EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 2, Article Number 6059
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Lower and Upper Acyclicities on Unitary Cayley Graphs of Finite Commutative Rings

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Abstract. A unitary Cayley graph Γ_n of a finite cyclic ring $\mathbb{Z}_n$ is a graph with vertex set $\mathbb{Z}_n$ and two vertices x and y are adjacent if and only if $x-y$ is a unit in $\mathbb{Z}_n$ or equivalently, $\gcd(x-y,n)=1$. A nonempty subset A of $\mathbb{Z}_n$ is called an acyclic set of Γ_n if a subgraph of Γ_n induced by A contains no cycles. The maximum cardinality among the acyclic sets of Γ_n is called the upper acyclic number of Γ_n and is denoted by $\Lambda(\Gamma_n)$. Moreover, the maximum number k of vertices of Γ_n in which every subgraph of Γ_n induced by k vertices contains no cycles is called the lower acyclic number of Γ_n and denoted by $\lambda(\Gamma_n)$. In this paper, we determine the lower and upper acyclic numbers for unitary Cayley graphs of $\mathbb{Z}_n$ and their complements.

2020 Mathematics Subject Classifications: 05C25, 05C69, 05C99

Key Words and Phrases: Acyclic sets, Lower acyclic numbers, Upper acyclic numbers, Unitary Cayley graphs

1. Introduction

In algebraic graph theory, the structure of algebraic methods are studied and then applied to problems about graphs. An interesting topic is to study properties of graphs in connection to algebraic systems. A well-known connection between graphs and algebraic system is the construction of graphs from algebras. Algebraic tools can be used to give elegant proofs of graph theoretic facts. For each $n \geq 2$, let Γ_n denote the unitary Cayley graph of a ring $\mathbb{Z}_n$, the ring of integers modulo n, whose vertex set is $\mathbb{Z}_n$ itself and two vertices x and y are joined by edge if $x-y$ is a unit in the ring $\mathbb{Z}_n$. Let us denote all elements in $\mathbb{Z}_n$ by integers $0, 1, 2, \ldots, n-1$. It is well known that all units in the ring $\mathbb{Z}_n$ are the integers a in which $\gcd(a,n)=1$. Therefore, the edge set of Γ_n can be expressed as $E(\Gamma_n) = \{\{x,y\} : x, y \in \mathbb{Z}_n \text{ and } \gcd(x-y,n)=1\}$. Clearly, if p is prime, then Γ_p is a complete graph. Moreover, all unitary Cayley graphs of order greater than 2 always contain cycles and their structures are highly symmetric. Moreover, there are

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i2.6059

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some remarkable properties between algebraic graph theory and number theory. Some prominent results of unitary Cayley graphs were studied by several researchers. In 1995, Dejter and Giudici [1] showed that unitary Cayley graphs are unions of disjoint hamiltonian cycles and presented the sufficient condition for being bipartite graphs. In 2007, Klotz and Sander [2] determined some invariant properties of unitary Cayley graphs and studied their perfectness. In 2012, Kiani and Aghaei [3] provided isomorphism theorems for unitary Cayley graphs of rings associated with Jacobson radicals. In 2014, Naghipour [4] considered some properties of induced subgraphs of unitary Cayley graphs of commutative rings.

One of interesting parameters of graphs is the *acyclic number* which is the maximum number of vertices that the subgraph induced by such vertices contains no cycles. There are many discussions of acyclic numbers, for instance, in 2009, Samodivkin [5] investigated the acyclic number of graphs with cut-vertices. Later in 2017, Petrusevski and Skrekovski [6] proposed a conjecture on this parameter. Moreover, they provided some conditions that make such the conjecture weaker for planar graphs.

Throughout the paper, the notation Γ_n stands for the unitary Cayley graph of a finite commutative ring $\mathbb{Z}_n$. In addition, let us denote by $\overline{\Gamma}_n$ the complement of Γ_n which is a simple graph obtained from Γ_n by deleting all edges in $E(\Gamma_n)$ and then adding all edges outside $E(\Gamma_n)$. In this paper, we study some properties on unitary Cayley graphs of finite commutative rings. Furthermore, we present the construction of cycles of lengths 3 and 4 which is useful for finding the maximum cardinality among acyclic sets of Γ_n and $\overline{\Gamma}_n$, such the number is called the upper acyclic number. Moreover, we provide some characterization of the structure of Γ_n and $\overline{\Gamma}_n$, where n is a power of a prime number,to find the maximum number k of vertices of Γ_n in which every subgraph of Γ_n induced by k vertices contains no cycles. This cardinality is called the lower acyclic number, and all sets mentioned in this research are considered to be finite sets.

2. Preliminaries

A graph G is a pair $(V(G), E(G))$ where $V(G)$ is the vertex set of G and $E(G)$ is an edge set of G. An edge of G joining between vertices $u, v \in V(G)$ is written as $\{u, v\}$, that is, $\{u, v\} \in E(G)$ means that u is adjacent to v in G. Let $v \in V(G)$. The *degree* of v, denoted by $deg(v)$, is the number of vertices adjacent to v in G. Furthermore, let C be a sequence $v_1, v_2, \ldots, v_k$ of distinct k vertices of G where $k \geq 3$. If v_i and v_{i+1} are adjacent in G for all $i = 1, 2, \ldots, k-1$ and v_1 is adjacent to v_k, then C is called a *cycle* in G with length k and denoted by C_k. Moreover, if k is odd (even), then C_k is said to be an *odd (even) cycle*. In particular, if $k = 3$, then the cycle C_3 is sometimes called a *triangle*. A graph G is said to be *complete* if $\{u, v\} \in E(G)$ for all $u, v \in V(G)$. In addition. a graph G will be called a *complete p-partite graph* if $V(G)$ is partitioned into p sets, called a *partite set*, provided that vertices in the same partite set are not mutually adjacent and every two vertices from different partite sets must be adjacent. Let $H = (V(H), E(H))$ be a graph. The graph H is called a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A *decomposition* of G is a collection of edge disjoint subgraphs of G in which every edge of G belongs to exactly one subgraph. Next, let W be a nonempty subset of $V(G)$. A *subgraph of* G *induced by* W, simply called an *induced subgraph* and denoted by $G[W]$, is a subgraph of G satisfying the condition that if $u, v \in W$ and $\{u, v\} \in E(G)$, then $\{u, v\} \in E(G[W])$, as well. Throughout the paper, all sets are finite sets and all graphs are simple. More information about graph theory can be found in [7].

We now provide some definitions and prominent facts which play a crucial role in the paper.

Definition 1. *[8] Let G be a group. A nonempty subset H of G is called a subgroup of G if H itself is a group under the group operation of G restricted to H. Moreover, a left coset of a subgroup H of G is a set of the form $gH := \{gh : h \in H\}$. The set of all cosets of H is denoted by G/H, that is, $G/H := \{gH : g \in G\}$.*

Definition 2. *[2] Let $n \geq 2$ be a positive integer. The* unitary Cayley graph $\Gamma_n = (\mathbb{Z}_n, U_n)$ *is defined by the additive group of the ring $\mathbb{Z}_n$ and the multiplicative group U_n of its units such that $V(\Gamma_n) = \mathbb{Z}_n$ and $E(\Gamma_n) = \{\{a, b\} : a, b \in \mathbb{Z}_n$ and $a - b \in U_n\}$ or equivalently, $E(\Gamma_n) = \{\{a, b\} : a, b \in \mathbb{Z}_n$ and $\gcd(a - b, n) = 1\}$.*

Definition 3. *[9] In number theory, the Euler's totient function counts the positive integers up to a given integer n that are relatively prime to n. Generally, a well-known notation written for the Euler's totient function is $\varphi(n)$ and may also be called the Euler's phi function. In other words, it is defined as the number of integers k such that $1 \leq k \leq n$ and $\gcd(k, n) = 1$.*

Theorem 1. *[10] Let n be an even positive integer such that $n \geq 8$. Then $\varphi(n) \geq 4$.*

Remark 1. *[2] Let Γ_n be the unitary Cayley graph with n vertices. Then $|E(G)| = \dfrac{n(\varphi(n))}{2}$ and the degree of a vertex $v \in V(\Gamma_n)$ is given by $\deg(v) = \varphi(n)$.*

Theorem 2. *[2] Let $n = p^k$ be such that p is prime and $k \in \mathbb{N} \setminus \{1\}$. Then Γ_n is a complete p-partite graph where each partite set has size p^{k-1}.*

Corollary 1. *[1] If n is an even positive integer, then the unitary Cayley graph Γ_n has no odd cycles. In particular, Γ_n has no triangles.*

Definition 4. *Let Γ_n be the unitary Cayley graph of $\mathbb{Z}_n$. The complement $\overline{\Gamma}_n$ of Γ_n is the graph in which $V(\overline{\Gamma}_n) = \mathbb{Z}_n$ and $E(\overline{\Gamma}_n) = \{\{a, b\} : a, b \in \mathbb{Z}_n \text{ and } \gcd(a - b, n) \neq 1\}$.*

Theorem 3. *[11] Let $n = p^k$ be such that p is prime and $k \in \mathbb{N} \setminus \{1\}$. Then $\overline{\Gamma}_n$ is decomposed into p complete graphs of order p^{k-1}.*

Lemma 1. *[7] If every vertex of a finite simple graph G has degree at least 2, then G contains a cycle.*

Definition 5. *[7] A nonempty subset X of $V(\Gamma_n)$ is an independent set of Γ_n if every pair of vertices in X is not adjacent in Γ_n.*

3. Lower acyclic numbers of Γ_n and $\overline{\Gamma}_n$

This section begins with the definition of a *lower acyclic number* of the graph G.

Definition 6. *A lower acyclic number of G, denoted by $\lambda(G)$, is the maximum number of k vertices in which every induced subgraph of k vertices contains no cycles.*

Example 1. *Let G be a graph with $V(G) = \{a, b, c, d, e, f, g, h, i\}$ and let $E(G)$ be defined as the following diagram:*

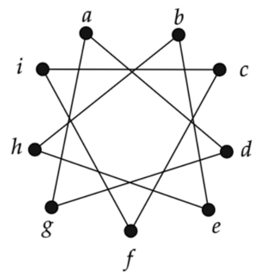


Figure 1: A graph G

We observe that an induced subgraph $G[\{a, d, g\}]$ forms a cycle of length 3. Therefore, $\lambda(G) = 2$.

Lemma 2. *Let n be an even positive integer such that $n \geq 8$. Then the unitary Cayley graph Γ_n contains a cycle C_4 of length 4 as a subgraph.*

Proof. Let $\Phi_n = \{k \in \mathbb{N} : 1 \leq k \leq n \text{ and } \gcd(k, n) = 1\}$. Then $|\Phi_n| = \varphi(n)$. For convenience, we write $\Phi_n = \{1, a_1, a_2, \ldots, a_{\varphi(n)-1}\}$, where $a_i < a_j$ and $1 \leq i < j \leq \varphi(n) - 1$. We claim that C_4 is contained in Γ_n. Firstly, since $\gcd(1 - 0, n) = \gcd(1, n) = 1$, there exists an edge between vertices 0 and 1 in Γ_n. Consider $a_1 \in \Phi_n$. We have $\gcd(a_1 - 0, n) = \gcd(a_1, n) = 1$. Then, there exists an edge between vertices 0 and a_1 in Γ_n. Next, by Theorem 1, we get that $a_1 + 1 < n$. Then $a_1 + 1 \in \mathbb{Z}_n$ and $\gcd((a_1 + 1) - a_1, n) = \gcd(1, n) = 1$. Hence, there is an edge between vertices a_1 and $a_1 + 1$ in Γ_n. Finally, we have $\gcd((a_1 + 1) - 1, n) = \gcd(a_1, n) = 1$. Then there exists an edge between vertices $a_1 + 1$ and 1 in Γ_n. Hence, 0, 1, a_1, and $a_1 + 1$ form a cycle of length 4 in Γ_n.

Before we present the lower acyclic number of Γ_n where n is even, the following example is needed for $n = 4, 6$. Further results for $n > 6$ will be proved in Theorem 5.

Example 2. *The unitary Cayley graphs Γ_4 and Γ_6 are shown as follows.*

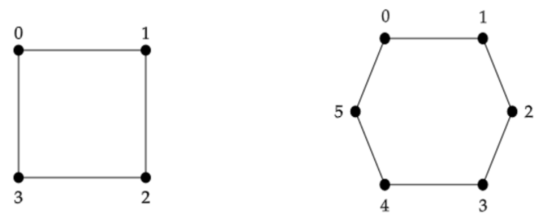


Figure 2: The unitary Cayley graphs Γ_4 and Γ_6

We see that Γ_4 is a cycle of length 4 and Γ_6 is a cycle of length 6. Therefore, $\lambda(\Gamma_4) = 3$ and $\lambda(\Gamma_6) = 5$.

Lemma 3. *Let n be an odd positive integer such that $n \geq 3$. Then the unitary Cayley graph Γ_n contains a cycle C_3 of length 3 as a subgraph.*

Proof. Let n be an odd positive integer. We claim that C_3 is contained in Γ_n. Since $\gcd(1-0, n) = 1$ and $\gcd(2-1, n) = 1$, there exist an edge between vertices 0 and 1 in Γ_n, and an edge between vertices 1 and 2 in Γ_n. Next, we show that $\gcd(2, n) = 1$. As the fact that n is an odd positive integer, then there exists an integer k such that $n = 2k + 1$. This means that n is not divisible by 2. Since the only positive divisors of 2 are 1 and 2, and n is odd, 2 does not divide n. So the only common positive divisor of 2 and n is 1. It follows that $1 = \gcd(2, n) = \gcd(2 - 0, n)$. Then there is an edge between vertices 0 and 2 in Γ_n. Hence, $0, 1$ and 2 form a cycle of length 3 in Γ_n.

In order to complete our results of the part of acyclic numbers of Γ_n, we present such the numbers as follows.

Theorem 4. *Let n be an odd positive integer such that $n \geq 3$. Then $\lambda(\Gamma_n) = 2$.*

Proof. By Lemma 3, the statement holds.

Theorem 5. *Let n be an even positive integer such that $n \geq 8$ and n is not prime. Then $\lambda(\Gamma_n) = 3$.*

Proof. By Corollary 1, Γ_n does not contain a triangle. It follows that $\lambda(\Gamma_n) \geq 3$. By Lemma 2, we can construct C_4 which is contained in Γ_n. Then $\lambda(\Gamma_n) \leq 3$. Therefore, $\lambda(\Gamma_n) = 3$.

For more results related to the acyclic numbers of the complement $\overline{\Gamma}_n$, we present these numbers in the following discussion. In particular, we explore their properties, characteristics, and the conditions under which they arise. This allows for a deeper understanding of how acyclic numbers behave in the context of the complement graph and their significance in graph theory. The results presented here provide a foundation for further investigation and possible extensions of this concept.

Example 3. *The complements* $\overline{\Gamma}_4$ *and* $\overline{\Gamma}_6$ *of the unitary Cayley graphs* Γ_4 *and* Γ_6 *are shown as follows.*

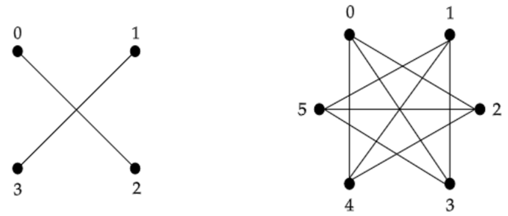


Figure 3: The complements $\overline{\Gamma}_4$ and $\overline{\Gamma}_6$

We observe that $\overline{\Gamma}_4$ does not contain any cycle. For $\overline{\Gamma}_6$, it is obvious that $\overline{\Gamma}_6$ contains triangles. Therefore, $\lambda(\overline{\Gamma}_4) = 4$ and $\lambda(\overline{\Gamma}_6) = 2$.

Lemma 4. *Let n be a positive integer such that $n \geq 8$ and n is not prime. Then the complement $\overline{\Gamma}_n$ of the unitary Cayley graph Γ_n contains a cycle C_3 of length 3 as a subgraph.*

Proof. Let n be a positive integer such that $n \geq 8$ and n is not prime.
Case 1 : n is an even integer. Then there exists an integer k such that $n = 2k$. Since $\gcd(4-2, 2k) = \gcd(2, 2k) \neq 1$, $\gcd(6-4, 2k) = \gcd(2, 2k) \neq 1$ and $\gcd(6-2, 2k) = \gcd(2(2), 2k) \neq 1$, there exist an edge between vertices 2 and 4 in $\overline{\Gamma}_n$, an edge between vertices 4 and 6 in $\overline{\Gamma}_n$, and an edge between vertices 2 and 6 in $\overline{\Gamma}_n$. So $2, 4$ and 6 induce a cycle C_3 in $\overline{\Gamma}_n$ as a subgraph.
Case 2 : n is an odd integer. Assume that $n = p_1^{m_1} \cdot p_2^{m_2} \cdots p_t^{m_t}$ where p_i and m_i are prime and positive, respectively such that $i = 1, 2, \ldots, t$ and $p_i < p_j$ for $i < j$. Since $\gcd(p_1 - 0, n) = \gcd(p_1, p_1^{m_1} \cdot p_2^{m_2} \cdots p_t^{m_t}) \neq 1$, $\gcd(2p_1 - p_1, n) = \gcd(p_1, p_1^{m_1} \cdot p_2^{m_2} \cdots p_t^{m_t}) \neq 1$, and $\gcd(2p_1 - 0, n) = \gcd(2p_1, p_1^{m_1} \cdot p_2^{m_2} \cdots p_t^{m_t}) \neq 1$, so there exist an edge between vertices 0 and p_1 in $\overline{\Gamma}_n$, an edge between vertices p_1 and $2p_1$ in $\overline{\Gamma}_n$, and an edge between vertices 0 and $2p_1$ in $\overline{\Gamma}_n$, respectively. Thus, 0, p_1 and $2p_1$ induce a cycle C_3 in $\overline{\Gamma}_n$.

Theorem 6. *Let n be a positive integer such that $n \geq 8$. Then*

$$\lambda(\overline{\Gamma}_n) = \begin{cases} 2 & \textit{if } n \textit{ is not prime}; \\ n & \textit{if } n \textit{ is prime.} \end{cases}$$

Proof. We consider the following two cases.
Case 1 : n is not prime. By Lemma 4, It follows that $\lambda(\overline{\Gamma}_n) = 2$.
Case 2 : n is prime. Then Γ_n is a complete graph which implies that $\overline{\Gamma}_n$ is an empty graph. Then $\lambda(\overline{\Gamma}_n) = n$, immediately.

4. Upper acyclic numbers of Γ_n and $\overline{\Gamma}_n$

This section begins with the definition of an *upper acyclic number* of the graph G.

Definition 7. *A nonempty subset A of the vertex set $V(G)$ of a graph G is called an acyclic set of G induced by A contains no cycles. An upper acyclic number of a graph G, denoted by $\Lambda(G)$, is the maximum cardinality among acyclic sets of G, that is,*

$$\Lambda(G) = \max\{|A| : A \text{ is an acyclic set of } G\}.$$

Example 4. *In Figure 1, we observe that the set $A = \{a, b, c, d, e, f\}$ is an acyclic set of G with the largest cardinality. Thus, $\Lambda(G) = 6$.*

Next, we find upper acyclic numbers of unitary Cayley graphs and their complements. The following two examples illustrate results for Γ_n and $\overline{\Gamma}_n$ where $n = 4, 6$. Other results will be presented in the sequel.

Example 5. *The set $\{0, 1, 2\}$ of vertices in the left diagram of Figure 2 and the set $\{0, 1, 2, 3\}$ of vertices in the left diagram of Figure 3 are acyclic sets of Γ_4 and $\overline{\Gamma}_4$, respectively. Therefore, it is not hard to conclude that $\Lambda(\Gamma_4) = 3$ and $\Lambda(\overline{\Gamma}_4) = 4$.*

Example 6. *It is easy to see that the set $\{0, 1, 2, 3, 4\}$ of vertices in the right diagram of Figure 2 is an acyclic set of Γ_6 with maximum cardinality. Furthermore, the set $\{0, 1, 2, 3\}$ of vertices in the right diagram of Figure 3 is an acyclic set of $\overline{\Gamma}_6$ with maximum cardinality. Therefore, $\Lambda(\Gamma_6) = 5$ and $\Lambda(\overline{\Gamma}_6) = 4$.*

Theorem 7. *Let p be a prime number such that $p \geq 3$. Then $\Lambda(\Gamma_p) = 2$ and $\Lambda(\overline{\Gamma}_p) = p$.*

Proof. Let p be a prime number such that $p \geq 3$. Then Γ_p is a complete graph and $\overline{\Gamma}_p$ is an empty graph. It follows that $\Lambda(\Gamma_p) = 2$ and $\Lambda(\overline{\Gamma}_p) = p$, respectively.

Theorem 8. *Let $n = p^k$ be such that p is prime and $k \in \mathbb{N} \setminus \{1\}$. Then $\Lambda(\Gamma_n) = p^{k-1} + 1$.*

Proof. Let $n = p^k$ be such that p is prime and $k \in \mathbb{N} \setminus \{1\}$. By Theorem 2, we obtain that Γ_n is a complete p-partite graph such that each partite set has size p^{k-1}.
Let P be a partite set of Γ_n. Thus the induced subgraph $\Gamma_n[P]$ is an empty graph.
Let $v \in V(\Gamma_n) \setminus P$. Hence $\Gamma_n[P \cup \{v\}]$ is a tree. It follows that $P \cup \{v\}$ is an acyclic set of Γ_n which leads to $\Lambda(\Gamma_n) \geq |P \cup \{v\}| = p^{k-1} + 1$.
We now suppose that there is an acyclic set, say A, of Γ_n in which $|A| > p^{k-1} + 1$. Then there exist $x, y, z \in A$ such that $x \in P_1$ and $y, z \notin P_1$ where P_1 is a partite set of Γ_n.
We now consider the following two cases.
Case 1 : $y, z \in P_2$ for some a partite set P_2 in which $P_1 \neq P_2$.
Since $|A| > p^{k-1} + 1$ where p is prime and $k \geq 2$, there exists $u \in A \setminus \{x, y, z\}$ such that $u \notin P_2$. If $u \in P_1$, then the sequence of edges uy, yx, xz, zu forms a cycle of length 4 in $\Gamma_n[A]$ (see Figure 4.) since Γ_n is a complete p-partite graph. This contradicts to the acyclicity of A. On the other hand, if $u \in P_3$ where $P_3 \neq P_1$, then the sequence of edges

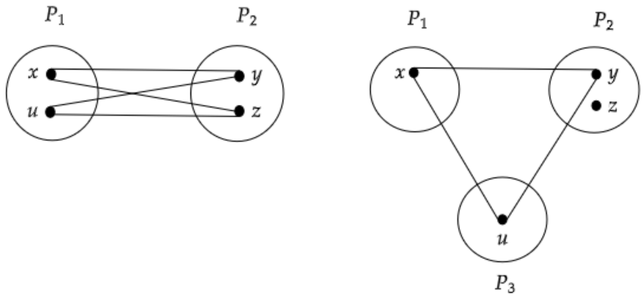


Figure 4: Two possible cases of the sequence of edges on cycles of length 4 and 3.

ux, xy, yu forms a cycle of length 3 in $\Gamma_n[A]$ (see Figure 4.) which is also a contradiction.
Case 2 : $y \in P_2$ and $z \in P_3$ for some partite sets P_2, P_3 in which P_1, P_2, P_3 are pairwise disjoint. We can observe that this case will generate a cycle of length 3 with the sequence of edges xy, yz, zx in $\Gamma_n[A]$ similar to Figure 4. This also contradicts to the acyclicity of A. From the above two cases, we can conclude that $\Lambda(\Gamma_n) = p^{k-1} + 1$.

Theorem 9. *Let n be a positive integer such that $n \geq 8$. If p is the least prime divisor of n, then $\frac{n}{p} + 1 \leq \Lambda(\Gamma_n) \leq n - \left(\frac{\varphi(n)}{2} + 1\right)$.*

Proof. Let n be a positive integer such that $n \geq 8$. Assume that p is the least prime divisor of n. To prove the lower bound of $\Lambda(\Gamma_n)$, consider the set $A = \{0, p, 2p, \ldots, n-p\}$. It is clear that $|A| = \frac{n}{p}$. For each $\{x, y\} \in A$, we have $x - y$ is the multiple of p which directly implies that $\gcd(x - y, n) \neq 1$, that is, $\{x, y\} \notin E(\Gamma_n)$. It follows that $\Gamma_n[A]$ is an empty graph. Now, let $v \in \mathbb{Z}_n \setminus A$. It is not hard to verify that $\Gamma_n[A \cup \{v\}]$ contains no cycles. Therefore, $A \cup \{v\}$ is an acyclic set in Γ_n. Hence $\Lambda(\Gamma_n) \geq |A \cup \{v\}| = \frac{n}{p} + 1$. For proving the upper bound of $\Lambda(\Gamma_n)$, let B be any subset of $\mathbb{Z}_n$ containing at least $n - \frac{\varphi(n)}{2}$ elements. By Remark 1, $\deg(u) = \varphi(n)$ for all $u \in V(\Gamma_n)$, we obtain that $\deg_{\Gamma_n[B]}(w) \geq \frac{\varphi(n)}{2}$ for all $w \in V(\Gamma_n[B])$. Since $n \geq 8$, we conclude by Theorem 1, that $\varphi(n) \geq 4$ which leads to $\deg_{\Gamma_n[B]}(w) \geq 2$ for all $w \in V(\Gamma_n[B])$. By Lemma 1, we get that $\Gamma_n[B]$ contains a cycle. It follows that B is not an acyclic set of Γ_n. Since B is arbitrary, we have that

$$\begin{aligned}\Lambda(\Gamma_n) \leq |B| - 1 &\leq \left(n - \frac{\varphi(n)}{2}\right) - 1 \\ &= n - \left(\frac{\varphi(n)}{2} + 1\right).\end{aligned}$$

Then the statement is proved, as required.

We now consider the unitary Cayley graph Γ_8 as shown in the following example. Actually, the lower bound and upper bound in Theorem 9 are sharp.

Example 7. *Consider the unitary Cayley graph* Γ_8 *follows.*

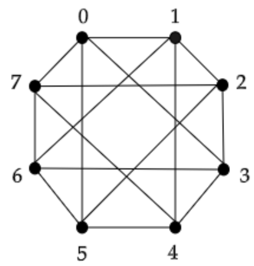


Figure 5: The unitary Cayley graph Γ_8

By Theorem 8, we obtain that $\Lambda(\Gamma_8) = 2^{3-1} + 1 = 5 = \frac{8}{2} + 1$ which achieves the lower bound in Theorem 9. Moreover, we can observe that $\Lambda(\Gamma_8) = 5 = 8 - \left(\frac{\varphi(8)}{2} + 1\right)$ which attains the upper bound mentioned in Theorem 9.

Theorem 10. *Let* $n = p^k$ *be such that* p *is prime and* $k \in \mathbb{N} \setminus \{1\}$*. Then* $\Lambda(\overline{\Gamma}_n) = 2p$*.*

Proof. By Theorem 3, we obtain that $\overline{\Gamma}_n$ is decomposed into p complete graphs of order p^{k-1}. Choose two vertices of each complete graph. We get that those vertices form an acyclic set of $\overline{\Gamma}_n$. Moreover, we can easily observe that there is no acyclic set containing more than two vertices from one complete graph. Therefore, $\Lambda(\overline{\Gamma}_n) = 2p$.

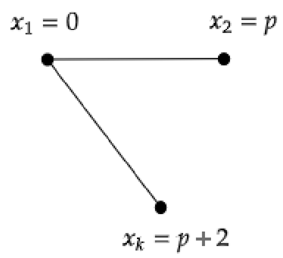
We mention a certain prominent fact that we will use in the next theorem. In combinatorics,the *Pigeonhole Principle* states that if n items are put into m containers with $n > m$, then at least one container must contain more than one item.

Theorem 11. *Let* n *be an even positive integer such that* $n \geq 4$*. Then* $\Lambda(\overline{\Gamma}_n) = 4$*.*

Proof. Let n be an even positive integer such that $n \geq 4$. Further, let $A = \{0, 1, 2, 3\}$. Clearly, $\overline{\Gamma}_n[A]$ contains no cycles. Hence, A is an acyclic set of $\overline{\Gamma}_n$ which implies that $\Lambda(\overline{\Gamma}_n) \geq 4$. Consider the set of any five vertices of $\overline{\Gamma}_n$, say $B = \{v_1, v_2, v_3, v_4, v_5\}$. By the Pigeonhole Principle, there exist at least three vertices $v_i, v_j, v_k \in B$ such that they are all even or odd. Hence the difference between any two vertices of v_i, v_j, v_k must be even. Then $\{v_l, v_m\} \notin E(\Gamma_n)$ for all $l, m \in \{i, j, k\}$. Thus, $\{v_l, v_m\} \in E(\overline{\Gamma}_n)$ for all $l, m \in \{i, j, k\}$. It follows that $\overline{\Gamma}_n[\{v_i, v_j, v_k\}]$ is a cycle of length 3 contained in $\overline{\Gamma}_n[B]$. Therefore, B is not acyclic in $\overline{\Gamma}_n$. Since B is arbitrary, we can conclude that $\Lambda(\overline{\Gamma}_n) < |B| = 5$. Thus $\Lambda(\overline{\Gamma}_n) = 4$, as required.

Theorem 12. *Let n be an odd positive integer such that $n \geq 5$ and n is not prime. If p is the least prime divisor of n, then $p+3 \leq \Lambda(\overline{\Gamma}_n) \leq 2p$.*

Proof. Let n be an odd positive integer such that $n > 5$ and n is not prime. Assume that $n = p_1^{m_1} \cdot p_2^{m_2} \cdots p_t^{m_t}$ where p_i and m_i are prime and positive numbers, respectively such that $i = 1, 2, \ldots, t$ and $p_i < p_j$ for $i < j$. Let p be the least prime divisor of n and $A = \{0, 1, 2, \ldots, p+1, p+2\}$. We show that $\overline{\Gamma}_n[A]$ contains no cycles. Suppose that $\Gamma_n[A]$ contains a cycle $C_k := \{v_1, v_2, \ldots, v_k, v_1\}$ such that $v_1, v_2, \ldots, v_k \in A$ and $3 \leq k \leq p+3$. Without loss of generality, we can assume that $v_1 < v_j$ for $2 \leq j \leq k$. Let $X := \{x_1, x_2, \ldots, x_k\}$ be such that $x_1 = 0$ and $x_l = v_l - v_1$ for all $l = 2, 3, \ldots, k$. Then $x_1, x_2, \ldots, x_k, x_1$ form a cycle in $\overline{\Gamma}_n[X]$. We now consider the following two cases.
Case 1 : $(p+2)|n$.
Consider $\{x_2, x_3\} \in E(\overline{\Gamma}_n[X])$, if $x_2 < x_3$, then $x_3 = 2p$ or $x_3 = 2p+2$ and so $x_3 \notin X$, a contradiction. If $x_3 < x_2$, then $x_3 = 0 = x_1$, a contradiction as $x_i \neq x_j$ for $i \neq j$.



Case 2 : $(p+2) \nmid n$.
Since $\{x_1, x_k\} \in E(\overline{\Gamma}_n[X])$, it follows that $x_k = p = x_2$, a contradiction. Hence $\overline{\Gamma}_n[A]$ contains no cycles. That is, $\Lambda(\overline{\Gamma}_n) \geq |A| = p+3$. Now, let $B := \{v_0, v_1, \ldots, v_{2p}\}$ be arbitrary. Let B_r be the set of elements in B having r as the remainder from dividing by p where $0 \leq r \leq p-1$. We show that $\overline{\Gamma}_n[B]$ contains a cycle. Now, we can place $v_0, v_1, \ldots, v_{2p}$ in such p sets and by the Pigeonhole Principle, we obtain that one of those sets must contain at least three vertices x, y, z. Without loss of generality, assume that $x, y, z \in B_s$ and $x > y > z$ where $0 \leq s \leq p-1$. Then $x = m_1 p + s$, $y = m_2 p + s$ and $z = m_3 p + s$ for some $m_1, m_2, m_3 \in \mathbb{Z}$. So $x - y = (m_1 - m_2)p$, $y - z = (m_2 - m_3)p$ and $x - z = (m_1 - m_3)p$, we obtain that $\gcd(x-y, n) \neq 1, \gcd(y-z, n) \neq 1$ and $\gcd(z-x, n) \neq 1$, respectively. Hence x, y, z, x form a cycle. It follows that $\overline{\Gamma}_n[B]$ contains a cycle.Therefore, $\Lambda(\overline{\Gamma}_n) \leq |B| - 1 = 2p$.Then the statement is proved, as required.

5. Conclusion

In this paper, we have provided certain structural properties and some invariant properties of unitary Cayley graphs Γ_n of finite commutative rings $\mathbb{Z}_n$. Such invariant properties consist of the lower acyclic number and the upper acyclic number. In the process of the study results, we have found that the decomposition of graphs is useful and plays an important role for determining those invariant parameters. Finally, we have presented the results of invariant parameters in the unitary Cayley graphs and their complements. For further works, one can investigate those invariant parameters for various types of algebraic graphs and their complements.

Acknowledgements

The authors are grateful to the referee(s) for suggestions on the manuscript. The first author would like to thank the Science Achievement Scholarship of Thailand (SAST). The corresponding author also thanks the Faculty of Science, Khon Kaen University.

Conflict of Interest

The authors declare that there are no conflicts of interest.

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