EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 2, Article Number 6063
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

2-Vertex Covering of a Graph

Javier A. Hassan[1,2,*], Sergio R. Canoy, Jr.[3,4], Anabel Gamorez[5], Eman C. Ahmad[5], An-Nadzwie S. Sappari[1]

[1] *Mathematics and Sciences Department, College of Arts and Sciences, MSU Tawi-Tawi College of Technology and Oceanography, Bongao, Tawi-Tawi, Philippines*
[2] *Department of Mathematics, College of Science, Korea University, Seoul, South Korea*
[3] *Department of Mathematics and Statistics, College of Science and Mathematics, MSU-Iligan Institute of Technology, Iligan City, Philippines*
[4] *Center of Mathematical and Theoretical Physical Sciences, PRISM, MSU-Iligan Institute of Technology, Iligan City, Philippines*
[5] *Department of Mathematics and Statistics, College of Science and Mathematics, Western Mindanao State University, Zamboanga City, Philippines*

Abstract. In this paper, we initiate the study on 2-vertex covering of a graph. We characterize the 2-vertex covering sets in some special graphs, join and corona of two graphs, and we derive some bounds or formulas of the said parameter of each of these graphs.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: Vertex cover, 2-domination, 2-vertex covering set, 2-vertex cover number

1. Introduction

Vertex cover of a graph is one the well-studied parameters in the theory of graphs. In optimization, the parameter can be used to model some real-world problems and in the elimination of repetitive DNA sequences for synthetic biology [1]. The parameter, as pointed out by Angel and Toregas et al. in [2] and [3], respectively, can also serve to model safety, defense strategy, and emergency facility location problems. It is well-known that the vertex cover problem is an NP-hard optimization problem. Karp in [4] used the NP-completeness of the clique problem to show that the vertex cover problem is NP-complete. NP-completeness of the vertex problem was also investigated by Garey et al.

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i2.6063

Email addresses: javierhassan@msutawi-tawi.edu.ph (J. Hassan)
sergio.canoy@g.msuiit.edu.ph (S. Canoy, Jr.)
anabel.gamorez@wmsu.edu.ph (A. Gamorez)
ahmad.eman@wmsu.edu.ph (E. Ahmad)
annadzwiesappari@msutawi-tawi.edu.ph (A. Sappari)

in [5] and [6]. Other studies on vertex cover and its variations can be found in [7], [8], [9], [10], [11], [12], and [13].

Vertex cover is closely related to the concept of domination. In fact, for a non-trivial connected graph, a vertex cover is a dominating set. Undoubtedly, a vertex cover can be made a dominating set in any graph by incorporating a domination-related concept in its definition. In this way, a variant of vertex cover emerges (see, for example, [8], [9], [10], [11], and [12]). Using the concept of 2-domination, we introduce the parameter called 2-vertex cover of a graph. As used to model a protection strategy in a network, the 2-vertex covering ensures that every node or vertex outside the cover has at least two neighbors coming from the covering. For studies that deal with 2-domination and its variants, readers may consider [14], [15],[16], [17], and [18].

2. Terminologies and Notations

The *open neighborhood* of a vertex v of a simple undirected graph G is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and its *closed neighborhood* is the set $N_G[v] = N_G(v) \cup \{v\}$. The *open neighborhood* of a set $S \subseteq V(G)$ is the set $N_G(S) = \cup_{v \in S} N_G(v)$, and its *closed neighborhood* is the set $N_G[S] = S \cup N_G(S)$. A vertex $v \in V(G)$ is an *isolated vertex* if $|N_G(v)| = 0$. The set containing all the isolated vertices in G will be denoted by $I(G)$. A vertex v is a *leaf* or an *endvertex* if $|N_G(v)| = 1$. The set $L(G)$ will denote the set consisting of all the leaves in G.

Let G and H be any two graphs. The *join* $G + H$ is the graph with vertex set $V(G+H) = V(G) \cup V(H)$ and edge set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The *corona* $G \circ H$ is the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the *ith* vertex of G to every vertex of the *ith* copy of H. We denote by H^v the copy of H in $G \circ H$ corresponding to the vertex $v \in G$ and write $v + H^v$ for $\langle \{v\} \rangle + H^v$.

A subset A of $V(G)$ is *independent* if for every pair of distinct vertices in A do not form an edge. The maximum cardinality of an independent set in G, denoted by $\alpha(G)$, is called the *independence number* of G. Any independent set with cardinality equal to $\alpha(G)$ is called an α-set in G.

A set $S \subseteq V(G)$ is a *dominating set* in G if $N_G[S] = V(G)$. It is a 2-*dominating set* if for every $v \in V(G) \setminus S$, $|N_G(v) \cap S| \geq 2$, i.e., v has at least two neighbors in S. The domination number (2-domination number) of G, denoted $\gamma(G)$ (resp. $\gamma_2(G)$), is the minimum cardinality of a dominating (resp. 2-dominating) set in G. Any dominating set (2-dominating set) with cardinality $\gamma(G)$ (resp. $\gamma_2(G)$) is called a γ-set (resp. γ_2-set).

A subset S of vertices of a graph G is called a *vertex cover* of G if for every edge $e = uv \in E(G)$, either $u \in S$ or $v \in S$. The minimum cardinality of a vertex cover of G is the *vertex cover number* of G and is denoted by $\beta(G)$. Any vertex cover of G with cardinality $\beta(G)$ is called a β-set. A set $S \subseteq V(G)$ is a 2-vertex cover (or covering) of G if S is both a vertex cover and a 2-dominating set in G. The 2-vertex covering number of G, denoted by $\beta_2(G)$, is the minimum cardinality of a 2-vertex covering of G. Any 2-vertex covering of G with cardinality $\beta_2(G)$ is called a β_2-set.

Readers are referred to [19] for other basic definitions that are not given here.

3. Main results

Theorem 1. *Let $G_1, G_2, \ldots, G_k$ be the components of G. Then $\beta_2(G) = \sum_{j=1}^{k} \beta_2(G_j)$.*

Proof. Let S be a β_2-set in G. For each $j \in [k] = \{1, 2, \cdots, k\}$, let $S_j = S \cap V(G_j)$. Then $S = \cup_{j=1}^{k} S_j$. Since S is a vertex cover of G, it follows that S_j is a vertex cover of G_j for each $j \in [k]$. Now, let $j \in [k]$ and let $v \in V(G_j) \setminus S_j$. Since S is a 2-dominating set in G, we have $|N_G(v) \cap S| \geq 2$. It follows that $|N_{G_j}(v) \cap S_j| \geq 2$. Therefore, S_j is a 2-vertex cover of G_j for each $j \in [k]$. Thus,

$$\beta_2(G) = |S| = |\cup_{j=1}^{k} S_j| = \sum_{j=1}^{k} |S_j| \geq \sum_{j=1}^{k} \beta_2(G_j).$$

For each $j \in [k]$, let D_j be a β_2-set in G_j. Clearly, $D = \cup_{j=1}^{k} D_j$ is a 2-vertex cover of G. Hence,

$$\beta_2(G) \leq |D| = |\cup_{j=1}^{k} D_j| = \sum_{j=1}^{k} |D_j| = \sum_{j=1}^{k} \beta_2(G_j).$$

This proves the assertion. □

Theorem 2. *Let G be a graph on n vertices. Then* $\max\{\gamma_2(G), \beta(G)\} \leq \beta_2(G) \leq n$. *Moreover, each of the following holds:*

- *(i) If G has vertex v with $|N_G(v)| \geq 2$, then $\beta_2(G) \leq n - 1$. In particular, if G is a connected graph and $n \geq 3$, then $\beta_2(G) \leq n - 1$.*
- *(ii) $\beta_2(G) = 1$ if and only if $G = K_1$.*
- *(iii) $\beta_2(G) = 2$ if and only if $G \in \{K_2, \overline{K}_2, K_2 + H, \overline{K}_2 + H\}$ for some graph H of order $n - 2$.*
- *(iv) $\beta_2(G) = n$ if and only if $G' \in \{K_1, K_2\}$ for every component G' of G.*

Proof. Since every 2-vertex covering of G is both a vertex cover and a 2-dominating set in G, it follows that $\max\{\gamma_2(G), \beta(G)\} \leq \beta_2(G)$. Clealy, $\beta_2(G) \leq n$.

(i) Suppose G has vertex v with $|N_G(v)| \geq 2$. Then clealy, $S = V(G) \setminus \{v\}$ is a 2-vertex cover of G. Hence, $\beta_2(G) \leq |S| = n - 1$. If G is connected and $n \geq 3$, then there exists $w \in V(G)$ with $|N_G(w)| \geq 2$. Therefore, $\beta_2(G) \leq n - 1$.

(ii) Suppose $\beta_2(G) = 1$, and let $S = \{v\}$ be a β_2-set of G. Since S is a 2-dominating set, there can be no vertex outside S. Hence, $G = K_1$.

The converse is clear.

(iii) Suppose $\beta_2(G) = 2$, say $S = \{x, y\}$ is a β_2-set of G. Suppose first that $xy \in E(G)$. If $n = 2$, then $G = K_2$. Suppose $n \geq 2$ and $z \in V(G) \setminus S$. Since S is a 2-dominating set in G, we have $z \in N_G(x) \cap N_G(y)$. This implies that $G = \langle\{x, y\}\rangle + H = K_2 + H$, where $H = \langle V(G) \setminus S\rangle$ is a graph of order $n - 2$. Next, suppose $xy \notin E(G)$. If $n = 2$, then $G = K_2$ by (ii) and Theorem 1. Suppose $n \geq 3$. Following an earlier argument, we have $V(G) \setminus S \subseteq N_G(x) \cap N_G(y)$. Therefore, $G = \langle\{x, y\}\rangle + H = \overline{K}_2 + H$, where $H = \langle V(G) \setminus S\rangle$ is a graph of order $n - 2$. Accordingly, $G \in \{K_2, \overline{K}_2, K_2 + H, \overline{K}_2 + H\}$ for some graph H of order $n - 2$.

The converse is clear.

(iv) Suppose $\beta_2(G) = n$. From (i), it follows that $|N_G(v)| \leq 1$ for every $v \in V(G)$. This implies that $G' \in \{K_1, K_2\}$ for every component G' of G.

For the converse, suppose that $G' \in \{K_1, K_2\}$ for every component G' of G. From (ii) and (iii), and by Theorem 1, it follows that $\beta_2(G) = n$. □

Theorem 3. *Let G be a graph on n vertices such that $|N_G(v)| \geq 2$ for some vertex $v \in V(G)$. Then $\beta_2(G) = n-1$ if and only if for every pair of non-adjacent vertices p and q of G, it holds that $p, q \in L(G) \cup I(G)$.*

Proof. Suppose $\beta_2(G) = n - 1$. Suppose, for a contradiction, that there exist non-adjacent vertices p and q such that $p, q \notin L(G) \cup I(G)$. Then $|N_G(p)| \geq 2$ and $|N_G(q)| \geq 2$. It follows that $S = V(G) \setminus \{p, q\}$ is a 2-vertex covering of G, implying that $\beta_(G) \leq |S| = n - 2$, a contradiction to our assumption. Therefore, G satisfies the given property.

For the converse, suppose that G satisfies the property and let S' be a β_2-set in G. Since $|N_G(v)| \geq 2$ for some vertex $v \in V(G)$, it follows from Theorem 2 that $\beta_2(G) = |S'| \leq n - 1$. Suppose for a contradiction that $\beta_2(G) \leq n - 2$. Then there exist distinct vertices $x, y \in V(G) \setminus S'$. Since S' is a vertex cover of G, we have $xy \notin E(G)$. This implies that $x, y \in L(G) \cup I(G)$ by the assumption. Therefore, S is not a 2-dominating set in G, a contradiction. Accordingly, $\beta_2(G) = |S'| = n - 1$. □

The next result is immediate from Theorem 3.

Corollary 1. *Let n be a positive integer such that $n \geq 3$. Then*

(i) $\beta_2(K_n) = n - 1$, *and*

(ii) $\beta_2(K_{1,n-1}) = n - 1$.

Theorem 4. *Let n be a positive integer. Then*

$$\beta_2(P_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil \ , & \textit{if } n \textit{ is odd} \\ \frac{n}{2} + 1 \ , & \textit{if } n \textit{ is even} \end{cases}$$

Proof. From (ii) and (iii) of Theorem 2, we have $\beta_2(P_1) = 1 = \frac{1+1}{2}$ and $\beta_2(P_2) = 2 = \frac{2}{2} + 1$. Let S be a β_2-set on P_n. Suppose first that n is odd and $n \geq 3$, say $n = 2r + 1$ where $r \geq 1$. Let $P_n = [a_1, a_2, \ldots, a_{2r}, a_{2r+1}]$. Since S is a 2-dominating set, we have $a_1, a_{2r+1} \in S$. Again, since S is a β_2-set of P_n, it follows that $S = \{a_1, a_3, \ldots, a_{2r-1}, a_{2r+1}\}$. Hence,

$$\beta_2(P_n) = |S| = \frac{n+1}{2} = \left\lceil \frac{n}{2} \right\rceil .$$

Next, suppose n is even and $n \geq 4$, say $n = 2t$ where $t \geq 2$. Let $P_n = [a_1, a_2, \ldots, a_{2r-1}, a_{2r}]$. Since S is 2-dominating, $a_1, a_{2r} \in S$. Again, since S is a β_2-set of P_n, $S = \{a_1, a_3, \ldots, a_{2r-1}\} \cup \{a_{2r+1}\}$. Hence,

$$\beta_2(P_n) = |S| = \frac{n}{2} + 1. \quad \square$$

Theorem 5. *Let n be a positive integer where $n \geq 3$. Then*

$$\beta_2(C_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil, \textit{if } n \textit{ is odd} \\ \frac{n}{2}, \textit{if } n \textit{ is even} \end{cases}$$

Proof. Let D be a β_2-set on P_n. Suppose n is odd. Clearly, $\beta_2(C_3) = 2 = \frac{1+1}{2}$. So suppose that $n \geq 5$, say $n = 2r + 1$ where $r \geq 2$. Let $C_n = [b_1, b_2, \ldots, b_{2r}, b_{2r+1}, b_1]$. We may assume that $b_1 \in D$. Since D is a β_2-set of C_n, it follows that $D = \{b_1, b_3, \ldots, b_{2r-1}, b_{2r+1}\}$. Hence,

$$\beta_2(C_n) = |D| = \frac{n+1}{2} = \left\lceil \frac{n}{2} \right\rceil .$$

Now, suppose that n is even and $n \geq 4$, say $n = 2m$ for $m \geq 2$. Let $C_n = [b_1, b_2, \ldots, b_{2m-1}, b_{2m}, b_1]$. Again, we may assume that $b_1 \in D$. Since D is a β_2-set of C_n, we have $D = \{b_1, b_3, \ldots, b_{2m-1}\}$. Hence,

$$\beta_2(P_n) = |D| = \frac{n}{2}. \quad \square$$

Theorem 6. *Let $G = K_{m_1, m_2, \cdots, m_k}$ be a complete k-partite graph with $2 \leq m_1 \leq m_2 \leq \cdots \leq m_k$. Then*

$$\beta_2(G) = \sum_{i \in [k] \setminus \{k\}} m_i$$

where $[k] = \{1, 2, \cdots, k\}$.

Proof. Let $S_1, S_2, \cdots, S_k$ be the partite sets of G and let S be a β_2-set of G. Suppose $v \in V(G) \setminus S$. Then there exists $j \in [k] = \{1, 2, \cdots, k\}$ such that $v \in S_j$. Since S is a vertex cover of G, it follows that $\cup_{i \in [k] \setminus \{j\}} S_i \subseteq S$. Moreover, since $V(G) \setminus S_j$ is a 2-vertex cover of G and S is a β_2-set of G, $S = V(G) \setminus S_j$. Again, because S is a β_2-set of G, we must have $j = k$. Therefore, $\beta_2(G) = |S| = \sum_{i \in [k] \setminus \{k\}} m_i$. $\square$

The next result follows from Theorem 6.

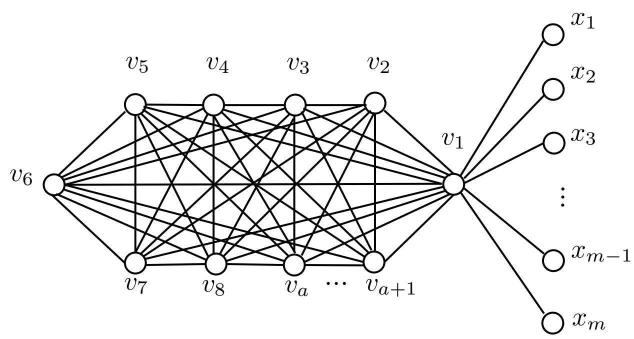
Corollary 2. *Let m and n be positive integers such that $2 \leq m \leq n$. For the complete bipartite $K_{m,n}$, we have $\beta_2(K_{m,n}) = m$.*

Remark 1. *Let G be a graph and let S be a 2-vertex cover of G. Then $L(G) \cup I(G) \subseteq S$.*

Theorem 7. *Let a and b be positive integers such that $3 \leq a \leq b$. Then there exists a connected graph G such that $\beta(G) = a$ and $\beta_2(G) = b$.*

Proof. If $a = b$, then consider $G = K_{a+1}$. Then $\beta(G) = a$ and, by Corollary 1(i), $\beta_2(G) = a$. Next, suppose $a < b$ and let $m = b-a$. Let G be the graph obtained from K_{a+1} by adding m pendant edges $v_1x_1, v_1x_2, \ldots, v_1x_m$, where $V(K_{a+1}) = \{v_1, v_2, \cdots, v_a, v_{a+1}\}$ (see Figure 2). Clearly, $S_1 = \{v_1, v_2, \cdots, v_a\}$ is a vertex cover of G. Hence, $\beta(G) \leq |S_1| = a$. Let S be a β-set in G. If $v_1 \notin S$, then $\{v_2, v_3, \cdots, v_{a+1}\} \subseteq S$ since S is a vertex cover of G. Again, since S is a vertex cover of G, it follows that $\{x_1, x_2, \cdots, x_m\} \subseteq S$. Thus, $S = \{x_1, x_2, \cdots, x_m, v_2, v_3, \cdots, v_{a+1}\}$. Consequently, $\beta(G) = |S| = m + a = b - a + a = b$, which is not possible. Thus, $v_1 \in S$. Suppose $|(V(K_{a+1}) \setminus \{v_1\}) \cap S| < a - 1$. Then there exist $r, t \in \{2, 3, \cdots, a+1\}$ such that $v_r, v_t \notin S$. This, however, is not possible because $v_rv_t \in E(G)$ and S is a vertex cover. Therefore, $|(V(K_{a+1}) \setminus \{v_1\}) \cap S| = a-1$. Therefore, since S is a β-set in G, $\beta(G) = |S| = a$.

Let D be a β_2-set in G. By Remark 1, $\{x_1, x_2, \cdots, x_m\} \subseteq S$. Now suppose $v_1 \notin S$. Since D is a vertex cover of G, it follows that $\{v_2, v_3, \cdots, v_{a+1}\} \subseteq D$. Hence, $D = \{x_1, x_2, \cdots, x_m, v_2, v_3, \cdots, v_{a+1}\}$. This implies that $\beta_2(G) = |D| = m + a = b$. Suppose $v_1 \in D$. Since D is a vertex cover of G, $|V(K_{a+1}) \setminus D| \leq 1$. The assumption that D is a β_2-set in G forces $|V(K_{a+1}) \setminus D| = 1$. Therefore, $\beta_2(G) = |D| = m + a = b$.



Therefore, the assertion holds. □

The next result is a consequence of Theorem 7.

Corollary 3. *Let n be a positive integer. Then there exists a connected graph G such that $\beta_2(G) - \beta(G) = n$. In other words, the difference $\beta_2(G) - \beta(G)$ can be made arbitrarily large.*

Theorem 8. *Let H be a non-trivial graph and let $G = K_1 + H$, where $K_1 = \langle\{v\}\rangle$. Then a set $S \subseteq V(G)$ is a 2-vertex cover of $G + H$ if and only if $S = V(H)$ or $S = \{v\} \cup D_H \cup I(H)$ where D_H is a vertex cover of H such that $D_H \cap I(G) = \varnothing$.*

Proof. Suppose S is a 2-vertex cover of G. Suppose $v \in S$ and let $w \in I(H)$. Since $N_G(w) \cap S = \{v\}$ and S is 2-dominating set in G, $w \in S$. Thus, $I(H) \subseteq S$. Let $D_H = [V(H) \setminus I(H)] \cap S$. Since S is a vertex cover of G, D_H is a vertex cover of H. Thus, (i) holds. If $v \notin S$, then $S = V(H)$ because $vp \in E(G)$ for all $p \in V(H)$ and S is a vertex cover of G. This implies that (ii) holds.

For the converse, suppose that (i) holds. Clearly, S is a vertex cover of G. Let $x \in V(G) \setminus S$. Then $x \in V(H) \setminus (D_H \cup I(G))$. Since $x \notin I(G)$, $xq \in E(H)$ for some $q \in V(H)$. Since D_H is a vertex cover of H, $q \in D_H$. It follows that $v, q \in N_G(x) \cap S$. Hence, S is a 2-dominating set in G. Therefore, S is a 2-vertex cover of G. If (ii) holds, then $S = V(H)$ is a 2-vertex cover of G because H is non-trivial. □

Lemma 1. *Let G be a graph of order n and let S be a β-set of G. Then each of the following holds:*

(i) $S \cap I(G) = \varnothing$.

(ii) $n = \beta(G) + |I(G)| + |(V(G) \setminus I(G)) \setminus S|$.

(iii) *If G is not the empty graph, then $|(V(G) \setminus I(G)) \setminus S| \geq 1$. Hence, $n = \beta(G) + |I(G)| + |(V(G) \setminus I(G)) \setminus S| \geq \beta(G) + |I(G)| + 1$.*

Proof. (i) Since a vertex cover only ensures that every edge is incident to a vertex inside the cover, S being a β-set of G implies that $S \cap I(G) = \varnothing$. Hence, (i) holds.

(ii) Since $V(G) = S \cup I(G) \cup [V(G) \setminus I(G)) \setminus S]$, (i) implies that $n = \beta(G) + |I(G)| + |(V(G) \setminus I(G)) \setminus S|$.

(iii) Suppose $G \neq \overline{K}_n$. Then $S \neq \varnothing$ and $\beta(G) = |S| \leq n - 1$. It follows that $|(V(G) \setminus I(G)) \setminus S| \geq 1$. Therefore,

$$n = \beta(G) + |I(G)| + |(V(G) \setminus I(G)) \setminus S| \geq \beta(G) + |I(G)| + 1. \quad \square$$

Corollary 4. *Let H be a non-trivial graph of order n and $G = K_1 + H$. Then each of the following holds:*

(i) *If $H \neq \overline{K}_n$, then $\beta_2(G) = \beta(H) + |I(H)| + 1$. Moreover, if H is connected, then $\beta_2(G) = \beta(H) + 1$.*

(ii) *If $H = \overline{K}_n$, then $\beta_2(G) = n$.*

Proof. (i) Suppose $H \neq \overline{K}_n$. Then $V(H) \setminus I(H) \neq \varnothing$. Let D_H be a β-set of H. Then $D_H \cap I(H) = \varnothing$. Let $S = \{v\} \cup D_H \cup I(H)$. Then S is a 2-vertex cover of G by Theorem 8. It follows that $\beta_2(G) \leq |S| = \beta(H) + |I(H)| + 1$.

Next, suppose S_0 is a β_2-set of G. If S_0 satisfies (ii) of Theorem 8, then $S_0 = \{v\} \cup D_H \cup I(H)$ where D_H is a vertex cover of H such that $D_H \cap I(G) = \varnothing$. Hence, $\beta_2(G) =$

$|S_0| \geq \beta(H) + |I(H)| + 1$. Suppose $S_0 = V(H)$. Then $\beta_2(G) = |S_0| = n$. By Lemma 1, $\beta_2(G) \geq \beta(H) + |I(H)| + 1$.

Therefore, $\beta_2(G) = \beta(H) + |I(H)| + 1$.

(ii) Suppose $H = \overline{K}_n$. Then $|I(H)| = n$ and $D_H = \varnothing$ is the only vertex cover of H. Hence, by Theorem 8, $S = V(H)$ is a β_2-set of G. Thus, $\beta_2(G) = n$. □

Theorem 9. *Let G and H be non-trivial graphs. Then $S \subseteq V(G+H)$ is a 2-vertex covering of $G+H$ if and only if $S = S_G \cup S_H$ and satisfies one of the following conditions:*

- (i) *$S_G = V(G)$ and S_H is a vertex cover of H.*
- (ii) *$S_H = V(H)$ and S_G is a vertex cover of G.*

Proof. Suppose S is a 2-vertex cover of $G+H$. Let $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$. Then $S = S_G \cup S_H$. Suppose $S_G \neq V(G)$ and $S_H \neq V(H)$. Pick any $x \in V(G) \setminus S_G$ and $p \in V(H) \setminus S_H$. Since $xp \in E(G+H)$, it follows that S is not a vertex cover of $G+H$, a contradiction. Thus, $S_G = V(G)$ or $S_H = V(H)$. Suppose $S_G = V(G)$ and let $st \in E(H)$. Since S is a vertex cover of $G+H$, $s \in S_H$ or $t \in S_H$. Hence, S_H is a vertex cover of H, showing that (i) holds. Similarly, S_G is a vertex cover of G whenever $S_H = V(H)$, showing that (ii) holds.

For the converse, suppose that $S = S_G \cup S_H$ and (i) holds. Let $pq \in E(G+H)$. If $p \in V(G)$ or $q \in V(G)$, then $p \in S$ or $q \in S$. Suppose $pq \in E(H)$. Since S_H is a vertex cover of H, $p \in S_H \subset S$ or $q \in S_H \subset S$. This implies that S is a vertex cover of $G+H$. Now let $z \in V(G+H) \setminus S$. Since $S_G = V(G)$, $z \in V(H) \setminus S_H$. The assumption that G is non-trivial assures that $|N_{G+H}(z) \cap S| \geq |N_{G+H}(z) \cap S_G| = |S_G| \geq 2$. Therefore, S is a 2-vertex covering of G. The same conclusion is true for S if (ii) holds. □

The next result is a consequence of Theorem 9

Corollary 5. *Let G and H be non-trivial graphs of orders m and n, respectively. Then*

$$\beta_2(G+H) = min\{m + \beta(H), n + \beta(G)\}.$$

Theorem 10. *Let G be a non-trivial connected graph and let H be any graph. Then $S \subseteq V(G \circ H)$ is a 2-vertex cover of $G \circ H$ if and only if $D = Q \cup (\cup_{v \in V(G)} R_v)$ and satisfies the following conditions:*

- (i) *Q is a vertex cover of G.*
- (ii) *$I(H^w) \subseteq R_w$ and $R_w \setminus I(H^w)$ is a vertex cover of H^v for each $w \in Q$.*
- (iii) *$S_v = V(H^v)$ for each $v \in V(G) \setminus Q$.*

Proof. Assume that S is a 2-vertex cover of $G \circ H$. Let $Q = D \cap V(G)$ and let $R_v = D \cap V(H^v)$ for each $v \in V(G)$. Clearly, $D = Q \cup (\cup_{v \in V(G)} R_v)$. Let $pq \in E(G) \subset E(G \circ H)$. The assumption that D is a vertex cover of $G \circ H$ implies that $p \in Q$ or $b \in Q$. This shows that Q is a vertex cover of G. This, in turn, shows that (i) holds. Let $w \in Q$.

Since D is a 2-dominating set, it follows that $I(H^w) \subseteq R_w$. Let $st \in E(H^w)$. Then $s, t \in V(H^w) \setminus I(H^w)$. Since D is a vertex cover of $G \circ H$), $s \in R_w$ or $t \in R_w$. Hence, $s \in R_w \setminus I(H^w)$ or $t \in R_w \setminus I(H^w)$. This implies that $R_w \setminus I(H^w)$ is a vertex cover of H^w. Thus, (ii) holds. Next, let $v \notin Q$ and let $q \in V(H^v)$. Since D is a vertex cover of $G \circ H$ and $vq \in E(G \circ H)$, $q \in R_v$. Since v was an arbitrary vertex of H^v, it follows that $R_v = V(H^v)$. This shows that (iii) also holds.

For the converse, suppose that D is as described and satisfies (i), (ii), and (iii). Let $vw \in E(G \circ H)$. If $v, w \in V(G)$, then $v \in Q$ or $v \in Q$ by (i). Suppose $v \in V(G)$ and $w \in V(H^v)$. If $v \in Q$, then vw is incident to $v \in D$. Suppose $v \notin Q$. Then $R_v = V(H^v)$ by (iii). Hence, $w \in S_v$ and vw is incident to $w \in D$. Next, suppose that that $v, w \in V(H^z)$ for some $z \in V(G)$. If $z \notin Q$, then $R_z = V(H^z)$. This implies that $v, w \in R_z \subset D$. Suppose that $z \in Q$. Since $vw \in E(G \circ H)$, $v, w \in V(H^z) \setminus I(H^z)$. By (ii), $R_z \setminus I(H^z)$ is a vertex cover of H^z. It follows that $v \in R_z \setminus I(H^z)$ or $w \in R_z \setminus I(H^z)$. Therefore, D is a vertex cover of $G \circ H$. Finally, let $x \in V(G \circ H) \setminus D$ and let $v \in V(G)$ such that $x \in V(v + H^v)$. If $x = v$, then $v \notin Q$. By (iii), $R_v = V(H^v)$. Since G is a non-trivial connected graph and Q is a vertex cover of G, it follows that $N_G(v) \cap Q \neq \varnothing$. Choose any $u \in N_G(v) \cap Q$ and $s \in R_v$. Then $u, s \in N_{G \circ H}(v) \cap D$. Suppose $x \in V(H^v) \setminus R_v$. Then $x \notin I(H^v)$ because $x \notin D$. Also, from (iii), it follows that $v \in Q$ (otherwise, $R_v = V(H^v)$ contrary to the fact that $x \in V(H^v) \setminus R_v$). Hence, from (ii), $R_v \setminus I(G)$ is a vertex cover of H^v. This implies that $N_{H^v}(x) \cap (R_v \setminus I(G)) \neq \varnothing$. Since $v \in N_{G \circ H}(x)$, it follows that $|N_{G \circ H}(x) \cap D| \geq |N_{H^v}(x) \cap (R_v \setminus I(G))| + 1 \geq 2$. This shows that D is a 2-dominating set in $G \circ H$. Therefore, D is a 2-vertex cover of $G \circ H$. □

Corollary 6. *Let G be a non-trivial connected graph of order m and let H be any graph of order n. Then*

$$\beta_2(G \circ H) = mn + (\beta(H) - n + |I(H)| + 1)\beta(G).$$

In particular, if H is a non-trivial connected graph, then

$$\beta_2(G \circ H) = mn + (\beta(H) - n + 1)\beta(G).$$

Proof. Let Q be a β-set in G, D_v a β-set in H^v and $R_v = D_v \cup I(H^v)$ for each $v \in Q$, and let $S_v = V(H^v)$ for each $v \in V(G) \setminus Q$. Then $D = Q \cup (\cup_{v \in V(G)} R_v)$ is a 2-vertex cover of $G \circ H$ by Theorem 10. It follows that

$$\begin{aligned}
\beta_2(G \circ H) &\leq |D| \\
&= |Q| + \sum_{v \in Q} |R_v| + \sum_{v \in V(G) \setminus Q} |R_v| \\
&= \beta(G) + \beta(G)[\beta(H) + |I(H)| + n(m - \beta(G)) \\
&= mn + (\beta(H) - n + |I(H)| + 1)\beta(G).
\end{aligned}$$

On the other hand, let D_0 be a β_2-set in $G \circ H$. Then $D_0 = X \cup (\cup_{v \in V(G)} T_v)$ and satisfies properties (i), (ii), and (iii) of Theorem 10. Hence, X is a vertex cover of G

by (i), $T_v = S_v \cup I(H^v)$, where S_v is a vertex cover of H^v, for each $v \in X$ by (ii), and $T_v = V(H^v)$ for each $v \in V(G) \setminus X$ by (iii). Thus,

$$\begin{aligned}
\beta_2(G \circ H) &= |D_0| \\
&= |X| + \sum_{v\in X} |T_v| + \sum_{v \in V(G)\setminus X} |T_v| \\
&\geq |X| + \sum_{v\in X} (\beta(H) + |I(H)|) + \sum_{v \in V(G)\setminus X} n \\
&= |X| + |X|(\beta(H) + |I(H)|) + (m - |X|)n \\
&= mn + (\beta(H) + |I(H)| - n + 1)|X| \\
&\geq mn + (\beta(H) + |I(H)| - n + 1)\beta(G).
\end{aligned}$$

This establishes the desired equality. If H is a non-trivial connected graph, then $|I(H)| = 0$. Hence, the additional assertion holds. □

4. Conclusion

The parameter 2-vertex cover, a variant of vertex cover, had been introduced and initially studied. This newly defined concept incorporates the concept of 2-domination in a graph. We gave bounds on the parameter, obtained the value of the parameter for some well-known classes of graphs, and characterized graphs that attain specific values such as 1, 2, $n-1$, and n, where n is the order of the graph. We also characterized the 2-vertex covering in the join and corona of two graphs and determined their 2-vertex cover numbers. We also showed that the difference between 2-vertex cover number and vertex cover number can be made arbitrarily large. The new parameter can be studied for other classes of graphs, say trees and other graphs resulting from some unary and binary operations, and bounds in terms of other parameters may be determined. Furthermore, since the vertex cover problem is NP-complete, the question as to whether the 2-vertex cover problem is also NP-complete remains unanswered.

Acknowledgements

The authors would like to thank the referees for reading the initial manuscript and for giving their comments and suggestions. Special thanks must go to **MSU-Iligan Institute of Technology, Iligan City, Mindanao State University - Tawi-Tawi College of Technology and Oceanography, Korea University,** and **Western Mindanao State University** for funding this research.

References

[1] A. Reis, S. Halper, G. Vezeau, D. Cetnar, A. Hossain, P. Clauer, and H. Salis. Simultaneous repression of multiple bacterial genes using nonrepetitive extra-long sgrna

arrays. *Nature Biotechnology*, 37(11):1294–1301, 2019.

[2] D. Angel and A. Amutha. Vertex covering and strong covering of flower like network structures. *Procedia Computer Science*, 87:164–171, 2016.

[3] C. Toregas, , R. Swain, C. Revelle, and L. Bercman. The location of emergency service facilities. *Journal of the Operations Research Society of America*, 19(6), 1971.

[4] R.M. Karp. Reducibility among combinatorial problems, complexity of computer computations. *Plenum Press, New York*, pages 85–103, 1972.

[5] M.R. Garey and D.S. Johnson. The rectilinear steiner tree problem is np-complete. *SIAM Journal on Applied Mathematics*, 32:826–834, 1977.

[6] M.R. Garey, D.S. Johnson, and L. Stockmeyer. Some simplified npcomplete problems. *Proceedings of the Sixth Annual ACM Symposium on Theory of Computing*, pages 47–63, 1974.

[7] B. Behsaz, P. Hatami, and E.S. Mahmoodian. On minimum vertex cover of generalized petersen graphs. *Australian Journal of Combinatorics*, 40:253–264, 2008.

[8] J. Hassan, M. A. Bonsocan, R. Rasid, and A. Sappari. Certified vertex cover of a graph. *ur. J. Pure Appl. Math.*, 17(2):1038–1045, 2024.

[9] M. Henning and A. Yeo. Identifying vertex covers in graphs. *The Electric Journal in Mathematics,*, 19(4):1038–1045., 2012.

[10] M. Marathe, R. Ravi, and C. P. Rangan. Generalized vertex covering in interval graphs. *Discrete Applied Mathematics*, 39:87–93, 1992.

[11] P. Pushpam and C. Suseendran. Secure vertex cover of a graph. *Discrete Mathematics, Algorithms and Applications*, 9(2), 2017.

[12] L. Sathikala, K. K. Basari, and K. Subramanian. Connected and total vertex covering in graphs. *Turkish Journal of Computer and Mathematics Education*, 12(2):2180–2185, 2021.

[13] J. Uy. Vertex cover of graphs. *Journal of Research in Science and Engineering*, 1:49–53, 2003.

[14] M. Blidia, M. Chellali, and O. Favaron. Independence and 2-domination in trees. *Australas. J. Combin.*, 33:317–327, 2005.

[15] M. Chellali. Bounds on the 2-domination number in cactus graphs. *Theoretical Computer Science*, 26:5–12, 2006.

[16] B. Domoloan and S. Canoy Jr. 2-domination and restrained 2-domination in graphs. *Applied Mathematical Sciences*, 9(114):5651–5659, 2015.

[17] S. Canoy Jr and B. Domoloan. Outer-connected 2-dominating sets of graphs. *Advances and Applications in Discrete Mathematics*, 20(1):25, 2019.

[18] S. Canoy Jr, F. Jamil, R.J. Fortosa, and J. Macalisang. Convex 2-domination in graphs. *European Journal of Pure and Applied Mathematics*, 17(3):1539–1552, 2024.

[19] F. Buckley and F. Harary. Distance in graphs. *Addison-Wesley, Redwood City*, 1990.