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Coefficient Results on Certain Subclasses of Sakaguchi Type Bi-univalent Functions

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Abstract. In this article, we study some subclasses of Sakaguchi type bi-univalent functions associated with Gegenbauer polynomials and Einstein function. We explore certain properties of functions belonging to these subclasses, including coefficient bounds and the Fekete–Szegö functionals. This research generalise and improves the related works of several earlier authors.

2020 Mathematics Subject Classifications: 30C10, 30C45, 30C50, 33C45

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1. Introduction and Preliminaries

Let $\mathcal{A}$ signify the class of analytic functions written in the form

$$\mathcal{F}(\xi) = \xi + \sum_{n=2}^{\infty} a_n \xi^n, \xi \in \mathbb{U} = \{\xi \in \mathbb{C} : |\xi| < 1\}, \tag{1}$$

with normalization $\mathcal{F}(0) = 0, \mathcal{F}'(0) = 1$. Define $\mathcal{S}$ as the subclass of univalent functions in $\mathcal{A}$.

An function $\mathcal{F} \in \mathcal{A}$ is said to be starlike, denoted as $\mathcal{S}^*$, if and only if

$$Re\left(\frac{\xi \mathcal{F}'(\xi)}{\mathcal{F}(\xi)}\right) > 0.$$

Similarly, $\mathcal{F}$ is convex, denoted as $\mathfrak{K}$, if and only if

$$Re\left(1 + \frac{\xi \mathcal{F}''(\xi)}{\mathcal{F}'(\xi)}\right) > 0.$$

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If a function $\mathcal{G} \in \mathcal{A}$ is given by

$$\mathcal{G}(\xi) = \xi + \sum_{n=2}^{\infty} b_n \xi^n, \xi \in \mathbb{U},$$

then the Hadamard product of two functions $\mathcal{F}$ and $\mathcal{G}$ is given by

$$(\mathcal{F} * \mathcal{G})(\xi) = \xi + \sum_{n=2}^{\infty} a_n b_n \xi^n, \xi \in \mathbb{U}.$$

If there exists an analytic Schwarz function Ω in $\mathbb{U}$, such that for all $\xi \in \mathbb{U}$, Ω satisfies $\Omega(0) = 0$, $|\Omega(\xi)| < 1$ and $\mathcal{F}(\xi) = \mathcal{G}(\Omega(\xi))$, then two analytic functions $\mathcal{F}$ and $\mathcal{G}$ can be described as $\mathcal{F}$ is subordinate to $\mathcal{G}$ or $\mathcal{G}$ is superordinate to $\mathcal{F}$. This relationship is written as $\mathcal{F} \prec \mathcal{G}$. If the function $\mathcal{G}$ is univalent in unit disk, then

$$\mathcal{F}(\xi) \prec \mathcal{G}(\xi) \Leftrightarrow \mathcal{F}(0) = \mathcal{G}(0) \text{ and } \mathcal{F}(\mathbb{U}) \subset \mathcal{G}(\mathbb{U}).$$

For more details on subordination principles, please refer to [1].

Koebe's one-quarter theorem [2] stated that, if $\mathbb{U}$ is mapped by $\mathcal{F}$ biholomorphically onto a domain Δ in the complex plane, then each of its tangent disks of radius r will be mapped onto a domain containing a disk of radius $\frac{1}{4}r$. Hence, the following conditions must be satisfied by function $\mathcal{F} \in \mathcal{A}$ for its inverse map $\mathcal{F}^{-1}$ to exist:

$$\mathcal{F}^{-1}(\mathcal{F}(\xi)) = \xi \,(\xi \in \mathbb{U}) \text{ and } \mathcal{F}(\mathcal{F}^{-1}(w)) = w \left(|w| < r_0(\mathcal{F}); r_0(\mathcal{F}) \geq \frac{1}{4}\right).$$

Inverse function $\mathcal{F}^{-1}$ of (1) can be conveyed as a power series of the form

$$\mathcal{G}(\omega) = \mathcal{F}^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \dots .$$

The class of bi-univalent functions, Σ, is a set of $\mathcal{F} \in \mathcal{A}$, in which both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are univalent in $\mathbb{U}$. Examples of functions in class Σ include

$$-log(1-\xi), \frac{\xi}{1-\xi}, \frac{1}{2}log\left(\frac{1+\xi}{1-\xi}\right), \dots .$$

It is noteworthy that the Koebe function is not bi-univalent, since it does not contain $\mathbb{U}$. To be precise, the image of Koebe function does not contain the slit of negative real axis from $-\frac{1}{4}$ to $-\infty$.

The coefficient bounds for functions of class Σ have been investigated since 1967, where Lewin [3] showed that $|a_2| < 1.51$. In addition to that, Brannan and Clunie [4] showed that $\underset{f\in\Sigma}{max}|a_2| = \sqrt{2}$. Subsequently, Netanyahu [5] improved this bound to $|a_2| \leq \frac{4}{3}$. The best non-sharp estimate $|a_2| < 1.485$ was obtained in 1984 by Tan [6]. On the other hand, the estimate for coefficients other than $|a_1|$ and $|a_2|$ for each $\mathcal{F} \in \Sigma$ is yet to be explored.

Introduced by Sakaguchi [7], a function $\mathcal{F} \in \mathcal{S}$ is described as starlike about symmetric points, provided it satisfies the inequality

$$Re\left(\frac{\xi\mathcal{F}'(\xi)}{\mathcal{F}(\xi)-\mathcal{F}(-\xi)}\right) > 0 \ (\xi \in \mathbb{U}).$$

This class of functions is denoted by $\mathcal{S}^*_{\mathcal{S}}$.

On the other hand, Das and Singh [8] established another class of univalent functions $\mathcal{F}$ which are convex about symmetric points. This family of functions, $\mathcal{K}_{\mathcal{S}}$ satisfy the inequality

$$Re\left(\frac{(\xi\mathcal{F}'(\xi))'}{(\mathcal{F}(\xi)-\mathcal{F}(-\xi))'}\right) > 0 \ (\xi \in \mathbb{U}).$$

Two polynomials $\mathfrak{P}_n$ and $\mathfrak{P}_m$ of order n and m are said to be orthogonal, provided

$$\int_a^b w(x)\mathfrak{P}_n(x)\mathfrak{P}_m(x)dx = 0 \text{ for } n \neq m,$$

where $w(x)$ is non-negative function in the interval (a, b). As described by [9], Gegenbauer polynomials are special orthogonal polynomials which are typically associated with typically real functions. For $\alpha \in \mathbb{R} - \{0\}$, Gegenbauer polynomial is defined by a generating function

$$\mathfrak{H}_\alpha(x,\xi) = \frac{1}{(1-2x\xi+\xi^2)^\alpha},$$

where $\xi \in \mathbb{U}$ and $x \in [-1, 1]$. When x is fixed, the function $\mathfrak{H}_\alpha$ is analytic in $\mathbb{U}$, therefore it is possible to express $\mathfrak{H}_\alpha$ in the form of Taylor series expansion

$$\mathfrak{H}_\alpha(x,\xi) = \sum_{n=0}^{\infty} C_n^\alpha(x)\xi^n,$$

where $C_n^\alpha(x)$ is a Gegenbauer polynomial of degree n.

Since $\mathfrak{H}_0$ generates nothing, it is presumed to be

$$\mathfrak{H}_0(x,\xi) = 1 - log(1-2x\xi+\xi^2) = \sum_{n=0}^{\infty} C_n^0(x)\xi^n.$$

By using the recurrence relations, Gegenbauer polynomial can also be represented as [10–12]

$$C_n^\alpha(x) = \frac{1}{n}[2x(n+\alpha-1)C_{n-1}^\alpha(x) - (n+2\alpha-2)C_{n-2}^\alpha(x)].$$

As indicated in [9], the Gegenbauer polynomial possesses initial values as follows

$$C_0^\alpha(x) = 1, C_1^\alpha(x) = 2\alpha x \text{ and } C_2^\alpha(x) = 2\alpha(1+\alpha)x^2 - \alpha.$$

Meanwhile, the name Einstein function is sometimes applied for one of the functions [13, 14]: $E_1(\xi) = \frac{\xi}{e^\xi-1}$, $E_2(\xi) = \frac{\xi^2 e^\xi}{(e^\xi-1)^2}$, $E_3(\xi) = log(1-e^{-\xi})$ or $E_4(\xi) = \frac{\xi}{e^\xi-1} - log(1-e^{-\xi})$.

Among these functions, $E_1(\xi) = \frac{\xi}{e^{\xi}-1}$, has some nice properties, such as E_1 is a convex function, with its real part, $Re(E_1(\xi)) > 0$, $\forall \xi \in \mathbb{U}$. Its image domain is starlike about $E_1(0) = 1$ and is symmetric along the real axis. However, since $E_1'(0) \not> 0$, a new function $\mathfrak{E}(\xi) = E_1(\xi) + \xi$ is defined to make $\mathfrak{E} \in \mathcal{P}$ (see [15]). This function has the series representation

$$\mathfrak{E}(\xi) = 1 + \xi + \sum_{n=1}^{\infty} \frac{\mathfrak{B}_n}{n!}\xi^n,$$

where $\mathfrak{B}_n$ is known as the nth Bernoulli number. By traversing the contour which possesses a radius which is less than $2\pi i$ and encloses the origin in positive (counterclockwise) direction, the values of $\mathfrak{B}_n$ can be determined by contour integral [16]

$$\mathfrak{B}_n = \frac{n!}{2\pi i} \oint \frac{\xi}{e^{\xi} - 1} \frac{d\xi}{\xi^{n+1}}.$$

It is known that the first few terms of $\mathfrak{B}_n$ are

$$\mathfrak{B}_0 = 1, \mathfrak{B}_1 = -\frac{1}{2}, \mathfrak{B}_2 = \frac{1}{6}, \mathfrak{B}_4 = -\frac{1}{30}, \mathfrak{B}_6 = \frac{1}{42} \text{ and } \mathfrak{B}_{2n+1} = 0, \forall n \in \mathbb{N}.$$

Some similar work has been done with regard to coefficient bounds for bi-univalent functions, including [17–19]. Several authors specifically studied this on Gegenbauer polynomial, such as [9, 20]. So far we have not seen any work with classes associated with Gegenbauer and Einstein functions of Sakaguchi type. Therefore, in this article, we are solving coefficient bounds and the Fekete-Szegö functional for the aforementioned classes.

Definition 1.1. *A bi-univalent function $\mathcal{F}$ is presumed to be in class $\mathfrak{H}\mathfrak{E}^*(\alpha)$ if for all $\xi, \omega \in \mathbb{U}$, the following subordinations hold,*

$$\frac{2\xi\mathcal{F}'(\xi)}{\mathcal{F}(\xi) - \mathcal{F}(-\xi)} \prec (\mathfrak{H} * \mathfrak{E})(\xi),$$

and

$$\frac{2\omega\mathcal{G}'(\omega)}{\mathcal{G}(\omega) - \mathcal{G}(-\omega)} \prec (\mathfrak{H} * \mathfrak{E})(\omega).$$

Definition 1.2. *A bi-univalent function $\mathcal{F}$ is presumed to be in class $\mathfrak{H}\mathfrak{E}_{\mathfrak{K}}(\alpha)$ if for all $\xi, \omega \in \mathbb{U}$, the following subordinations hold,*

$$\frac{2(\xi\mathcal{F}'(\xi))'}{(\mathcal{F}(\xi) - \mathcal{F}(-\xi))'} \prec (\mathfrak{H} * \mathfrak{E})(\xi),$$

and

$$\frac{2(\omega\mathcal{G}'(\omega))'}{(\mathcal{G}(\omega) - \mathcal{G}(-\omega))'} \prec (\mathfrak{H} * \mathfrak{E})(\omega).$$

Remark 1. *If $\mathcal{F}(-\xi) = -\mathcal{F}(\xi)$, then $\mathfrak{H}\mathfrak{E}^*(\alpha)$ and $\mathfrak{H}\mathfrak{E}_{\mathfrak{K}}(\alpha)$ is a class of starlike and convex bi-univalent functions associated with $\mathfrak{H} * \mathfrak{E}$, respectively.*

2. Coefficient Bounds for the Class $\mathfrak{H}\mathfrak{E}^*(\alpha)$

Within this section and Section 3, coefficient bounds will be investigated for bi-univalent functions in the families $\mathfrak{H}\mathfrak{E}^*(\alpha)$ and $\mathfrak{H}\mathfrak{E}_{\mathfrak{K}}(\alpha)$, respectively.

Theorem 2.1. *Suppose that $\mathcal{F} \in \Sigma$ belongs to the class $\mathfrak{H}\mathfrak{E}^*(\alpha)$, then*

$$|a_2| \le \frac{|\alpha| x\sqrt{6x}}{\sqrt{|8\alpha x^2 - 4x^2 + 2|}},$$

and

$$|a_3| \le \frac{\alpha^2 x^2}{4} + \frac{\alpha x}{2}.$$

Proof. Let $\mathfrak{u}, \mathfrak{v}$ be Schwarz functions such that $\mathfrak{u}(\xi) = \sum_{k=1}^{\infty} d_k \xi^k$, $\mathfrak{v}(\omega) = \sum_{k=1}^{\infty} \mathfrak{d}_k \omega^k$, then

$$(\mathfrak{H} * \mathfrak{E})(\mathfrak{u}(\xi)) = 1 + \frac{C_1^\alpha(x)}{2} d_1 \xi + \left(\frac{C_1^\alpha(x)}{2} d_2 + \frac{C_2^\alpha(x)}{12} d_1^2\right) \xi^2 + \dots , \tag{2}$$

and

$$(\mathfrak{H} * \mathfrak{E})(\mathfrak{v}(\omega)) = 1 + \frac{C_1^\alpha(x)}{2} \mathfrak{d}_1 \omega + \left(\frac{C_1^\alpha(x)}{2} \mathfrak{d}_2 + \frac{C_2^\alpha(x)}{12} \mathfrak{d}_1^2\right) \omega^2 + \dots . \tag{3}$$

A Sakaguchi type function of the class $\mathcal{A}$ can be expanded as follows

$$\frac{2\xi \mathcal{F}'(\xi)}{\mathcal{F}(\xi) - \mathcal{F}(-\xi)} = 1 + 2a_2 \xi + 2a_3 \xi^2 + \dots , \tag{4}$$

and

$$\frac{2\omega \mathcal{G}'(\omega)}{\mathcal{G}(\omega) - \mathcal{G}(-\omega)} = 1 - 2a_2 \omega + (4a_2^2 - 2a_3)\omega^2 + \dots . \tag{5}$$

Comparing coefficients of (2) with (4), and (3) with (5), the followings are obtained

$$2a_2 = \frac{C_1^\alpha(x)}{2} d_1, \tag{6}$$

$$2a_3 = \frac{C_1^\alpha(x)}{2} d_2 + \frac{C_2^\alpha(x)}{12} d_1^2, \tag{7}$$

$$-2a_2 = \frac{C_1^\alpha(x)}{2} \mathfrak{d}_1, \tag{8}$$

and

$$4a_2^2 - 2a_3 = \frac{C_1^\alpha(x)}{2} \mathfrak{d}_2 + \frac{C_2^\alpha(x)}{12} \mathfrak{d}_1^2. \tag{9}$$

From (6) and (8), we have

$$d_1 = -\mathfrak{d}_1. \tag{10}$$

By adding the squares of (6) and (8),

$$8a_2^2 = \frac{[C_1^\alpha(x)]^2}{4} (d_1^2 + \mathfrak{d}_1^2), \tag{11}$$

and

$$d_1^2 + \mathfrak{d}_1^2 = \frac{32a_2^2}{[C_1^\alpha(x)]^2}. \tag{12}$$

Sum up (7) and (9), we have

$$4a_2^2 = \frac{C_1^\alpha(x)}{2}(d_2 + \mathfrak{d}_2) + \frac{C_2^\alpha(x)}{12}(d_1^2 + \mathfrak{d}_1^2). \tag{13}$$

Substituting (12) into (13), we obtain

$$4a_2^2 = \frac{C_1^\alpha(x)}{2}(d_2 + \mathfrak{d}_2) + C_2^\alpha(x)\left(\frac{8a_2^2}{3[C_1^\alpha(x)]^2}\right),$$

and so

$$\left(8 - \frac{16C_2^\alpha(x)}{3[C_1^\alpha(x)]^2}\right)a_2^2 = C_1^\alpha(x)(d_2 + \mathfrak{d}_2). \tag{14}$$

Substitute the values of C_n^α, (14) becomes

$$\left(8 - \frac{32\alpha(1+\alpha)x^2 - 16\alpha}{12\alpha^2x^2}\right)a_2^2 = 2\alpha x(d_2 + \mathfrak{d}_2)$$

and

$$a_2^2 = \frac{3\alpha^2x^3}{8\alpha x^2 - 4x^2 + 2}(d_2 + \mathfrak{d}_2). \tag{15}$$

Since $|\mathfrak{u}(\xi)| < 1$ and $|\mathfrak{v}(\omega)| < 1$, we have

$$|d_k| \leq 1 \text{ and } |\mathfrak{d}_k| \leq 1, \forall k \in \mathbb{N}. \tag{16}$$

Therefore,

$$|a_2| \leq \frac{|\alpha|x\sqrt{6x}}{\sqrt{|8\alpha x^2 - 4x^2 + 2|}}.$$

Now we proceed to evaluate the bounds of $|a_3|$. Taking (7)$-$(9),

$$4a_3 - 4a_2^2 = \frac{C_1^\alpha(x)}{2}(d_2 - \mathfrak{d}_2) + \frac{C_2^\alpha(x)}{12}(d_1^2 - \mathfrak{d}_1^2). \tag{17}$$

Substituting (10) into (17) and rearranging the terms, the following result is obtained

$$a_3 = a_2^2 + \frac{C_1^\alpha(x)}{8}(d_2 - \mathfrak{d}_2). \tag{18}$$

Substitute (11) and the values of $C_1^\alpha(x)$,

$$a_3 = \frac{4\alpha^2x^2}{32}(d_1^2 + \mathfrak{d}_1^2) + \frac{2\alpha x}{8}(d_2 - \mathfrak{d}_2).$$

Taking (16) into consideration, we have

$$|a_3| \leq \frac{\alpha^2x^2}{4} + \frac{\alpha x}{2}.$$

3. Coefficient Bounds for the Class $\mathfrak{H}\mathfrak{E}_{\mathfrak{K}}(\alpha)$

Theorem 3.1. *Suppose that $\mathcal{F} \in \Sigma$ belongs to the class $\mathfrak{H}\mathfrak{E}_{\mathfrak{K}}(\alpha)$, then*

$$|a_2| \leq \frac{|\alpha|x\sqrt{3x}}{\sqrt{|10\alpha x^2 - 8x^2 + 4|}},$$

and

$$|a_3| \leq \frac{\alpha^2 x^2}{16} + \frac{\alpha x}{6}.$$

Proof. Let $\mathfrak{u}, \mathfrak{v}$ be Schwarz functions such that $\mathfrak{u}(\xi) = \sum_{k=1}^{\infty} d_k \xi^k$, $\mathfrak{v}(\omega) = \sum_{k=1}^{\infty} \mathfrak{d}_k \omega^k$, then functions $\mathcal{F}$ and $\mathcal{G}$ that are convex about symmetric points can be expanded as follows

$$\frac{2(\xi \mathcal{F}'(\xi))'}{(\mathcal{F}(\xi) - \mathcal{F}(-\xi))'} = 1 + 4a_2\xi + 6a_3\xi^2 + ... , \tag{19}$$

and

$$\frac{2(\omega \mathcal{G}'(\omega)')'}{(\mathcal{G}(\omega) - \mathcal{G}(-\omega))'} = 1 - 4a_2\omega + (12a_2^2 - 6a_3)\omega^2 + ... \tag{20}$$

Comparing coefficients of (2) with (19), and (3) with (20), the followings are obtained

$$4a_2 = \frac{C_1^{\alpha}(x)}{2} d_1, \tag{21}$$

$$6a_3 = \frac{C_1^{\alpha}(x)}{2} d_2 + \frac{C_2^{\alpha}(x)}{12} d_1^2, \tag{22}$$

$$-4a_2 = \frac{C_1^{\alpha}(x)}{2} \mathfrak{d}_1, \tag{23}$$

and

$$12a_2^2 - 6a_3 = \frac{C_1^{\alpha}(x)}{2} \mathfrak{d}_2 + \frac{C_2^{\alpha}(x)}{12} \mathfrak{d}_1^2. \tag{24}$$

From (21) and (23), we obtain

$$d_1 = -\mathfrak{d}_1. \tag{25}$$

By adding the squares of (21) and (23),

$$32a_2^2 = \frac{[C_1^{\alpha}(x)]^2}{4} (d_1^2 + \mathfrak{d}_1^2), \tag{26}$$

and

$$d_1^2 + \mathfrak{d}_1^2 = \frac{128a_2^2}{[C_1^{\alpha}(x)]^2}. \tag{27}$$

Sum up (22) and (24), we obtain

$$12a_2^2 = \frac{C_1^{\alpha}(x)}{2} (d_2 + \mathfrak{d}_2) + \frac{C_2^{\alpha}(x)}{12} (d_1^2 + \mathfrak{d}_1^2). \tag{28}$$

Substituting (27) into (28), we have

$$12a_2^2 = \frac{C_1^\alpha(x)}{2}(d_2 + \mathfrak{d}_2) + C_2^\alpha(x)\left(\frac{32a_2^2}{3[C_1^\alpha(x)]^2}\right),$$

and

$$\left(24 - \frac{64C_2^\alpha(x)}{3[C_1^\alpha(x)]^2}\right)a_2^2 = C_1^\alpha(x)(d_2 + \mathfrak{d}_2). \tag{29}$$

Substitute the values of C_n^α, (29) become

$$\left(24 - \frac{128\alpha(1+\alpha)x^2 - 64\alpha}{12\alpha^2x^2}\right)a_2^2 = 2\alpha x(d_2 + \mathfrak{d}_2)$$

and

$$a_2^2 = \frac{3\alpha^2x^3}{20\alpha x^2 - 16x^2 + 8}(d_2 + \mathfrak{d}_2). \tag{30}$$

By (16),

$$|a_2| \le \frac{|\alpha|x\sqrt{3x}}{\sqrt{|10\alpha x^2 - 8x^2 + 4|}}.$$

To evaluate the bounds of $|a_3|$, take (22)$-$(24),

$$12a_3 - 12a_2^2 = \frac{C_1^\alpha(x)}{2}(d_2 - \mathfrak{d}_2) + \frac{C_2^\alpha(x)}{12}(d_1^2 - \mathfrak{d}_1^2). \tag{31}$$

Substituting (25) into (31) and rearranging the terms,

$$a_3 = a_2^2 + \frac{C_1^\alpha(x)}{24}(d_2 - \mathfrak{d}_2). \tag{32}$$

Substitute (26) and the values of $C_1^\alpha(x)$,

$$a_3 = \frac{\alpha^2x^2}{32}(d_1^2 + \mathfrak{d}_1^2) + \frac{\alpha x}{12}(d_2 - \mathfrak{d}_2).$$

Taking (16) into consideration, we obtain

$$|a_3| \le \frac{\alpha^2x^2}{16} + \frac{\alpha x}{6}.$$

4. Fekete-Szegö Inequality for the Class $\mathfrak{H}\mathfrak{E}^*(\alpha)$

One of the most prominent problem affiliated to coefficient estimates of univalent functions is Fekete-Szegö inequality. First investigated in [21], it states that for univalent functions $\mathcal{F}$, the inequality

$$|a_3 - \eta a_2^2| \leq 1 + 2e^{-2\eta/(1-\mu)}$$

is sharp when $\eta \in \mathbb{R}$.

Within this section and Section 5, the sharp bounds of Fekete-Szegö functional for the class $\mathfrak{H}\mathfrak{E}^*(\alpha)$ and $\mathfrak{H}\mathfrak{E}_{\mathfrak{K}}(\alpha)$ are to be evaluated.

Theorem 4.1. *Suppose that $\mathcal{F} \in \Sigma$ belongs to the class $\mathfrak{H}\mathfrak{E}^*(\alpha)$, then*

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{|\alpha| x}{2}, & |1-\eta| \leq |\frac{4\alpha x^2 - 2x^2 + 1}{6\alpha x^2}| \\ \frac{3\alpha^2 x^3 (1-\eta)}{4\alpha x^2 - 2x^2 + 1}, & |1-\eta| \geq |\frac{4\alpha x^2 - 2x^2 + 1}{6\alpha x^2}| \end{cases}.$$

Proof. Let $\mathcal{F} \in \mathfrak{H}\mathfrak{E}^*(\alpha)$. Using (15) and (18), for some $\eta \in \mathbb{R}$,

$$\begin{aligned} a_3 - \eta a_2^2 &= a_2^2 + \frac{C_1^\alpha(x)}{8}(d_2 - \mathfrak{d}_2) - \eta a_2^2 \\ &= \frac{2\alpha x}{8}(d_2 - \mathfrak{d}_2) + (1-\eta)\left(\frac{3\alpha^2 x^3}{8\alpha x^2 - 4x^2 + 2}\right)(d_2 + \mathfrak{d}_2) \\ &= \alpha x \left\{ \left[\frac{1}{4} + \frac{3\alpha x^2(1-\eta)}{8\alpha x^2 - 4x^2 + 2}\right] d_2 + \left[\frac{3\alpha x^2(1-\eta)}{8\alpha x^2 - 4x^2 + 2} - \frac{1}{4}\right] \mathfrak{d}_2 \right\} \\ &= \alpha x \left\{ \left[\mathfrak{h}_1(\eta) + \frac{1}{4}\right] d_2 + \left[\mathfrak{h}_1(\eta) - \frac{1}{4}\right] \mathfrak{d}_2 \right\} \end{aligned}$$

where $\mathfrak{h}_1(\eta) = \frac{3\alpha x^2(1-\eta)}{8\alpha x^2 - 4x^2 + 2}$.

Using triangle inequality and considering (16), we are able to conclude that

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{|\alpha| x}{2}, & |\mathfrak{h}_1(\eta)| \leq \frac{1}{4} \\ 2|\alpha|\ x\ |\mathfrak{h}_1(\eta)|, & |\mathfrak{h}_1 \eta)| \geq \frac{1}{4} \end{cases}.$$

Hence,

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{|\alpha| x}{2}, & |1-\eta| \leq |\frac{4\alpha x^2 - 2x^2 + 1}{6\alpha x^2}| \\ \frac{3\alpha^2 x^3 (1-\eta)}{4\alpha x^2 - 2x^2 + 1}, & |1-\eta| \geq |\frac{4\alpha x^2 - 2x^2 + 1}{6\alpha x^2}| \end{cases}.$$

Corollary 4.1. *Suppose that $\mathcal{F} \in \Sigma$ belongs to the class $\mathfrak{H}\mathfrak{E}^*(\alpha)$, then*

$$|a_3 - a_2^2| \leq \frac{|\alpha| x}{2}.$$

Proof. Take $\eta = 1$ in Theorem 4.1.

5. Fekete-Szegö Inequality for the Class $\mathfrak{H}\mathfrak{E}_{\mathfrak{K}}(\alpha)$

Theorem 5.1. *Suppose that $\mathcal{F} \in \Sigma$ belongs to the class $\mathfrak{H}\mathfrak{E}_{\mathfrak{K}}(\alpha)$, then*

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{|\alpha| x}{6}, & |1-\eta| \leq |\frac{5\alpha x^2 - 4x^2 + 2}{9\alpha x^2}| \\ \frac{3\alpha^2 x^3 (1-\eta)}{10\alpha x^2 - 8x^2 + 4}, & |1-\eta| \geq |\frac{5\alpha x^2 - 4x^2 + 2}{9\alpha x^2}| \end{cases}.$$

Proof. Let $\mathcal{F} \in \mathfrak{H}\mathfrak{E}_{\mathfrak{K}}(\alpha)$. Using (30) and (32), for some $\eta \in \mathbb{R}$,

$$\begin{aligned} a_3 - \eta a_2^2 &= a_2^2 + \frac{C_1^{\alpha}(x)}{24}(d_2 - \mathfrak{d}_2) - \eta a_2^2 \\ &= \frac{\alpha x}{12}(d_2 - \mathfrak{d}_2) + (1-\eta)\left(\frac{3\alpha^2 x^3}{20\alpha x^2 - 16x^2 + 8}\right)(d_2 + \mathfrak{d}_2) \\ &= \alpha x \left\{ \left[\frac{1}{12} + \frac{3\alpha x^2(1-\eta)}{20\alpha x^2 - 16x^2 + 8}\right] d_2 + \left[\frac{3\alpha x^2(1-\eta)}{20\alpha x^2 - 16x^2 + 8} - \frac{1}{12}\right] \mathfrak{d}_2 \right\} \\ &= \alpha x \left\{ \left[\mathfrak{h}_2(\eta) + \frac{1}{12}\right] d_2 + \left[\mathfrak{h}_2(\eta) - \frac{1}{12}\right] \mathfrak{d}_2 \right\} \end{aligned}$$

where $\mathfrak{h}_2(\eta) = \frac{3\alpha x^2(1-\eta)}{20\alpha x^2 - 16x^2 + 8}$.

Using triangle inequality and considering (16), we are able to conclude that

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{|\alpha| x}{6}, & |\mathfrak{h}_2(\eta)| \leq \frac{1}{12} \\ 2|\alpha|\ x\ |\mathfrak{h}_2(\eta)|, & |\mathfrak{h}_2(\eta)| \geq \frac{1}{12} \end{cases}.$$

Hence,

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{|\alpha| x}{6}, & |1-\eta| \leq |\frac{5\alpha x^2 - 4x^2 + 2}{9\alpha x^2}| \\ \frac{3\alpha^2 x^3 (1-\eta)}{10\alpha x^2 - 8x^2 + 4}, & |1-\eta| \geq |\frac{5\alpha x^2 - 4x^2 + 2}{9\alpha x^2}| \end{cases}.$$

Corollary 5.1. *Suppose that $\mathcal{F} \in \Sigma$ belongs to the class $\mathfrak{H}\mathfrak{E}_{\mathfrak{K}}(\alpha)$, then*

$$|a_3 - a_2^2| \leq \frac{|\alpha| x}{6}.$$

Proof. Take $\eta = 1$ in Theorem 5.1.

6. Conclusion

By convoluting Gegenbauer polynomials and Einstein functions, some new subclasses of Sakaguchi type bi-univalent functions are introduced in this paper. Some coefficient bounds are evaluated and the Fekete-Szegö inequalities are assessed for these subclasses.

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