}\mathbb{C}(\xi)} \|\mathcal{X} - \overline{\mathcal{X}}\| + \frac{\mathscr{K}_{\mathcal{F}} T^{\xi}}{\Gamma(\xi)\mathbb{A}\mathbb{B}\mathbb{C}(\xi)} \|\mathcal{X} - \overline{\mathcal{X}}\| \\ &= \left[\frac{\mathscr{K}_{\mathcal{F}}}{\mathbb{A}\mathbb{B}\mathbb{C}(\xi)} + \frac{\mathscr{K}_{\mathcal{F}} T^{\xi}}{\Gamma(\xi)\mathbb{A}\mathbb{B}\mathbb{C}(\xi)} \right] \|\mathcal{X} - \overline{\mathcal{X}}\|. \end{aligned} \tag{32}$$

From (31) and (32), one can concludes that $\mathbb{T}$ satisfies the criteria of Banach theorem. Thus our proposed model (3) has a unique solution.

4. Simulations Methodology

Here, we develop a numerical algorithm for the piecewise model (3) that has been suggested. For our model, we expand the numerical scheme of [28]. In this case, the domain $\mathbb{J}$ is split into two sub-intervals. First from the solution (29) of (28), using $t = t_{m+1}$, we get

$$\mathcal{X}(t_{m+1}) = \begin{cases} \mathcal{X}_0 + \dfrac{1}{\Gamma(\xi)} \displaystyle\int_0^{t_m} \mathcal{F}(\ell, \mathcal{X}(\ell))(t_m - \ell)^{\xi-1} d\ell, \ t \in \mathbb{J}_1, \\ \mathcal{X}(t_1) + \dfrac{1-\xi}{\mathbb{A}\mathbb{B}\mathbb{C}(\xi)} \mathcal{F}(t_m, \mathcal{X}(t_m)) + \dfrac{\xi}{\mathbb{A}\mathbb{B}\mathbb{C}\xi\Gamma(\xi)} \displaystyle\int_{t_1}^{t_{m+1}} (t_{m+1} - \ell)^{\xi-1} \mathcal{F}(\ell, \mathcal{X}(\ell)) d(\ell), \ t \in \mathbb{J}_2. \end{cases} \tag{33}$$

If we use for simplicity

$$\mho_1 = \begin{bmatrix} (1+m-q)^{\xi} \Big(2(m-q)^2 + (3\xi+10)(m-q) + 2\xi^2 + 9\xi + 12 \Big) \\ -(m-q) \Big(2(m-q)^2 + (5\xi+10)(-q+m) + 6\xi^2 + 18\xi + 12 \Big) \end{bmatrix},$$

$$\mho_2 = \begin{bmatrix} (1+m-q)^{\xi} \Big(3 + 2\xi - q + m \Big) \\ -(m-q) \Big(m - q + 3\xi + 3 \Big) \end{bmatrix}, \quad \mho_3 = \begin{bmatrix} (1+m-q)^{\xi} - (m-q)^{\xi} \end{bmatrix}$$

and step size $\mathrm{h} = \Delta t$, using interpolation approximation in (33), we have

$$\mathcal{X}(t_{m+1}) = \begin{cases} \mathcal{X}_0 + \begin{cases} \dfrac{\mathrm{h}^{\xi-1}}{\Gamma(\xi+1)} \displaystyle\sum_{q=2}^{i} \Big[\mathcal{F}(\mathcal{X}(t_{q-2}), t_{q-2})\Big] \mho_1 \\ + \dfrac{\mathrm{h}^{\xi-1}}{\Gamma(\xi+2)} \displaystyle\sum_{q=2}^{i} \Big[\mathcal{F}(\mathcal{X}(t_{q-1}), t_{q-1}) - \mathcal{F}(\mathcal{X}(t_{q-2}), t_{q-2})\Big] \mho_2 \\ + \dfrac{\xi \mathrm{h}^{\xi-1}}{2\Gamma(\xi+3)} \displaystyle\sum_{q=2}^{i} \Big[\mathcal{F}(\mathcal{X}(t_{q-2}), t_{q-2})\Big] \mho_3, \end{cases} \\ \mathcal{X}(t_1) + \begin{cases} \dfrac{1-\xi}{\mathbb{ABC}(\xi)} \mathcal{F}(\mathcal{X}(t_m) t_m) + \dfrac{\xi}{\mathbb{ABC}(\xi)} \dfrac{\mathrm{h}^{\xi-1}}{\Gamma(\xi+1)} \displaystyle\sum_{q=i+3}^{m} \Big[\mathcal{F}(\mathcal{X}(t_{q-2}), t_{q-2})\Big] \mho_1 \\ + \dfrac{\xi}{\mathbb{ABC}(\xi)} \dfrac{\mathrm{h}^{\xi-1}}{\Gamma(\xi+2)} \displaystyle\sum_{q=i+3}^{m} \Big[\mathcal{F}(\mathcal{X}_(t_{q-1}), t_{q-1}) \\ + \mathcal{F}(\mathcal{X}_(t_{q-2}), t_{q-2})\Big] \mho_2 \\ + \dfrac{1}{\mathbb{ABC}(\xi)} \dfrac{\xi \mathrm{h}^{\xi-1}}{\Gamma(\xi+3)} \displaystyle\sum_{q=i+3}^{m} \Big[\mathcal{F}(\mathcal{X}_(t_q), t_q) - 2\mathcal{F}(\mathcal{X}(t_{q-1}), t_{q-1}) \\ + \mathcal{F}(\mathcal{X}(t_{q-2}), t_{q-2})\Big] \mho_3. \end{cases} \end{cases} \tag{34}$$

According to the numerical formula given in (34), we can write our proposed model (3) as

$$\mathcal{S}(t_{m+1}) = \begin{cases} \mathcal{S}_0 + \begin{cases} \dfrac{\mathrm{h}^{\xi-1}}{\Gamma(\xi+1)} \displaystyle\sum_{q=2}^{i} \Big[\Lambda^{\xi} - (\rho^{\xi} + \epsilon_1^{\xi}\mathcal{I}_{pre}(t_{q-2}) + \epsilon_2^{\xi}\mathcal{I}_{post}(t_{q-2}))\mathcal{S}(t_{q-2})\Big]\mho_1 \\ + \dfrac{\mathrm{h}^{\xi-1}}{\Gamma(\xi+2)} \displaystyle\sum_{q=2}^{i} \Big[\Lambda^{\xi} - \rho^{\xi}\mathcal{S}(t_{q-1}) - \epsilon_1^{\xi}\mathcal{S}(t_{q-1})\mathcal{I}_{pre}(t_{q-1}) - \epsilon_2^{\xi}\mathcal{S}(t_{q-1})\mathcal{I}_{post}(t_{q-1}) \\ - \Big(\Lambda^{\xi} - \rho^{\xi}\mathcal{S}(t_{q-2}) - \epsilon_1^{\xi}\mathcal{S}(t_{q-2})\mathcal{I}_{pre}(t_{q-2}) - \epsilon_2^{\xi}\mathcal{S}(t_{q-2})\mathcal{I}_{post}(t_{q-2})\Big)\Big]\mho_2 \\ + \dfrac{\xi\mathrm{h}^{\xi-1}}{2\Gamma(\xi+3)} \displaystyle\sum_{q=2}^{i} \Big[\Lambda^{\xi} - \mathcal{S}(t_{q-2})(\rho^{\xi} + \epsilon_1^{\xi}\mathcal{I}_{pre}(t_{q-2}) + \epsilon_2^{\xi}\mathcal{I}_{post}(t_{q-2}))\Big]\mho_3, \end{cases} \\ \mathcal{S}(t_1) + \begin{cases} \dfrac{1-\xi}{\mathbb{ABC}(\xi)}\Big[\Lambda^{\xi} - \rho^{\xi}\mathcal{S}(t_m) - \epsilon_1^{\xi}\mathcal{S}(t_m)\mathcal{I}_{pre}(t_m) - \epsilon_2^{\xi}\mathcal{S}(t_m)\mathcal{I}_{post}(t_m)\Big] \\ + \dfrac{\xi}{\mathbb{ABC}(\xi)}\dfrac{\mathrm{h}^{\xi-1}}{\Gamma(\xi+1)} \displaystyle\sum_{q=i+3}^{m} \Big[\Lambda^{\xi} - \rho^{\xi}\mathcal{S}(t_{q-2}) \\ - \epsilon_1^{\xi}\mathcal{S}(t_{q-2})\mathcal{I}_{pre}(t_{q-2}) - \epsilon_2^{\xi}\mathcal{S}(t_{q-2})\mathcal{I}_{post}(t_{q-2})\Big]\mho_1 \\ + \dfrac{\xi}{\mathbb{ABC}(\xi)}\dfrac{\mathrm{h}^{\xi-1}}{\Gamma(\xi+2)} \displaystyle\sum_{q=i+3}^{m} \Big[\Lambda^{\xi} - \rho^{\xi}\mathcal{S}(t_{q-2}) - \epsilon_1^{\xi}\mathcal{S}(t_{q-1})\mathcal{I}_{pre}(t_{q-1}) \\ - \epsilon_2^{\xi}\mathcal{S}(t_{q-1})\mathcal{I}_{post}(t_{q-1}) \\ + \Lambda^{\xi} - \rho^{\xi}\mathcal{S}(t_{q-2}) - \epsilon_1^{\xi}\mathcal{S}(t_{q-2})\mathcal{I}_{pre}(t_{q-2}) - \epsilon_2^{\xi}\mathcal{S}(t_{q-2})\mathcal{I}_{post}(t_{q-2})\Big]\mho_2 \\ + \dfrac{1}{\mathbb{ABC}(\xi)}\dfrac{\xi\mathrm{h}^{\xi-1}}{\Gamma(\xi+3)} \displaystyle\sum_{q=i+3}^{m} \Big[\Lambda^{\xi} - \rho^{\xi}\mathcal{S}(t_q) - \epsilon_1^{\xi}\mathcal{S}(t_q)\mathcal{I}_{pre}(t_q) - \epsilon_2^{\xi}\mathcal{S}(t_q)\mathcal{I}_{post}(t_q) \\ - 2\Big[\Lambda^{\xi} - \rho^{\xi}\mathcal{S}(t_{q-1}) - \epsilon_1^{\xi}\mathcal{S}(t_{q-1})\mathcal{I}_{pre}(t_{q-1}) - \epsilon_2^{\xi}\mathcal{S}(t_{q-1})\mathcal{I}_{post}(t_{q-1})\Big] \\ + \Lambda^{\xi} - \rho^{\xi}\mathcal{S}(t_{q-2}) - \epsilon_1^{\xi}\mathcal{S}(t_{q-2})\mathcal{I}_{pre}(t_{q-2}) - \epsilon_2^{\xi}\mathcal{S}(t_{q-2})\mathcal{I}_{post}(t_{q-2})\Big]\mho_3. \end{cases} \end{cases} \tag{35}$$

$$\mathcal{L}_{\mathcal{E}}(t_{m+1}) = \begin{cases} \mathcal{L}_{\mathcal{E}0} + \begin{cases} \dfrac{\mathrm{h}^{\xi-1}}{\Gamma(\xi+1)} \displaystyle\sum_{q=2}^{i} \Big[\mathcal{S}(t_{q-2})(\epsilon_1^{\xi}\mathcal{I}_{pre}(t_{q-2}) + \epsilon_2^{\xi}\mathcal{I}_{post}(t_{q-2})) - (\varepsilon^{\xi}+\rho^{\xi})\mathcal{L}_{\mathcal{E}}(t_{q-2}) \Big] \mho_1 \\ + \dfrac{\mathrm{h}^{\xi-1}}{\Gamma(\xi+2)} \displaystyle\sum_{q=2}^{i} \Big[\epsilon_1^{\xi}\mathcal{S}(t_{q-1})\mathcal{I}_{pre}(t_{q-1}) + \epsilon_2^{\xi}\mathcal{S}(t_{q-1})\mathcal{I}_{post}(t_{q-1}) - (\varepsilon^{\xi}+\rho^{\xi})\mathcal{L}_{\mathcal{E}}(t_{q-1}) \\ - \Big(\epsilon_1^{\xi}\mathcal{S}(t_{q-2})\mathcal{I}_{pre}(t_{q-2}) + \epsilon_2^{\xi}\mathcal{S}(t_{q-2})\mathcal{I}_{post}(t_{q-2}) - (\varepsilon^{\xi}+\rho^{\xi})\mathcal{L}_{\mathcal{E}}(t_{q-2}) \Big) \Big] \mho_2 \\ + \dfrac{\xi\mathrm{h}^{\xi-1}}{2\Gamma(\xi+3)} \displaystyle\sum_{q=2}^{i} \Big[(\epsilon_1^{\xi}\mathcal{I}_{pre}(t_{q-2}) + \epsilon_2^{\xi}\mathcal{I}_{post}(t_{q-2}))\mathcal{S}(t_{q-2}) - (\varepsilon^{\xi}+\rho^{\xi})\mathcal{L}_{\mathcal{E}}(t_{q-2}) \Big] \mho_3, \end{cases} \\ \mathcal{L}_{\mathcal{E}}(t_1) + \begin{cases} \dfrac{1-\xi}{\mathbb{ABC}(\xi)} \Big[(\epsilon_1^{\xi}\mathcal{I}_{pre}(t_m) + \epsilon_2^{\xi}\mathcal{I}_{post}(t_m))\mathcal{S}(t_m) - (\varepsilon^{\xi}+\rho^{\xi})\mathcal{L}_{\mathcal{E}}(t_m) \Big] \\ + \dfrac{\xi}{\mathbb{ABC}(\xi)} \dfrac{\mathrm{h}^{\xi-1}}{\Gamma(\xi+1)} \displaystyle\sum_{q=i+3}^{m} \Big[\Big(\epsilon_1^{\xi}\mathcal{I}_{pre}(t_{q-2}) \\ + \epsilon_2^{\xi}\mathcal{I}_{post}(t_{q-2}) \Big) \mathcal{S}(t_{q-2}) - (\varepsilon^{\xi}+\rho^{\xi})\mathcal{L}_{\mathcal{E}}(t_{q-2}) \Big] \mho_1 \\ + \dfrac{\xi}{\mathbb{ABC}(\xi)} \dfrac{\mathrm{h}^{\xi-1}}{\Gamma(\xi+2)} \displaystyle\sum_{q=i+3}^{m} \Big[\epsilon_1^{\xi}\mathcal{S}(t_{q-1})\mathcal{I}_{pre}(t_{q-1}) + \epsilon_2^{\xi}\mathcal{S}(t_{q-1})\mathcal{I}_{post}(t_{q-1}) \\ - (\varepsilon^{\xi}+\rho^{\xi})\mathcal{L}_{\mathcal{E}}(t_{q-1}) \\ + \epsilon_1^{\xi}\mathcal{S}(t_{q-2})\mathcal{I}_{pre}(t_{q-2}) + \epsilon_2^{\xi}\mathcal{S}(t_{q-2})\mathcal{I}_{post}(t_{q-2}) - (\varepsilon^{\xi}+\rho^{\xi})\mathcal{L}_{\mathcal{E}}(t_{q-2}) \Big] \mho_2 \\ + \dfrac{1}{\mathbb{ABC}(\xi)} \dfrac{\xi\mathrm{h}^{\xi-1}}{\Gamma(\xi+3)} \displaystyle\sum_{q=i+3}^{m} \Big[\epsilon_1^{\xi}\mathcal{S}(t_q)\mathcal{I}_{pre}(t_q) + \epsilon_2^{\xi}\mathcal{S}(t_q)\mathcal{I}_{post}(t_q) - (\varepsilon^{\xi}+\rho^{\xi})\mathcal{L}_{\mathcal{E}}(t_q) \\ - 2\Big(\epsilon_1^{\xi}\mathcal{S}(t_{q-1})\mathcal{I}_{pre}(t_{q-1}) + \epsilon_2^{\xi}\mathcal{S}(t_{q-1})\mathcal{I}_{post}(t_{q-1}) - (\varepsilon^{\xi}+\rho^{\xi})\mathcal{L}_{\mathcal{E}}(t_{q-1}) \Big) \\ + \epsilon_1^{\xi}\mathcal{S}(t_{q-2})\mathcal{I}_{pre}(t_{q-2}) + \epsilon_2^{\xi}\mathcal{S}(t_{q-2})\mathcal{I}_{post}(t_{q-2}) - (\varepsilon^{\xi}+\rho^{\xi})\mathcal{L}_{\mathcal{E}}(t_{q-2}) \Big] \mho_3. \end{cases} \end{cases} \tag{36}$$

$$\mathcal{I}_{pre}(t_{m+1}) = \begin{cases} \mathcal{I}_{pre_0} + \begin{cases} \dfrac{\mathrm{h}^{\xi-1}}{\Gamma(\xi+1)} \displaystyle\sum_{q=2}^{i} \Big[\varepsilon^{\xi}\mathcal{L}_{\mathcal{E}}(t_{q-2}) - (\rho^{\xi}+\kappa_1^{\xi})\mathcal{I}_{pre}(t_{q-2}) - \alpha^{\xi}\mathcal{I}_{pre}(t_{q-2}-\theta)\Big]\mho_1 \\ + \dfrac{\mathrm{h}^{\xi-1}}{\Gamma(\xi+2)} \displaystyle\sum_{q=2}^{i} \Big[\varepsilon^{\xi}\mathcal{L}_{\mathcal{E}}(t_{q-1}) - (\rho^{\xi}+\kappa_1^{\xi})\mathcal{I}_{pre}(t_{q-1}) - \alpha^{\xi}\mathcal{I}_{pre}(t_{q-1}) \\ - \Big(\varepsilon^{\xi}\mathcal{L}_{\mathcal{E}}(t_{q-2}) - (\rho^{\xi}+\kappa_1^{\xi})\mathcal{I}_{pre}(t_{q-2}) - \alpha^{\xi}\mathcal{I}_{pre}(t_{q-2})\Big)\Big]\mho_2 \\ + \dfrac{\xi\mathrm{h}^{\xi-1}}{2\Gamma(\xi+3)} \displaystyle\sum_{q=2}^{i} \Big[\varepsilon^{\xi}\mathcal{L}_{\mathcal{E}}(t_{q-2}) - (\rho^{\xi}+\kappa_1^{\xi})\mathcal{I}_{pre}(t_{q-2}) - \alpha^{\xi}\mathcal{I}_{pre}(t_{q-2})\Big]\mho_3, \end{cases} \\ \mathcal{I}_{pre}(t_1) + \begin{cases} \dfrac{1-\xi}{\mathbb{ABC}(\xi)}\Big[\varepsilon^{\xi}\mathcal{L}_{\mathcal{E}}(t_m) - (\rho^{\xi}+\kappa_1^{\xi})\mathcal{I}_{pre}(t_m) - \alpha^{\xi}\mathcal{I}_{pre}(t_m)\Big] + \dfrac{\xi}{\mathbb{ABC}(\xi)}\dfrac{\mathrm{h}^{\xi-1}}{\Gamma(\xi+1)} \\ \times \displaystyle\sum_{q=i+3}^{m} \Big[\varepsilon^{\xi}\mathcal{L}_{\mathcal{E}}(t_{q-2}) - (\rho^{\xi}+\kappa_1^{\xi})\mathcal{I}_{pre}(t_{q-2}) - \alpha^{\xi}\mathcal{I}_{pre}(t_{q-2})\Big]\mho_1 \\ + \dfrac{\xi}{\mathbb{ABC}(\xi)}\dfrac{\mathrm{h}^{\xi-1}}{\Gamma(\xi+2)} \displaystyle\sum_{q=i+3}^{m} \Big[\varepsilon^{\xi}\mathcal{L}_{\mathcal{E}}(t_{q-1}) - (\rho^{\xi}+\kappa_1^{\xi})\mathcal{I}_{pre}(t_{q-1}) - \alpha^{\xi}\mathcal{I}_{pre}(t_{q-1}) \\ + \varepsilon^{\xi}\mathcal{L}_{\mathcal{E}}(t_{q-2}) - (\rho^{\xi}+\kappa_1^{\xi})\mathcal{I}_{pre}(t_{q-2}) - \alpha^{\xi}\mathcal{I}_{pre}(t_{q-2})\Big]\mho_2 \\ + \dfrac{1}{\mathbb{ABC}(\xi)}\dfrac{\xi\mathrm{h}^{\xi-1}}{\Gamma(\xi+3)} \displaystyle\sum_{q=i+3}^{m} \Big[\varepsilon^{\xi}\mathcal{L}_{\mathcal{E}}(t_q) - (\rho^{\xi}+\kappa_1^{\xi})\mathcal{I}_{pre}(t_q) - \alpha^{\xi}\mathcal{I}_{pre}(t_q) \\ - 2\Big[\varepsilon^{\xi}\mathcal{L}_{\mathcal{E}}(t_{q-1}) - (\rho^{\xi}+\kappa_1^{\xi})\mathcal{I}_{pre}(t_{q-1}) - \alpha^{\xi}\mathcal{I}_{pre}(t_{q-1})\Big] \\ + \varepsilon^{\xi}\mathcal{L}_{\mathcal{E}}(t_{q-2}) - (\rho^{\xi}+\kappa_1^{\xi})\mathcal{I}_{pre}(t_{q-2}) - \alpha^{\xi}\mathcal{I}_{pre}(t_{q-2})\Big]\mho_3. \end{cases} \end{cases} \tag{37}$$

$$\mathcal{I}_{post}(t_{m+1}) = \begin{cases} \mathcal{I}_{post_0} + \begin{cases} \dfrac{\mathrm{h}^{\xi-1}}{\Gamma(\xi+1)} \displaystyle\sum_{q=2}^{i} \Big[\alpha^{\xi}\mathcal{I}_{pre}(t_{q-2}) - (\rho^{\xi}+\kappa_2^{\xi})\mathcal{I}_{post}(t_{q-2})\Big]\mho_1 \\ + \dfrac{\mathrm{h}^{\xi-1}}{\Gamma(\xi+2)} \displaystyle\sum_{q=2}^{i} \Big[\mathcal{F}(\mathcal{X}(t_{q-1}),t_{q-1}) - \mathcal{F}(\mathcal{X}(t_{q-2}),t_{q-2})\Big]\mho_2 \\ + \dfrac{\xi\mathrm{h}^{\xi-1}}{2\Gamma(\xi+3)} \displaystyle\sum_{q=2}^{i} \Big[\mathcal{F}(\mathcal{X}(t_q),t_{q-2})\Big]\mho_3, \end{cases} \\ \mathcal{I}_{post}(t_1) + \begin{cases} \dfrac{1-\xi}{\mathbb{ABC}(\xi)} \Big[\alpha^{\xi}\mathcal{I}_{pre}(t_m) - (\rho^{\xi}+\kappa_2^{\xi})\mathcal{I}_{post}(t_m)\Big] \\ + \dfrac{\xi}{\mathbb{ABC}(\xi)} \dfrac{\mathrm{h}^{\xi-1}}{\Gamma(\xi+1)} \displaystyle\sum_{q=i+3}^{m} \Big[\alpha^{\xi}\mathcal{I}_{pre}(t_{q-2}) - (\rho^{\xi}+\kappa_2^{\xi})\mathcal{I}_{post}(t_{q-2})\Big]\mho_1 \\ + \dfrac{\xi}{\mathbb{ABC}(\xi)} \dfrac{\mathrm{h}^{\xi-1}}{\Gamma(\xi+2)} \displaystyle\sum_{q=i+3}^{m} \Big[\alpha^{\xi}\mathcal{I}_{pre}(t_{q-1}) - (\rho^{\xi}+\kappa_2^{\xi})\mathcal{I}_{post}(t_{q-1}) \\ + \alpha^{\xi}\mathcal{I}_{pre}(t_{q-2}) - (\rho^{\xi}+\kappa_2^{\xi})\mathcal{I}_{post}(t_{q-2})\Big]\mho_2 \\ + \dfrac{1}{\mathbb{ABC}(\xi)} \dfrac{\xi\mathrm{h}^{\xi-1}}{\Gamma(\xi+3)} \displaystyle\sum_{q=i+3}^{m} \Big[\alpha^{\xi}\mathcal{I}_{pre}(t_q) - (\rho^{\xi}+\kappa_2^{\xi})\mathcal{I}_{post}(t_q) \\ - 2\Big[\alpha^{\xi}\mathcal{I}_{pre}(t_{q-1}) - (\rho^{\xi}+\kappa_2^{\xi})\mathcal{I}_{post}(t_{q-1})\Big] \\ + \alpha^{\xi}\mathcal{I}_{pre}(t_{q-2}) - (\rho^{\xi}+\kappa_2^{\xi})\mathcal{I}_{post}(t_{q-2})\Big]\mho_3. \end{cases} \end{cases} \tag{38}$$

$$\mathcal{R}(t_{m+1}) = \begin{cases} \mathcal{R}_0 + \begin{cases} \dfrac{\mathrm{h}^{\xi-1}}{\Gamma(\xi+1)} \displaystyle\sum_{q=2}^{i} \Big[\kappa_1^{\xi}\mathcal{I}_{pre}(t_{q-2}) + \kappa_2^{\xi}\mathcal{I}_{post}(t_{q-2}) - \rho^{\xi}\mathcal{R}(t_{q-2})\Big]\mho_1 \\ + \dfrac{\mathrm{h}^{\xi-1}}{\Gamma(\xi+2)} \displaystyle\sum_{q=2}^{i} \Big[\kappa_1^{\xi}\mathcal{I}_{pre}(t_{q-1}) + \kappa_2^{\xi}\mathcal{I}_{post}(t_{q-1}) - \rho^{\xi}\mathcal{R}(t_{q-1}) \\ - \Big(\kappa_1^{\xi}\mathcal{I}_{pre}(t_{q-2}) + \kappa_2^{\xi}\mathcal{I}_{post}(t_{q-2}) - \rho^{\xi}\mathcal{R}(t_{q-2})\Big)\Big]\mho_2 \\ + \dfrac{\xi\mathrm{h}^{\xi-1}}{2\Gamma(\xi+3)} \displaystyle\sum_{q=2}^{i} \Big[\kappa_1^{\xi}\mathcal{I}_{pre}(t_{q-2}) + \kappa_2^{\xi}\mathcal{I}_{post}(t_{q-2}) - \rho^{\xi}\mathcal{R}(t_{q-2})\Big]\mho_3, \end{cases} \\ \mathcal{R}(t_1) + \begin{cases} \dfrac{1-\xi}{\mathbb{ABC}(\xi)} \Big[\kappa_1^{\xi}\mathcal{I}_{pre}(t_m) + \kappa_2^{\xi}\mathcal{I}_{post}(t_m) - \rho^{\xi}\mathcal{R}(t_m)\Big] \\ + \dfrac{\xi}{\mathbb{ABC}(\xi)} \dfrac{\mathrm{h}^{\xi-1}}{\Gamma(\xi+1)} \displaystyle\sum_{q=i+3}^{m} \Big[\kappa_1^{\xi}\mathcal{I}_{pre}(t_{q-2}) + \kappa_2^{\xi}\mathcal{I}_{post}(t_{q-2}) - \rho^{\xi}\mathcal{R}(t_{q-2})\Big]\mho_1 \\ + \dfrac{\xi}{\mathbb{ABC}(\xi)} \dfrac{\mathrm{h}^{\xi-1}}{\Gamma(\xi+2)} \displaystyle\sum_{q=i+3}^{m} \Big[\kappa_1^{\xi}\mathcal{I}_{pre}(t_{q-1}) + \kappa_2^{\xi}\mathcal{I}_{post}(t_{q-1}) - \rho^{\xi}\mathcal{R}(t_{q-1}) \\ + \kappa_1^{\xi}\mathcal{I}_{pre}(t_{q-2}) + \kappa_2^{\xi}\mathcal{I}_{post}(t_{q-2}) - \rho^{\xi}\mathcal{R}(t_{q-2})\Big]\mho_2 \\ + \dfrac{1}{\mathbb{ABC}(\xi)} \dfrac{\xi\mathrm{h}^{\xi-1}}{\Gamma(\xi+3)} \displaystyle\sum_{q=i+3}^{m} \Big[\kappa_1^{\xi}\mathcal{I}_{pre}(t_q) + \kappa_2^{\xi}\mathcal{I}_{post}(t_q) - \rho^{\xi}\mathcal{R}(t_q) \\ - 2\Big[\kappa_1^{\xi}\mathcal{I}_{pre}(t_{q-1}) + \kappa_2^{\xi}\mathcal{I}_{post}(t_{q-1}) - \rho^{\xi}\mathcal{R}(t_{q-1})\Big] \\ + \kappa_1^{\xi}\mathcal{I}_{pre}(t_{q-2}) + \kappa_2^{\xi}\mathcal{I}_{post}(t_{q-2}) - \rho^{\xi}\mathcal{R}(t_{q-2})\Big]\mho_3. \end{cases} \end{cases} \tag{39}$$

The formulae deduced in (35)-(39) will be used to simulate the proposed model (3) graphically in next section.

5. Discussions of Numerical Simulations

For numerical illustration, we use the scheme designed in the above section. Here, we provide some numerical values from [30] for the nomenclatures involve in our suggested model (3) as: $\mathcal{S} = 2.465104$, $\mathcal{L}_{\mathcal{E}} = 0.001664$, $\mathcal{I}_{pre} = 0.001751$, $\mathcal{I}_{post} = 0.000333$, $\mathcal{R} = 0.244729$ and $\Lambda = 0.0119$, $\rho = 0.0005$, $\epsilon_1 = 0.51$, $\epsilon_2 = 0.25$, $\varepsilon = 0.07$, $\kappa_1 = 0.31$, $\kappa_2 = 0.51$, $\alpha = 0.04$. Here it should be kept in mind that initial data is considered in millions. We take $t_1 < 100$, $T \leq 400$, and using the numerical scheme established afore three different cases are discussed here.

5.1. *When* $\xi \in [0.85, 1.0]$

We present the results graphically using various values of fractional orders in figure 2.

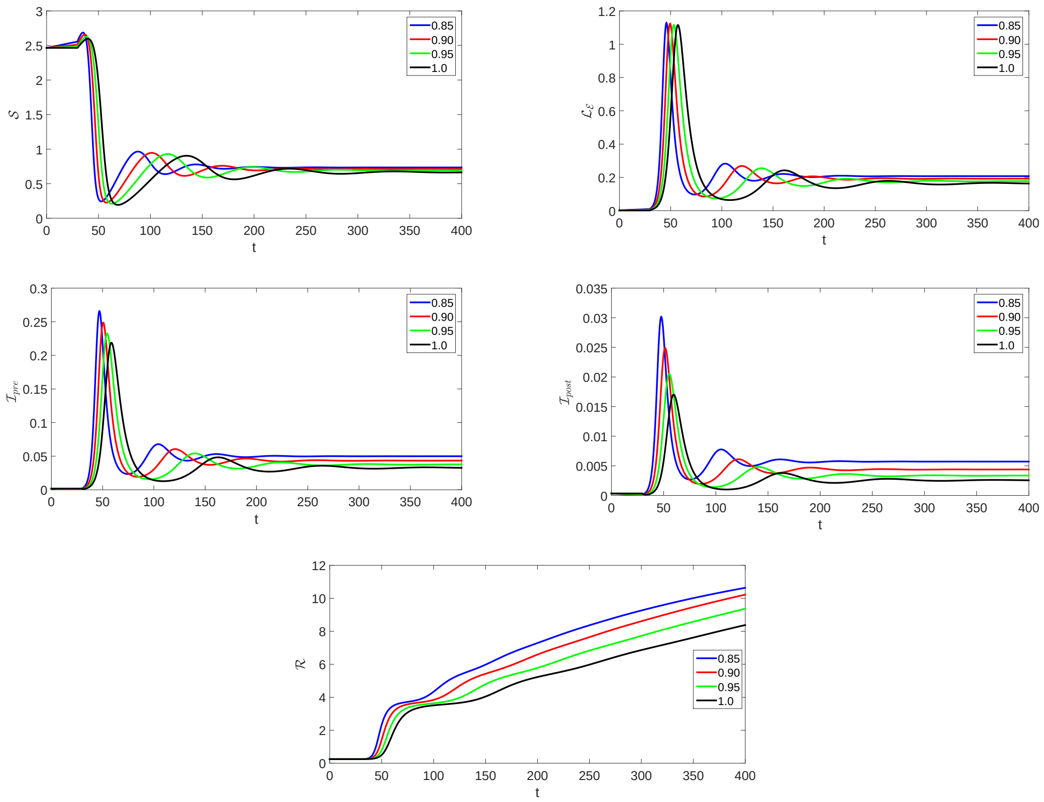

Figure 2: Numerical results for different classes at various fractional orders.

We see from 2 that as the population of susceptible class is decreasing with different scenarios. As a results the population density of latently individuals class who is also exposed to capture virus increasing and then goes on reducing as shown in figure 2. In addition, the population density of pre-mutation and post-mutation individuals also increases with rapid growth for some time after that they both go to reduce (figures 2). Consequently, the population density of recovered class is increasing in figure 2. Moreover, the crossover dynamics of each compartments can be observed in the neighborhood of t_1.

5.2. *When* $\xi \in [0.65, 0.80]$

Here we re simulate the results for slighter lower fractional orders in sub figures 3.

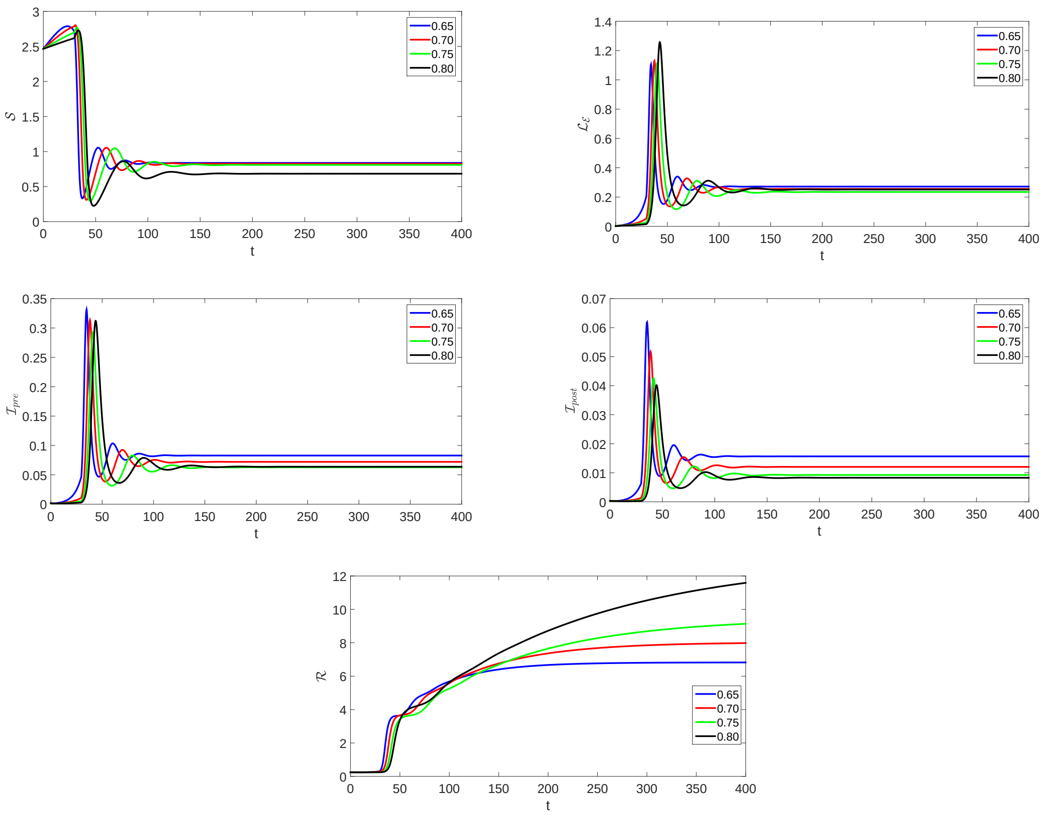

Figure 3: Numerical results for different classes at various fractional orders.

Here we see in figures 3, the significant impact of fractional order derivatives on the dynamics. There is clear description of crossover behaviors in each and every compartments dynamics. The decline in the susceptible populations implies the raise in latently, infected individuals, pre and post mutations patient populations which after some time decrease and produce they have no impact on the recovered class population density which is decreasing.

5.3. *When* $\xi \in [0.45, 0.60]$

Here we re simulate the results for slighter lower fractional orders in figures 4.

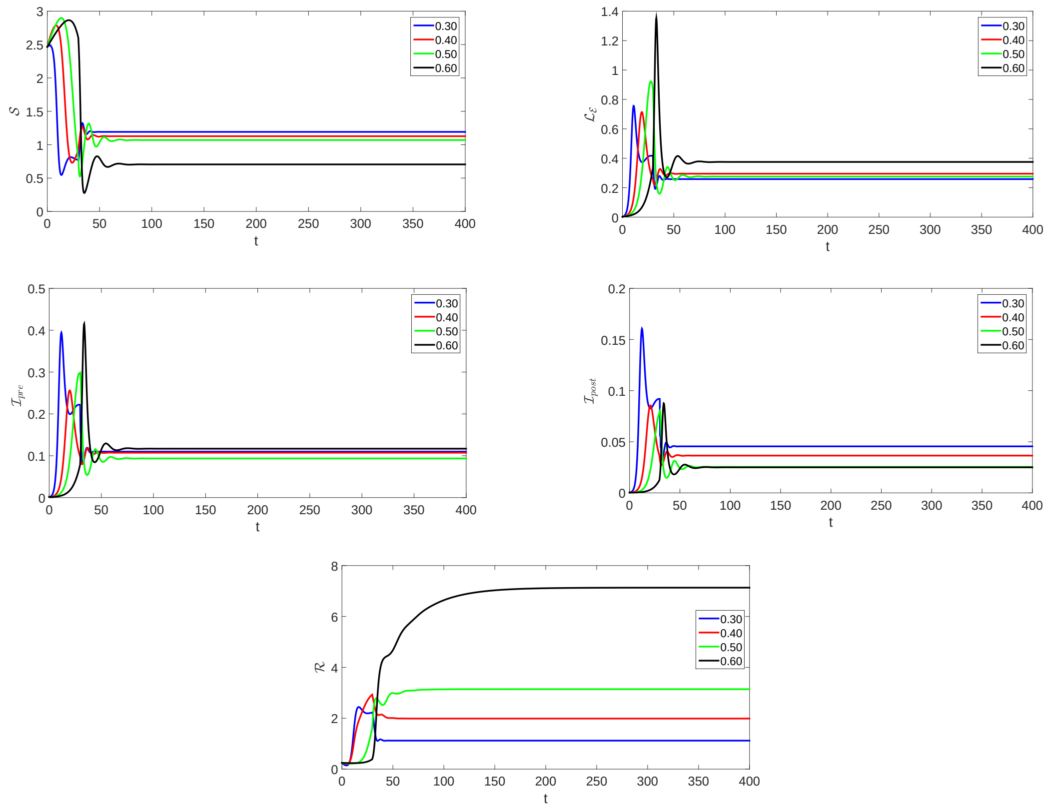

Figure 4: Numerical results for different classes at various fractional orders.

Here we see in sub figures of 4, lower fractional order produce significant impact on the dynamics of all compartments of the proposed model. There is clear description of crossover behaviors in each and every compartments dynamics. The decline in the susceptible populations implies the raise in latently, infected individuals, pre and post mutations patient populations. Both pre and post mutation classes produce no impact on the recovered class population density which is increasing.

6. Conclusion

In the presented research work, we have established a model under the concept of piecewise equations with fractional order derivatives. extending the usual analysis tools, we have computed boundedness feasible region, and positivity of solution by using the adopted derivatives. Previously in literatures, researchers have deduced the concerned results by using traditional integer order derivatives. Also, we have computed the fundamental reproductive ratio and deduced the local asymptotical stability for trivial equilibrium. For global stability of the model non-trivial unique equilibrium, we have constructed a Voltera-Lyapunov function and proved the concerned results. For the existence of unique solution, we have established sufficient results using Banach theorem. Finally a sophisticated numerical scheme based on interpolation method has also derived. In the last, we have simulated the results graphically for various fractional orders of different compartments of the proposed model. Our simulations results revealed that piecewise derivatives have significantly described the multi-behaviors in the dynamics of mutation model. In the future, the aforesaid analysis can be used to investigate complex dynamical systems of infectious disease with multiple delays terms. Also, we can investigate the considered model by using fractals fractional or stochastic concepts. Also, a comparative study of piecewise and stochastic will help readers to further understand mutation process with more informative details.

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Competing interest

Does not exist.

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