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Fekete–Szegő Inequalities for a New Class of Bi-Univalent Functions Defined via the Mittag-Leffler Function

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Abstract. In this paper, we introduce a new subclass of analytic functions denoted by $\mathcal{M}_{\Sigma}^{p,q}(\propto,\beta)$, where we use the subordination relationship between the Mittag-Leffler function and the (p,q)-derivative of $\mathfrak{F}(z)$ to define this new class. By employing the Taylor-Maclaurin series expansion, we focus on estimating the bounds for the coefficients $|a_2|$ and $|a_3|$. Moreover, we establish Fekete–Szegő inequalities for functions within this class.

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Key Words and Phrases: Mittag-Leffler function, Fekete-Szegő inequalities, Analytic Functions, (p,q)-derivative

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1. Introduction

Quantum calculus has become an essential tool across various fields, including mathematics, physics, and computer science. A significant development in this area is the (p,q)-calculus, which extends the concept of (p,q)-numbers. Since its inception in 1991, it has garnered significant interest from researchers [1–4]. Notably, Fibonacci oscillators were introduced in [1], and [2] explored the use of (p,q)-numbers to create a (p,q)-Harmonic oscillator. In [3], this approach was used to generalize certain q-oscillator algebras, while [4] utilized it in the calculation of (p,q)-Stirling numbers.

Let $\mathcal{A}$ denote the class of all functions $\mathfrak{F}$ that are analytic within the open unit disk

$$\Theta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

and normalized by the conditions:

$$\mathfrak{F}(0) = 0 \quad \text{and} \quad \mathfrak{F}'(0) = 1.$$

Thus, the function $\mathfrak{F} \in \mathcal{A}$ has the following Taylor-Maclaurin series representation:

$$\mathfrak{F}(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \Theta). \tag{1}$$

For two functions $\mathfrak{F}, \mathcal{G} \in \mathcal{A}$, the function $\mathfrak{F}$ is said to be subordinate to the function $\mathcal{G}$ in Θ, denoted by

$$\mathfrak{F}(z) \prec \mathcal{G}(z) \quad (z \in \Theta), \tag{2}$$

if there exists a function $w \in \mathcal{B}_0 := \{w : w \in \mathcal{A}, w(0) = 0 \text{ and } |w(z)| < 1\,(z \in \Theta)\}$ such that

$$\mathfrak{F}(z) = \mathcal{G}(w(z)) \quad (z \in \Theta). \tag{3}$$

In the case when the function $\mathcal{G}$ is univalent in Θ, the following equivalence is established:

$$\mathfrak{F}(z) \prec \mathcal{G}(z) \quad (z \in \Theta) \quad \Leftrightarrow \quad \mathfrak{F}(0) = \mathcal{G}(0) \text{ and } \mathfrak{F}(\Theta) \subset \mathcal{G}(\Theta). \tag{4}$$

It is well known that every univalent function $\mathfrak{F}$ has an inverse $\mathfrak{F}^{-1}$, defined by

$$\mathfrak{F}^{-1}(\mathfrak{F}(z)) = z = \mathfrak{F}(\mathfrak{F}^{-1}(z)) \quad (z \in \Theta),$$

and

$$\mathfrak{F}(\mathfrak{F}^{-1}(w)) = w \quad \left(|w| < r_0(\mathfrak{F}); \quad r_0(\mathfrak{F}) \geq \frac{1}{4}\right),$$

where

$$\mathfrak{F}^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_3^2 - 5a_2 a_3 + a_4)w^4 + \cdots \tag{5}$$

A function $\mathfrak{F} \in \mathcal{A}$ is said to be bi-univalent in the domain Θ if both $\mathfrak{F}$ and its inverse $\mathfrak{F}^{-1}$ are univalent in Θ. The set of such functions is represented by Σ. The influential study conducted by Srivastava *et al.* [5] has significantly renewed interest in the exploration of various subclasses within the analytic and bi-univalent function class Σ. Following this foundational work [6], a considerable number of research papers have focused on defining and analyzing different subclasses of the bi-univalent class Σ, as evidenced in several contributions (see, for instance, [5–26]).

One of the interesting problems in Geometric Function Theory is the Fekete–Szegő problem. This problem deals with the coefficients of functions $\mathfrak{F} \in \mathcal{S}$, and in [27], Fekete and Szegő established the following sharp result for such functions:

$$\left|a_3 - \varsigma a_2^2\right| \leq \begin{cases} 4\varsigma - 3, & \varsigma \geq 1, \\ 1 + 2e^{\frac{-2\varsigma}{1-\varsigma}}, & 0 \leq \varsigma < 1, \\ 3 - 4\varsigma, & \varsigma < 0. \end{cases}$$

The fundamental inequality $\left|a_3 - \varsigma a_2^2\right| \leq 1$ is achieved when $\varsigma \to 1$. The combination $F_\varsigma(\mathfrak{F}) = a_3 - \varsigma a_2^2$ plays an important role in the theory, and finding sharp bounds for $|F_\varsigma(\mathfrak{F})|$ is a notable maximization problem.

In geometric function theory, a wide range of analytic function subclasses has been explored through diverse analytical approaches. One of the essential frameworks facilitating this investigation is fractional q-calculus, which has emerged as a valuable tool in understanding the structure and properties of these function classes. The integration of q-calculus into geometric function theory notably began with the incorporation of basic (or q-) hypergeometric functions, as first introduced in a foundational work by Srivastava (see [28]).

The framework of univalent function theory is particularly well-suited for formulation using concepts from (q)-calculus. Recently, various researchers have utilized fractional (q)-integral and fractional (q)-differential operators to define and explore new subclasses of analytic functions (see [28, 29]). In this paper, we summarize the key ideas and operator definitions from (q)-calculus that are relevant to our analysis.

Unless otherwise specified, we assume that $0 < q < p \leq 1$. The definitions for fractional q-calculus operators, applicable to complex-valued functions $\mathfrak{F}(z)$, are given in alignment with the notation adopted in [30].

Definition 1. *([31]). The (p, q)-derivative of the function $\mathfrak{F}$, given by (1.1), is defined as:*

$$D_{p,q}\mathfrak{F}(z) = \begin{cases} \frac{\mathfrak{F}(pz) - \mathfrak{F}(qz)}{(p-q)z}, & z \neq 0, \\ \mathfrak{F}'(0), & z = 0, \text{ provided } \mathfrak{F}'(0) \text{ exists.} \end{cases} \tag{6}$$

From Definition 1.1, we deduce that:

$$D_{p,q}\mathfrak{F}(z) = 1 + \sum_{n=2}^{\infty} [n]_{p,q} a_n z^{n-1}, \tag{7}$$

where the symbol $[n]_{p,q}$ denotes the so-called (p,q)-bracket or twin-basic number:

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}.$$

It is clear that:

$$D_{p,q} z^n = [n]_{p,q} z^{n-1}.$$

Note also that for $p = 1$, the Jackson (p,q)-derivative reduces to the Jackson q-derivative given by (see [32]):

$$D_q \mathfrak{F}(z) = \begin{cases} \frac{\mathfrak{F}(z) - \mathfrak{F}(qz)}{(1-q)z}, & z \neq 0, \\ \mathfrak{F}'(0), & z = 0. \end{cases} \tag{8}$$

The twin-basic number is a natural generalization of the q-number, that is:

$$\lim_{p \to 1} [n]_{p,q} = [n]_q = \frac{1 - q^n}{1 - q}, \quad q \neq 1.$$

The familiar Mittag-Leffler function $E_{\gimel}(z)$, introduced by Mittag-Leffler [33], and its generalization $E_{\gimel,\beth}(z)$, introduced by Wiman (see [34, 35]), are defined as follows:

$$E_{\gimel}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\gimel n + 1)} = E_{\gimel,1}(z), \tag{9}$$

and

$$E_{\gimel,\beth}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\gimel n + \beth)}. \tag{10}$$

where $\gimel, \beth, z \in \mathbb{C}; \Re(\gimel) > 0$ These functions appear in various fields, including solutions to fractional differential equations, random walks, super-diffusive transport problems, and studies of complex systems.

Several properties of the Mittag-Leffler functions $E_{\gimel}(z)$ and $E_{\gimel,\beth}(z)$, along with their generalizations, can be found in a number of recent works (see [36], [37],[38], and [39–41]).

Since the Mittag-Leffler function $E_{\gimel,\beth}(z)$ does not belong to the class $\mathcal{A}$, The following normalized form of the Mittag-Leffler function is considered see [39] :

$$\Xi_{\gimel,\beth}(z) = \Gamma(\beth) z E_{\gimel,\beth}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beth) z^n}{\Gamma(\gimel(n-1) + \beth)}.$$

$$= z + \frac{\Gamma(\beth) z^2}{\Gamma(\gimel + \beth)} + \frac{\Gamma(\beth) z^3}{\Gamma(2\gimel + \beth)} + ... \tag{11}$$

where $\gimel, \beth, z \in \mathbb{C}; \Re(\gimel) > 0; \beth \neq 0, -1, -2, ...$ Whilst the definition (3) holds true for complex-valued parameters $\gimel$ and $\beth$ and $z \in \mathbb{C}$, yet (for the purpose of this paper) we shall restrict our attention to the case of real-valued parameters $\gimel$ and $\beth$ and $z \in \mathbb{D}$.

We observe that the normalized Mittag-Leffler function $\Xi_{\gimel,\beth}$ in (3) contains such well-known functions as its special cases given below:

$$\Xi_{2,1}(z) = z\cosh(\sqrt{z}), \quad \Xi_{2,2}(z) = \sqrt{z}\sinh(\sqrt{z}),$$

$$\Xi_{2,3}(z) = 2[\cosh(\sqrt{z}) - 1], \quad \text{and} \quad \Xi_{2,4}(z) = \frac{6[\sinh(\sqrt{z}) - \sqrt{z}]}{\sqrt{z}}.$$

Definition 2. *A function $\mathfrak{F} \in \Sigma$ given by (1) is said to be in the class $\mathcal{M}_{\Sigma}^{p,q}(\gimel, \beth)$, if the following conditions are satisfied:*

$$D_{p,q}\mathfrak{F}(z) \prec \Xi_{\gimel,\beth}(z), \tag{12}$$

and

$$D_{p,q}\mathcal{G}(w) \prec \Xi_{\gimel,\beth}(w) \tag{13}$$

where $z, w \in \Theta$, and $\mathcal{G} = \mathfrak{F}^{-1}$

We can derive the following corollaries:

For $p = 1$ on the class $\mathcal{M}_{\Sigma}^{p,q}(\gimel, \beth)$, we have that $\mathcal{M}_{\Sigma}^{1,q}(\gimel, \beth)$:

Corollary 1. *A function $\mathfrak{F} \in \Sigma$ given by (1) is said to be in the class $\mathcal{M}_{\Sigma}^{1,q}(\gimel, \beth)$, if the following conditions are satisfied:*

$$D_{q}\mathfrak{F}(z) \prec \Xi_{\gimel,\beth}(z), \tag{14}$$

and

$$D_{q}\mathcal{G}(w) \prec \Xi_{\gimel,\beth}(w) \tag{15}$$

where $z, w \in \Theta$, and $\mathcal{G} = \mathfrak{F}^{-1}$

For $p = 1$ and $q = 1$ on the class $\mathcal{M}_{\Sigma}^{p,q}(\gimel, \beth)$, we have that $\mathcal{M}_{\Sigma}(\gimel, \beth)$:

Corollary 2. *A function $\mathfrak{F} \in \Sigma$ given by (1) is said to be in the class $\mathcal{M}_{\Sigma}(\gimel, \beth)$, if the following conditions are satisfied:*

$$\mathfrak{F}'(z) \prec \Xi_{\gimel,\beth}(z), \tag{16}$$

and

$$\mathcal{G}'(w) \prec \Xi_{\gimel,\beth}(w) \tag{17}$$

where $z, w \in \Theta$, and $\mathcal{G} = \mathfrak{F}^{-1}$

Lemma 1.5 [42] If $h \in H$, where H represents all analytic functions in Θ and satisfy $\Re(h(z)) > 0$, where

$$h(z) = 1 + h_1 z + h_2 z^2 + \dots, \tag{18}$$

then $|h_i| \leq 2$ for each index i.

2. Estimating Coefficients for the Class $\mathcal{M}_{\Sigma}^{p,q}(\gimel,\beth)$

This section is devoted to deriving coefficient bounds for the class $\mathcal{M}_{\Sigma}^{p,q}(\gimel,\beth)$. We present several estimates for the initial coefficients and establish related results. In the concluding part of this section, some of these results are highlighted as special cases in the form of corollaries.

Theorem 1. *Suppose* $\mathfrak{F}$ *defined by* (1) *is in the class* $\mathcal{M}_{\Sigma}^{p,q}(\gimel,\beth)$ *, where* $z,w\in\Theta,\gimel,\beth\in\mathbb{C};\Re(\gimel)>0;\beth\neq 0,-1,-2,...,0<q<p\leq 1$ *Then:*

$$|a_2|\leq\frac{\Gamma(\beth)\sqrt{2\Gamma(2\gimel+\beth)}}{\sqrt{\left|2\left([3]_{p,q}\Gamma(\beth)\Gamma(2\gimel+\beth)-[2]_{p,q}^2\left(\Gamma(\gimel+\beth)\right)^2\right)\Gamma(\gimel+\beth)\right|}}$$

and

$$|a_3|\leq\frac{2\Gamma(\beth)}{|[3]_{p,q}\Gamma(\gimel+\beth)|}+\frac{4\left(\Gamma(\beth)\right)^2}{|[2]_{p,q}^2\left(\Gamma(\gimel+\beth)\right)^2|}$$

Proof: Suppose $\mathfrak{F}\in\mathcal{M}_{\Sigma}^{p,q}(\gimel,\beth)$ and let $\mathcal{G}$ be the analytic extension of $\mathfrak{F}^{-1}$ to Θ. Then, there exist two functions s and t, which are analytic in Θ, satisfying $s(0)=t(0)=0$, $|s(z)|<1$, and $|t(w)|<1$ for all $z,w\in\Theta$, such that:

$$D_{p,q}\mathfrak{F}(z)=\Xi_{\gimel,\beth}(s(z)) \tag{19}$$

$$D_{p,q}\mathcal{G}(w)=\Xi_{\gimel,\beth}(t(w)). \tag{20}$$

Next, let the functions s and t be defined as:

$$s(z)=s_1z+s_2z^2+\cdots$$

and

$$t(w)=t_1w+t_2w^2+\cdots$$

By combining equations (19) (20) :

$$1+[2]_{p,q}a_2z+[3]_{p,q}a_3z^2+...=\Xi_{\alpha,\beth}(s_1z+s_2z^2+\cdots)$$

$$1+[2]_{p,q}a_2z+[3]_{p,q}a_3z^2+...=1+\frac{\Gamma(\beth)}{\Gamma(\gimel+\beth)}s_1z+\left(\frac{\Gamma(\beth)}{\Gamma(\gimel+\beth)}s_2+\frac{\Gamma(\beth)}{\Gamma(2\gimel+\beth)}s_1^2\right)z^2+.... \tag{21}$$

and similarly

$$1-[2]_{p,q}a_2w+[3]_{p,q}(2a_2^2-a_3)w^2+...=\Xi_{\alpha,\beth}(t_1w+t_2w^2+\cdots)$$

$$1-[2]_{p,q}a_2w+[3]_{p,q}(2a_2^2-a_3)w^2+...=1+\frac{\Gamma(\beth)}{\Gamma(\gimel+\beth)}t_1w+\left(\frac{\Gamma(\beth)}{\Gamma(\gimel+\beth)}t_2+\frac{\Gamma(\beth)}{\Gamma(2\gimel+\beth)}t_1^2\right)w^2+.... \tag{22}$$

Based on (21) and (22), it is obtained that:

$$[2]_{p,q}a_2 = \frac{\Gamma(\beth)}{\Gamma(\gimel+\beth)}s_1, \tag{23}$$

$$[3]_{p,q}a_3 = \frac{\Gamma(\beth)}{\Gamma(\gimel+\beth)}s_2 + \frac{\Gamma(\beth)}{\Gamma(2\gimel+\beth)}s_1^2, \tag{24}$$

$$-[2]_{p,q}a_2 = \frac{\Gamma(\beth)}{\Gamma(\gimel+\beth)}t_1, \tag{25}$$

$$[3]_{p,q}(2a_2^2 - a_3) = \frac{\Gamma(\beth)}{\Gamma(\gimel+\beth)}t_2 + \frac{\Gamma(\beth)}{\Gamma(2\gimel+\beth)}t_1^2. \tag{26}$$

From equations (23) and (25), it is derived that:

$$s_1 = -t_1, \tag{27}$$

$$2[2]_{p,q}^2a_2^2 = \left(\frac{\Gamma(\beth)}{\Gamma(\gimel+\beth)}\right)^2\left(s_1^2+t_1^2\right). \tag{28}$$

By adding (24) to (26), it is obtained that:

$$2[3]_{p,q}a_2^2 = \frac{\Gamma(\beth)}{\Gamma(\gimel+\beth)}\left(s_2+t_2\right) + \frac{\Gamma(\beth)}{\Gamma(2\gimel+\beth)}\left(s_1^2+t_1^2\right). \tag{29}$$

Substituting (28) into equation (29), it has been found that:

$$a_2^2 = \frac{(\Gamma(\beth))^2\Gamma(2\gimel+\beth)\left(s_2+t_2\right)}{2\left([3]_{p,q}\Gamma(\beth)\Gamma(2\gimel+\beth) - [2]_{p,q}^2\left(\Gamma(\gimel+\beth)\right)^2\right)\Gamma(\gimel+\beth)}, \tag{30}$$

From Lemmas 1.5 and (30) we get:

$$|a_2| \leq \frac{\Gamma(\beth)\sqrt{2\Gamma(2\gimel+\beth)}}{\sqrt{\left|2\left([3]_{p,q}\Gamma(\beth)\Gamma(2\gimel+\beth) - [2]_{p,q}^2\left(\Gamma(\gimel+\beth)\right)^2\right)\Gamma(\gimel+\beth)\right|}}$$

The result of subtracting (24) from (26) is:

$$2[3]_{p,q}(a_3 - a_2^2) = \frac{\Gamma(\beth)}{\Gamma(\gimel+\beth)}\left(s_2-t_2\right) + \frac{\Gamma(\beth)}{\Gamma(2\gimel+\beth)}\left(s_1^2-t_1^2\right). \tag{31}$$

In view of (27) and (28), it is obtained that (31):

$$a_3 = \frac{\Gamma(\beth)\left(s_2-t_2\right)}{2[3]_{p,q}\Gamma(\gimel+\beth)} + a_2^2, \tag{32}$$

$$a_3 = \frac{\Gamma(\beth)\left(s_2-t_2\right)}{2[3]_{p,q}\Gamma(\gimel+\beth)} + \frac{(\Gamma(\beth))^2\left(s_1^2+t_1^2\right)}{2[2]_{p,q}^2\left(\Gamma(\gimel+\beth)\right)^2}. \tag{33}$$

As indicated by Lemma 1.5:

$$|a_3| \leq \frac{2\Gamma(\beth)}{|[3]_{p,q}\Gamma(\gimel+\beth)|} + \frac{4\left(\Gamma(\beth)\right)^2}{|[2]_{p,q}^2\left(\Gamma(\gimel+\beth)\right)^2|}$$

We can derive the following corollaries:
For $p = 1$ on the class $\mathcal{M}_\Sigma^{p,q}(\gimel,\beth)$, we have that $\mathcal{M}_\Sigma^{1,q}(\gimel,\beth)$:

Corollary 3. *Suppose* $\mathfrak{F}$ *defined by* (1) *is in the class* $\mathcal{M}_\Sigma^{p,1}(\gimel,\beth)$ *, where* $z, w \in \Theta, \gimel, \beth \in \mathbb{C}; \Re(\gimel) > 0; \beth \neq 0, -1, -2, ..., 0 < q < 1$ *Then:*

$$|a_2| \leq \frac{\Gamma(\beth)\sqrt{2\Gamma(2\gimel+\beth)}}{\sqrt{|2\left([3]_q\Gamma(\beth)\Gamma(2\gimel+\beth) - [2]_q^2\left(\Gamma(\gimel+\beth)\right)^2\right)\Gamma(\gimel+\beth)|}}$$

and

$$|a_3| \leq \frac{2\Gamma(\beth)}{|[3]_q\Gamma(\gimel+\beth)|} + \frac{4\left(\Gamma(\beth)\right)^2}{|[2]_q^2\left(\Gamma(\gimel+\beth)\right)^2|}$$

For $p = 1$ and $q = 1$ on the class $\mathcal{M}_\Sigma^{p,q}(\gimel,\beth), we have that \mathcal{M}_\Sigma(\gimel,\beth)$:

Corollary 4. *Suppose* $\mathfrak{F}$ *defined by* (1) *is in the class* $\mathcal{M}_\Sigma(\gimel,\beth)$ *, where* $z, w \in \Theta, \gimel, \beth \in \mathbb{C}; \Re(\gimel) > 0; \beth \neq 0, -1, -2, ...$ *Then:*

$$|a_2| \leq \frac{\Gamma(\beth)\sqrt{2\Gamma(2\gimel+\beth)}}{\sqrt{|2\left(3\Gamma(\beth)\Gamma(2\gimel+\beth) - 4\left(\Gamma(\gimel+\beth)\right)^2\right)\Gamma(\gimel+\beth)|}}$$

and

$$|a_3| \leq \frac{2\Gamma(\beth)}{|3\Gamma(\gimel+\beth)|} + \frac{4\left(\Gamma(\beth)\right)^2}{|4\left(\Gamma(\gimel+\beth)\right)^2|}$$

3. Fekete–Szegő Inequalities for the Function Class $\mathcal{M}_\Sigma^{p,q}(\gimel,\beth)$

In this section, we focus on the Fekete–Szegő inequalities for the function class $\mathcal{M}_\Sigma^{p,q}(\gimel,\beth)$. We also present some special cases in the form of corollaries.

Theorem 2. *Suppose* $\mathfrak{F}$*, as defined by equation* (1)*, belongs to the class* $\mathcal{M}_\Sigma^{p,q}(\gimel,\beth)$*, where* $\gimel, \beth \in \mathbb{C}; \Re(\gimel) > 0; \beth \neq 0, -1, -2, ..., 0 < q < p \leq 1, \varsigma \in \mathbb{R}$*. Then:*

$$|a_3 - \varsigma a_2^2| \leq \begin{cases} \frac{2\Gamma(\beth)}{[3]_{p,q}\Gamma(\alpha+\beth)} & \text{for } |h(\varsigma)| \leq \frac{1}{2[3]_{p,q}} \\ 4|h(\varsigma)| & \text{for } |h(\varsigma)| \geq \frac{1}{2[3]_{p,q}} \end{cases} \tag{34}$$

Proof:
From equations (30) and (32), it is derived that:

$$a_3 - \varsigma a_2^2 = \frac{\Gamma(\beth)\,(s_2 - t_2)}{2[3]_{p,q}\Gamma(\daleth + \beth)} + (1-\varsigma)a_2^2$$

Also,

$$a_3 - \varsigma a_2^2 = \frac{\Gamma(\beth)\,(s_2 - t_2)}{2[3]_{p,q}\Gamma(\daleth + \beth)} + \frac{(\Gamma(\beth))^2\Gamma(2\daleth + \beth)(1-\varsigma)\,(s_2 + t_2)}{2\left([3]_{p,q}\Gamma(\beth)\Gamma(2\daleth + \beth) - [2]_{p,q}^2\,(\Gamma(\daleth + \beth))^2\right)\Gamma(\daleth + \beth)}$$

Simplify to:

$$a_3 - \eta a_2^2 = \frac{\Gamma(\beth)}{\Gamma(\daleth + \beth)}\left[\left(h(\varsigma) + \frac{1}{2[3]_{p,q}}\right)s_2 + \left(h(\varsigma) - \frac{1}{2[3]_{p,q}}\right)t_2\right] \tag{35}$$

where

$$h(\varsigma) = \frac{\Gamma(\beth)\Gamma(2\daleth + \beth)(1-\varsigma)}{2\left([3]_{p,q}\Gamma(\beth)\Gamma(2\daleth + \beth) - [2]_{p,q}^2\,(\Gamma(\daleth + \beth))^2\right)} \tag{36}$$

We can derive the following corollaries:

For $p = 1$ on the class $\mathcal{M}_{\Sigma}^{p,q}(\daleth, \beth), we have that \mathcal{M}_{\Sigma}^{1,q}(\daleth, \beth)$:

Corollary 5. *Suppose $\mathfrak{F}$, as defined by equation (1), belongs to the class $\mathcal{M}_{\Sigma}^{p,1}(\daleth, \beth)$, where $\daleth, \beth \in \mathbb{C}; \Re(\daleth) > 0; \beth \neq 0, -1, -2, ..., 0 < q < p \leq 1, \varsigma \in \mathbb{R}$. Then:*

$$|a_3 - \varsigma a_2^2| \leq \begin{cases} \frac{2\Gamma(\beth)}{[3]_q\Gamma(\alpha+\beth)} & for\ \left|\frac{\Gamma(\beth)\Gamma(2\daleth+\beth)(1-\varsigma)}{2\left([3]_q\Gamma(\beth)\Gamma(2\daleth+\beth)-[2]_q^2(\Gamma(\daleth+\beth))^2\right)}\right| \leq \frac{1}{2[3]_q} \\ 4\left|\frac{\Gamma(\beth)\Gamma(2\daleth+\beth)(1-\varsigma)}{2\left([3]_q\Gamma(\beth)\Gamma(2\daleth+\beth)-[2]_q^2(\Gamma(\daleth+\beth))^2\right)}\right| & for\ \left|\frac{\Gamma(\beth)\Gamma(2\daleth+\beth)(1-\varsigma)}{2\left([3]_q\Gamma(\beth)\Gamma(2\daleth+\beth)-[2]_q^2(\Gamma(\daleth+\beth))^2\right)}\right| \geq \frac{1}{2[3]_q} \end{cases} \tag{37}$$

For $p = 1$ and $q = 1$ on the class $\mathcal{M}_{\Sigma}^{p,q}(\daleth, \beth), we have that \mathcal{M}_{\Sigma}(\daleth, \beth)$:

Corollary 6. *Suppose $\mathfrak{F}$, as defined by equation (1), belongs to the class $\mathcal{M}_{\Sigma}(\daleth, \beth)$, where $\daleth, \beth \in \mathbb{C}; \Re(\daleth) > 0; \beth \neq 0, -1, -2, ..., \varsigma \in \mathbb{R}$. Then:*

$$|a_3 - \varsigma a_2^2| \leq \begin{cases} \frac{2\Gamma(\beth)}{3\Gamma(\alpha+\beth)} & for\ \left|\frac{\Gamma(\beth)\Gamma(2\daleth+\beth)(1-\varsigma)}{2\left(3\Gamma(\beth)\Gamma(2\daleth+\beth)-4(\Gamma(\daleth+\beth))^2\right)}\right| \leq \frac{1}{6} \\ 4\left|\frac{\Gamma(\beth)\Gamma(2\daleth+\beth)(1-\varsigma)}{2\left(3\Gamma(\beth)\Gamma(2\daleth+\beth)-4(\Gamma(\daleth+\beth))^2\right)}\right| & for\ \left|\frac{\Gamma(\beth)\Gamma(2\daleth+\beth)(1-\varsigma)}{2\left(3\Gamma(\beth)\Gamma(2\daleth+\beth)-4(\Gamma(\daleth+\beth))^2\right)}\right| \geq \frac{1}{6} \end{cases} \tag{38}$$

In conclusion, we have estimated the basic Taylor coefficients $|a_2|$ and $|a_3|$, and derived upper bounds for the Fekete–Szegő problem $|a_3 - \varsigma a_2^2|$. Additionally, we presented some results as special cases.

4. Conclusions

In this paper, we introduced a new subclass of analytic functions defined by $\mathcal{M}_{\Sigma}^{p,q}(\gimel, \beth)$. A key element of our approach is the use of the Mittag-Leffler function, which plays a fundamental role in defining and analyzing the functions in this class. The sharp bounds we obtained for the Fekete-Szegő functional within this subclass yield improved results and pave the way for further exploration in this area. The subclass offers a robust framework for studying bi-univalent functions, particularly in relation to coefficient estimates, distortion theorems, and other central topics in geometric function theory. The constructions and methodologies we presented can be employed by researchers to investigate more complex problems associated with coefficient bounds and to expand the applicability of Mittag-Leffler functions in practical contexts. Ultimately, these results may contribute to the formation of new subclasses, enhancing our understanding of bi-univalent function properties and providing a solid foundation for novel developments and applications in this vibrant area of research.

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