EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 2, Article Number 6074
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

2-Step Movability of Hop Dominating Sets in Graphs

Roger L. Estrella[1,*], Gina M. Malacas[1,2], Sergio R. Canoy, Jr.[1,2]

[1] *Department of Mathematics and Statistics, College of Science and Mathematics, MSU-Iligan Institute of Technology, 9200 Iligan City, Philippines*
[2] *Center for Mathematical and Theoretical Physical Sciences- PRISM, MSU-Iligan Institute of Technology, 9200 Iligan City, Philippines*

Abstract. Let G be an undirected connected graph with vertex and edge sets $V(G)$ and $E(G)$, respectively. A hop dominating set S in G is 2-step movable hop dominating if for each $v \in S$, $S \setminus \{v\}$ or $[S \setminus \{v\}] \cup \{w\}$ for some $w \in [V(G) \setminus S] \cap N_G^2(v)$ is a hop dominating set in G. The minimum cardinality of a 2-step movable hop dominating set in G, denoted by $\gamma_{mh}^2(G)$, is called the 2-step movable hop domination number of G. In this paper, we characterize those graphs which admit a 2-step movable hop dominating set. We give bounds on the 2-step movable hop domination number and give necessary and sufficient conditions for those graphs that attain these bounds. We also characterize the 2-step movable hop dominating sets in the shadow graph and determine the 2-step movable hop domination numbers of the shadow graph and complementary prism.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: hop domination, 2-step movable hop dominating, 2-step movable hop domination number

1. Introduction

Movability of dominating sets was introduced and studied by Blair et al. in [1]. Apparently, this is a variation on dominating sets in which vertices in a dominating set are either removed or replaced. A motivation of this study can be seen, for example, in a network with sensors located at some nodes or vertices to serve their purpose (e.g. monitor activities in the network). It may happen that malfunctioning of some of these sensors occurs due to loss of battery supply or destruction by natural calamities. When such a case happens, a new nearby location for a sensor can be chosen appropriately so as to preserve the desired activity, connectivity, or security these sensors are purposely designed

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i2.6074

Email addresses: roger.estrella@g.msuiit.edu.ph (R. Estrella)
gina.malacas@g.msuiit.edu.ph (G. Malacas)
sergio.canoy@g.msuiit.edu.ph (S. Canoy)

in the network. Movability of different types of dominating sets had been considered in [2], [3], [4], [5], [6], and [7].

Hop domination, a kind of domination introduced by Natarajan et al. in [8], has also gained popularity and interest among the researchers in the field. Through the years, a great number of variants of hop domination have already been considered and studied (see, for example, [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], and [19]). Since hop domination and domination have similar applications in networks, it is also worthwhile to consider movability of hop dominating sets in graphs. In this paper, we introduce 2-step movability of hop dominating sets. Since these types of sets need not be present in some graphs, we characterize those graphs that admit such sets. We also give bounds on the 2-step movable hop domination number and determine the values of the parameter in some known graphs including the shadow graph and complementary prism.

2. Terminology and Notation

Let $G = V(G), E(G))$ be an undirected graph. For any two vertices u and v of G, the distance $d_G(u, v)$ is the length of a shortest path joining u and v. Any u-v path of length $d_G(u, v)$ is called a *u-v geodesic*. The interval $I_G[u, v]$ consists of u, v, and all vertices lying on a u-v geodesic. The interval $I_G(u, v) = I_G[u, v] \setminus \{u, v\}$. Vertices u and v are adjacent (or neighbors) if $uv \in E(G)$. The set of neighbors of a vertex u in G, denoted by $N_G(u)$, is called the *open neighborhood* of u. The *closed neighborhood* of u is the set $N_G[u] = N_G(u) \cup \{u\}$. If $X \subseteq V(G)$, the *open neighborhood* of X is the set $N_G(X) = \bigcup_{u\in X} N_G(u)$. The *closed neighborhood* of X is the set $N_G[X] = N_G(X) \cup X$.

A set $D \subseteq V(G)$ is a *dominating set* (resp. *total dominating set*) of G if for every $v \in V(G) \setminus D$ (resp. $v \in V(G)$), there exists $u \in D$ such that $uv \in E(G)$, that is, $N_G[D] = V(G)$ (resp. $N_G(D) = V(G)$). The *domination number* (resp. *total domination number*) of G, denoted by $\gamma(G)$ (resp. $\gamma_t(G)$), is the minimum cardinality of a dominating (resp. total dominating) set in G. Any dominating (resp. total dominating) set in G with cardinality $\gamma(G)$ (resp. $\gamma_t(G)$), is called a γ-set (resp. γ_t-set) in G. If $\gamma(G) = 1$ and $\{v\}$ is a dominating set in G, then we call v a *dominating vertex* in G.

A vertex v in G is a *hop neighbor* of vertex u in G if $d_G(u, v) = 2$. The set $N^2_G(u) = \{v \in V(G) : d_G(v, u) = 2\}$ is called the *open hop neighborhood* of u. The *closed hop neighborhood* of u is given by $N^2_G[u] = N^2_G(u) \cup \{u\}$. The *open hop neighborhood* of $X \subseteq V(G)$ is the set $N^2_G(X) = \bigcup_{u\in X} N^2_G(u)$. The *closed hop neighborhood* of X is the set $N^2_G[X] = N^2_G(X) \cup X$.

A set $S \subseteq V(G)$ is a *hop dominating set* in G if $N^2_G[S] = V(G)$, that is, for every $v \in V(G)\setminus S$, there exists $u \in S$ such that $d_G(u, v) = 2$. The minimum cardinality among all hop dominating sets in G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. Any hop dominating set with cardinality equal to $\gamma_h(G)$ is called a γ_h-set. A hop dominating set S is 2-*step movable hop dominating* if for each $v \in S$, $S \setminus \{v\}$ is hop dominating or there exists $w \in (V(G) \setminus S) \cap N^2_G(v)$ such that $(S \setminus \{v\}) \cup \{w\}$ is hop dominating in G. The minimum cardinality among all 2-step movable hop dominating sets in G, denoted by $\gamma^2_{mh}(G)$, is called the 2-*step movable hop domination number* of G. Any 2-step movable hop dominating set with cardinality equal to $\gamma^2_{mh}(G)$ is called a γ^2_{mh}-set.

The *shadow graph* $D_2(G)$ of graph G is constructed by taking two copies of G, say G_1 and G_2, and then joining each vertex $u \in V(G_1)$ to the neighbors of its corresponding vertex $u' \in V(G_2)$.

For a graph G, the *complementary prism* $G\overline{G}$, is the graph formed from the disjoint union of G and its complement $\overline{G}$ by adding a perfect matching between corresponding vertices of G and $\overline{G}$. For each $v \in V(G)$, let $\overline{v}$ denote the vertex in $\overline{G}$ corresponding to v. In simple terms, the graph $G\overline{G}$ is form from $G \cup \overline{G}$ by adding the edge $v\overline{v}$ for every vertex $v \in V(G)$.

3. Results

Our first result characterizes those connected graphs which admit a 2-step movable hop dominating set.

Theorem 1. *Let G be a connected graph. Then G admits a 2-step movable hop dominating set if and only if $\gamma(G) \neq 1$.*

Proof. Suppose G admits a 2-step movable hop dominating set, say S. Suppose $\gamma(G) = 1$, say v is a dominating vertex in G. Then $v \in S$ because S is a hop dominating set in G. Since every hop dominating set contains all dominating vertices of G where v is one of them, it follows that $S \setminus \{v\}$ is not a hop dominating set. Also, since $N^2_G(v) = \varnothing$, there exists no $w \in [V(G) \setminus S] \cap N^2_G(v)$ such that $[S \setminus \{v\}] \cup \{w\}$ is a hop dominating set in G. This implies that S is not a 2-step movable hop dominating set, contrary to our assumption. Thus, $\gamma(G) \neq 1$.

For the converse, suppose that $\gamma(G) \neq 1$. Let $x \in V(G)$. Since x is not a dominating vertex of G, there exists $y \in N^2_G(x)$. It follows that $V(G) \setminus \{x\}$ is a hop dominating set in G. This implies that $V(G)$ is a 2-step movable hop dominating set in G. □

Remark 1. *Let $G_1, G_2, \ldots, G_k$ be the components of a graph G. Then S is a hop dominating set in G if and only if $S_j = S \cap V(G_j)$ is a hop dominating set in G_j for each $j \in [k] = \{1, 2, \cdots, k\}$. Moreover, $\gamma_h(G) = \sum_{j=1}^{k} \gamma_h(G_j)$.*

Theorem 2. *Let $G_1, G_2, \ldots, G_k$ be the components of G. Then G admits a 2-step movable hop dominating set if and only if $\gamma(G_j) \neq 1$ for every $j \in [k] = \{1, 2, \cdots, k\}$. In this case, $\gamma^2_{mh}(G) = \sum_{j=1}^{k} \gamma^2_{mh}(G_j)$.*

Proof. Suppose G admits a 2-step movable hop dominating set, say S. Let $S_j = S \cap V(G_j)$ for each $j \in [k]$. By Remark 1, each set S_j is a hop dominating set in G_j. For an arbitrary $j \in [k]$, let $x \in S_j$. Then $x \in S$. Since S a 2-step movable hop dominating set in G, $S \setminus \{x\}$ or $[S \setminus \{x\}] \cup \{y\}$ for some $y \in [V(G) \setminus S] \cap N^2_G(x)$, is a hop dominating set in

G. If $S \setminus \{x\}$ is a hop dominating set in G, then $S_j \setminus \{x\}$ is a hop dominating set in G_j by Remark 1. If $(S \setminus \{x\}) \cup \{y\}$ is a hop dominating set in G for some $y \in (V(G) \setminus S) \cap N_G^2(x)$, then $y \in (V(G_j) \setminus S_j) \cap N_{G_j}^2(x)$ and $(S_j \setminus \{x\}) \cup \{y\}$ is a hop dominating set in G_j. Hence, S_j is a 2-step movable hop dominating set in G_j. By Theorem 1, $\gamma(G_j) \neq 1$. Note that if, in particular, S is a γ_{mh}^2-set in G, then

$$\gamma_{mh}^2(G) = |S| = |\cup_{j \in [k]} S_j| \geq \sum_{j \in [k]} \gamma_{mh}^2(G_j).$$

Conversely, suppose $\gamma(G_j) \neq 1$ for each $j \in [k]$. Then each G_j admits a 2-step movable hop dominating set D_j by Theorem 1. Clearly, $D = \cup_{j \in [k]} D_j$ is a 2-step movable hop dominating set in G. Moreover, if D_j is a γ_{mh}^2-set in G_j for each $j \in [k]$, then we have

$$\gamma_{mh}^2(G) \leq |D| = |\cup_{j \in [k]} D_j| = \sum_{j \in [k]} \gamma_{mh}^2(G_j).$$

Therefore, the assertion holds. □

Corollary 1. *If G admits a 2-step movable hop dominating set, then $|V(G)| \geq 4$.*

Proof. Suppose G admits a 2-step movable hop dominating set. Let G' be a component of G. Then $\gamma(G') \neq 1$ by Theorem 2. It follows that $G' \notin \{K_1, K_2, K_3, P_3\}$. Thus, $4 \leq |V(G')| \leq |V(G)|$. □

Throughout this section, unless specified, it is assumed that every component of a graph does not have a dominating vertex, i.e., every graph admits a 2-step movable hop dominating set.

Remark 2. *Let G be any graph. Then $\gamma_h(G) \leq \gamma_{mh}^2(G)$. Moreover, for each positive integer n, there exists a connected graph G such that $\gamma_{mh}^2(G) - \gamma_h(G) = n$. In other words, the difference $\gamma_{mh}^2(G) - \gamma_h(G)$ can be made arbitrarily large.*

Note that for a graph that admits a 2-step movable hop dominating set, every 2-step movable hop dominating set is hop dominating. Thus, $\gamma_h(G) \leq \gamma_{mh}^2(G)$.

To see that the second part of Remark 2 holds, let n be a positive integer and consider the graph G in Figure 1 obtained from K_{n+2} by adding the edges ab and bx_1, where $V(K_{n+1}) = \{x_1, x_2, \cdots, x_{n+2}\}$. Clearly, $\{a, b\}$ is a γ_h-set in G. Hence, $\gamma_h(G) = 2$. Let S be a γ_{mh}^2-set in G. Suppose $b \notin S$. If $x_1 \in S$, then $S = \{x_1, x_2, \cdots, x_{n+2}\}$ is a γ_{mh}^2-set in G. If $x_1 \notin S$, then $S = \{a, x_2, \cdots, x_{n+2}\}$. Suppose $b \in S$. Suppose $|S \cap \{x_2, \cdots, x_{n+2}\}| \leq n-1$. We may assume $x_2, x_3 \notin S$. Since for each $j \in \{2, 3\}$ the set $(S \setminus \{b\}) \cup \{x_j\}$ is not hop dominating, it follows that S is not 2-step hop dominating, a contradiction.

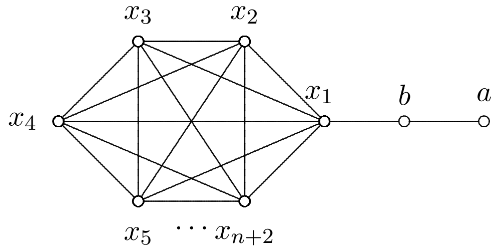

Figure 1: Graph G with $\gamma^2_{mh}(G) - \gamma_h(G) = n$

Thus, $|S \cap \{x_2, \cdots, x_{n+2}\}| = n$. Moreover, $|S \cap \{a, x_1\}| = 1$. Therefore, in any case, $\gamma^2_{mh}(G) = n + 2$. Accordingly, $\gamma^2_{mh}(G) - \gamma_h(G) = n$. □

Theorem 3. *Let G be any graph of order $n \geq 4$ and let S be a hop dominating set in G. Then S is a 2-step movable hop dominating set in G if and only if for each $v \in S$ such that $S \setminus \{v\}$ is not hop dominating, it holds that there exists $w \in (V(G) \setminus S) \cap N^2_G(v)$ such that $ephn(v; S) \subseteq N^2_G[w]$.*

Proof. Suppose S is a 2-step movable hop dominating set in G. Let $v \in S$ such that $S \setminus \{v\}$ is not hop dominating. Since S is 2-step movable hop dominating, there exists $w \in (V(G) \setminus S) \cap N^2_G(v)$ such that $S_w = (S \setminus \{v\}) \cup \{w\}$ is a hop dominating set in G. Now let $z \in ephn(v; S)$. Then $N^2_G(z) \cap S = \{v\}$. Since S_w is a hop dominating set in G, it follows that $z \in N^2_G[w]$. Thus, $ephn(v; S) \subseteq N^2_G[w]$.

For the converse, suppose that the given property holds. Let $v \in S$ such that $S \setminus \{v\}$ is not hop dominating. Then by assumption, there exists $w \in (V(G) \setminus S) \cap N^2_G(v)$ such that $ephn(v; S) \subseteq N^2_G[w]$. Let $S_w = (S \setminus \{v\}) \cup \{w\}$ and let $x \in V(G) \setminus S_w$. If $x = v$, then $x \in N^2_G(w)$. Suppose $x \neq v$. If $x \notin ephn(v; S)$, then there exists $u \in (S \setminus \{v\}) \cap N^2_G(x)$ since S is a hop dominating set in G. Hence, $x \in N^2_G(S_w)$. Next, suppose that $x \in ephn(v; S)$. Then $x \in N^2_G(w)$ because $ephn(v; S) \subseteq N^2_G[w]$ and $x \neq w$. Therefore, S_w is a hop dominating set in G. Since this is true for every $v \in S$ such that $S \setminus \{v\}$ is not hop dominating, it follows that S is a 2-step movable hop dominating set in G. □

Corollary 2. *Let G be a non-trivial graph and let S be a hop dominating set in G. If each $v \in S$ satisfies the property that $|ephn(v; S)| \leq 1$ or $|ephn(v; S)| \geq 2$ such that there exists $q \in ephn(v; S)$ with the property that $d_G(q, w) = 2$ for every $w \in ephn(v; S) \setminus \{q\}$, then S is 2-step movable hop dominating in G.*

Proof. Suppose S satisfies the given property. Let $v \in S$ and suppose $S \setminus \{v\}$ is not hop dominating. Since $\gamma(H) \neq 1$ for every component H of G, $|N^2_G(v)| \neq 0$. Suppose $N^2_G(v) \cap (V(G) \setminus S) = \varnothing$, i.e., $N^2_G(v) \subseteq S \setminus \{v\}$. This and the fact that S is hop dominating imply that $S \setminus \{v\}$ is hop dominating, a contradiction. Thus, $N^2_G(v) \cap (V(G) \setminus S) \neq \varnothing$. If $|ephn(v; S)| = 0$, then $ephn(v; S) \subseteq N^2_G[u]$ for each $u \in N^2_G(v) \cap (V(G) \setminus S)$. Suppose $|ephn(v; S)| = 1$, say $x_v \in ephn(v; S)$. Then $x_v \in (V(G) \setminus S) \cap N^2_G(v)$ and $ephn(v; S) = \{x_v\} \subseteq N^2_G[x_v]$. Finally, suppose $|ephn(v; S)| \geq 2$ such that there exists $q \in ephn(v : S)$ such that $d_G(q, w) = 2$ for every $w \in ephn(v; S) \setminus \{q\}$. Then $q \in (V(G) \setminus S) \cap N^2_G(v)$.

Moreover, since $d_G(q, w) = 2$ for every $w \in ephn(v; S) \setminus \{q\}$, it follows that $ephn(v; S) \subseteq N^2_G[q]$. Therefore, S is a 2-step movable hop dominating set in G by Theorem 3. □

Theorem 4. *Let G be a graph of order $n \geq 4$. Then $2 \leq \gamma^2_{mh}(G) \leq n - 2$. Moreover, each of the following holds:*

(i) $\gamma^2_{mh}(G) = 2$ if and only if there exist distinct vertices $p, q, v_p, v_q \in V(G)$ satisfying the following conditions:

(i_1) $N^2_G[\{p, q\}] = V(G)$, i.e., $\{p, q\}$ is hop dominating in G.

(i_2) $d_G(p, v_p) = d_G(q, v_q) = 2$, $V(G) \setminus (N^2_G[q] \cup \{p\}) \subseteq N^2_G[v_p]$, and $V(G) \setminus (N^2_G[p] \cup \{q\}) \subseteq N^2_G[v_q]$.

(ii) $\gamma^2_{mh}(G) = n - 2$ if and only if G satisfies the following conditions:

(j_1) there exist distinct vertices v and w of G such that $d_G(v, w) = 2$ whenever $|ephn(x; V(G) \setminus \{v, w\})| = 2$ for some $x \notin \{v, w\}$; and

(j_2) for each hop dominating set S in G with $|S| < n - 2$, $|ephn(p; S)| \geq 2$ for some $p \in S$ and there exists no $q \in ephn(p; S)$ such that $d_G(q, s) = 2$ for all $s \in ephn(p; S) \setminus \{q\}$.

Proof. Since $\gamma_h(G) \geq 2$, it follows from Remark 2 that $\gamma^2_{mh}(G) \geq 2$. Next, let $v \in V(G)$. Since $\gamma(G) \neq 1$, $|N^2_G(v)| \geq 1$. Let $z \in N^2_G(v)$ and let $w \in N_G(v) \cap N_G(w)$. Set $S = V(G) \setminus \{v, w\}$. Then S is a hop dominating set in G. Let $x \in S$. If $x \notin N^2_G(v) \cup N^2_G(w)$, then let $S_1 = S \setminus \{x\} = V(G) \setminus \{x, v, w\}$. Clearly, S_1 is hop dominating in G. Suppose $x \in N^2_G(v) \cup N^2_G(w)$. If $x \in N^2_G(w) \setminus N^2_G(v)$ or $x \in N^2_G(v) \cap N^2_G(w)$, then $x \neq z$ because $zw \in E(G)$. Then $(S \setminus \{x\}) \cup \{w\} = V(G) \setminus \{x, v\}$ is hop dominating in G because $x \in N^2_G(w)$ and $v \in N^2_G(z)$. If $x \in N^2_G(v) \setminus N^2_G(w)$, then $(S \setminus \{x\}) \cup \{v\} = V(G) \setminus \{x, w\}$ is a hop dominating in G. This implies that S is a 2-step movable hop dominating set in G. Therefore, $\gamma^2_{mh}(G) \leq |S| = n - 2$.

(i) Suppose $\gamma^2_{mh}(G) = 2$, say $S = \{p, q\}$ is a γ^2_{mh}-set of G. Then S is a hop dominating set and $\gamma_h(G) = 2$. It follows that $S \setminus \{p\}$ and $S \setminus \{q\}$ are not hop dominating sets. By Theorem 3, there exist vertices $v_p \in (V(G) \setminus S) \cap N^2_G(p)$ and $v_q \in (V(G) \setminus S) \cap N^2_G(q)$ such that $ephn(p; S) \subseteq N^2_G[v_p]$ and $ephn(q; S) \subseteq N^2_G[v_q]$. Now let $a \in ephn(p; S)$. Then $a \in V(G) \setminus \{p, q\}$ and $N^2_G(a) \cap \{p, q\} = \{p\}$. This shows that $a \in V(G) \setminus (N^2_G[q] \cup \{p\})$. It follows that $ephn(p; S) \subseteq V(G) \setminus (N^2_G[q] \cup \{p\})$. Next, let $b \in V(G) \setminus (N^2_G[q] \cup \{p\})$. Then $b \in V(G) \setminus N^2_G(q)$. Since S is hop dominating and $b \notin S$, it follows that $b \in N^2_G(p)$. Thus, $b \in ephn(p; S)$, showing that $ephn(p; S) = V(G) \setminus (N^2_G[q] \cup \{p\})$. Similarly, $ephn(q; S) = V(G) \setminus (N^2_G[p] \cup \{q\})$. It remains to show that $v_p \neq v_q$. To this end, suppose $v_p = v_q$. Let $[p, s, v_p]$ and $[q, t, v_q]$ be p-v_p and q-v_q geodesics, respectively. Since S is hop dominating, $t \neq s$. It follows that $t \in ephn(p; S) \setminus N^2_G[v_p]$, a contradiction. Therefore, p, q, v_p, and v_q are distinct vertices of G satisfying conditions (i_1) and (i_2).

For the converse, suppose there exist distinct vertices $p, q, v_p, v_q \in V(G)$ satisfying conditions (i_1) and (i_2). Set $D = \{p, q\}$ and let $D_p = (D \setminus \{p\}) \cup \{v_p\} = \{v_p, q\}$. Then D

is a hop dominating set by condition (i_1). Let $v \in V(G) \setminus D_p$. If $v = p$, then $d_G(v, v_p) = 2$ by assumption. So suppose $v \neq p$. If $v \in N_G^2[q] \cup \{p\}$, then $v \in N_G^2(q)$ because $v \neq q$. If $v \in V(G) \setminus (N_G^2[q] \cup \{p\})$, then $v \in N_G^2(v_p)$ by (i_2). This shows that D_p is a hop dominating set in G. Similarly, $D_q = \{v_q, p\}$ is a hop dominating set in G. Therefore, D is a 2-step movable hop dominating set in G. By Remark 1, $\gamma_{mh}^2(G) = 2$.

(ii) Suppose $\gamma_{mh}^2(G) = n - 2$. Let $D = V(G) \setminus \{v, w\}$ be a γ_{mh}^2-set in G. Suppose there exists $z \in D$ such that $ephn(z; D) = \{v, w\}$. Suppose further that $d_G(v, w) \neq 2$. Then $D \setminus \{z\}$, $(D \setminus \{z\}) \cup \{v\}$, and $(D \setminus \{z\}) \cup \{w\}$ are not hop dominating sets. This implies that D is not a 2-step movable hop dominating set, contrary to our assumption. Therefore, (j_1) holds.

Let S be a hop dominating set in G with $|S| < n - 2$. Since $\gamma_h^1(G) = n - 2$, S is not 2-step movable hop dominating in G. Hence, there exists $p \in S$ such that none of $S \setminus \{p\}$ and the sets $(S \setminus \{p\}) \cup \{s\}$ for $s \in (V(G) \setminus S) \cap N_G^2(p)$ is a hop dominating set in G. By the contrapositive of Corollary 2, this implies that $ephn(p; S) \geq 2$ and there exists no $q \in ephn(p; S)$ such that $d_G(q, t) = 2$ for all $t \in ephn(p; S) \setminus \{q\}$. This shows that (j_2) holds.

For the converse, suppose G satisfies (j_1) and (j_2). Let $Q = V(G) \setminus \{v, w\}$. Then Q is a 2-step hop dominating set in G by (j_1) and Corollary 2. By (j_2), it follows that Q is a γ_{mh}^2-set in G. Thus, $\gamma_{mh}^2(G) = |Q| = n - 2$. □

Theorem 5. *Let n be any positive integer such that $n \geq 4$. Then*

$$\gamma_{mh}^2(P_n) = \begin{cases} \left\lfloor \dfrac{n}{3} \right\rfloor + 2, & \text{if } n \in \{5, 6, 7, 9, 10\} \\ 2t, & \text{if } n = 4t \ (t \geq 1) \\ & \text{or } n = 4t + 1 \ (t \geq 3) \\ & \text{or } n = 4t + 2 \ (t \geq 3) \\ 2t + 1, & \text{if } n = 4t + 3 \ (t \geq 2). \end{cases}$$

Proof. Let $P_n = [v_1, v_2, \ldots, v_n]$ and consider the following cases:

Case 1. $n = 4t$.

Clearly, if $n = 4$, then $\gamma_{mh}^2(P_4) = 2$. If $n = 8$, then $R = \{v_1, v_4, v_7, v_8\}$ is a γ_{mh}^2-set of P_8. Hence, $\gamma_{mh}^2(P_8) = 4$. Next, let $t \geq 3$. For each $j \in \{1, 2, \ldots, \lfloor \frac{t-1}{2} \rfloor$, set $S_j = \{v_{8j-1}, v_{8j}, v_{8j+1}, v_{8j+2}\}$. Let $S = \{v_1, v_2\} \cup \left(\bigcup_{j=1}^{\lfloor \frac{t-1}{2} \rfloor} S_j \right)$. Since S is hop dominating and $|ephn(v; S)| \leq 1$ for each $v \in S$, S is a 2-step movable hop dominating set in P_n by Corollary 2. Moreover, $|S| = 2 + \sum_{j=1}^{\lfloor \frac{t-1}{2} \rfloor} |S_j| = 2 + 4(\frac{t-1}{2}) = 2t$. It can be shown that if S' is a hop dominating set in P_n with $|S'| < |S|$, then there exists a vertex $w \in S'$ with $|ephn(w; S')| = 2$. This implies that S' is not a 2-step movable hop dominating set.

Therefore, $\gamma^2_{mh}(P_n) = |S| = 2t$.

Case 2. $n = 4t + 1$ or $n = 4t + 2$.

If $n = 5$, then $S = \{v_1, v_2, v_5\}$ is a γ^2_{mh}-set in P_5. Hence, $\gamma^2_{mh}(P_5) = 3$. If $n = 6$, then $S = \{v_1, v_2, v_5, v_6\}$ is a γ^2_{mh}-set in P_6. Thus, $\gamma^2_{mh}(P_6) = 4$. If $n = 9$, then $S = \{v_1, v_2, v_5, v_6, v_7\}$ is a γ^2_{mh}-set in P_9 and if $n = 10$, then $S = \{v_1, v_4, v_7, v_8, v_9\}$ is a γ^2_{mh}-set in P_{10}. Hence, $\gamma^2_{mh}(P_9) = \gamma^2_{mh}(P_{10}) = 5$. Next, let $t \geq 3$. Let $S_j = \{v_{4j+3}, v_{4j+4}\}$ for each $j \in \{1, \ldots, t-1\}$. Let $S = \{v_1, v_4\} \cup \bigcup_{j=1}^{t-1} S_j$. Since S is hop dominating and $|ephn(v; S)| \leq 1$ for each $v \in S$, S is a 2-step movable hop dominating set in P_n by Corollary 2. Again, it can be verified that every hop dominating set S' in P_n with $|S'| < |S|$ has a vertex $w \in S'$ with $|ephn(w; S')| = 2$ and so cannot be a 2-step movable hop dominating set in P_n. Therefore,

$$\gamma^2_{mh}(P_n) = |S| = 2 + \sum_{j=1}^{t-1} |S_j| = 2 + 2(t-1) = 2t.$$

Case 3: $n = 4t + 3$.

If $n = 7$, then $S = \{v_1, v_2, v_5, v_6\}$ is a γ^2_{mh}-set in P_7. Thus, $\gamma^2_{mh}(P_7) = 4$. Next, let $t \geq 2$ and $D_j = \{v_{4j+3}, v_{4j+4}\}$ for each $j \in \{1, 2, \ldots, t-1\}$. Let $D = \{v_1, v_4, v_{4t+3}\} \cup \bigcup_{j=1}^{t-1} S_j$. By Corollary 2, D is a 2-step movable hop dominating set in P_n because it is hop dominating and $|ephn(v; D| \leq 1$ for each $v \in D$. Moreover, D is a γ^2_{mh}-set; hence,

$$\gamma^2_{mh}(P_n) = |S| = 2 + \sum_{j=1}^{t-1} |S_j| = 3 + 2(t-1) = 2t + 1.$$

This proves the assertion. □

Theorem 6. *Let n be any positive integer such that $n \geq 4$. Then*

$$\gamma^2_{mh}(C_n) = \begin{cases} 2t, & \text{if } n = 4t,\ t \geq 1 \\ & \text{or } n = 4t+1,\ t \leq 3 \\ & \text{or } n = 4t+2,\ t \geq 1 \\ 2t-1, & \text{if } n = 4t+1,\ t \geq 4 \\ 2t+1, & \text{if } n = 4t+3,\ t \geq 1. \end{cases}$$

Proof. Let $C_n = [v_1, v_2, \ldots, v_n, v_1]$, where $n \geq 4$. Consider the following cases:

Case 1. $n = 4t$, where $t \geq 1$.

Consider the following subcases:

Subcase 1: t is odd.

If $n = 4$, then $S = \{v_1, v_2\}$ is a γ^2_{mh}-set in C_4. Hence, $\gamma^2_{mh}(C_4) = 2$. Next, suppose that $t \geq 3$. Let $S_j = \{v_{8j-1}, v_{8j}, v_{8j+1}, v_{8j+2}\}$ for $j = 1, \ldots, \frac{t-1}{2}$. Let $S = \{v_1, v_2\} \cup \left(\bigcup_{j=1}^{\frac{t-1}{2}} S_j\right)$. Then S is a 2-step movable hop dominating set in C_n. If S' is a hop dominating set with $|S'| \leq |S|$, then $\exists$ a vertex $v \in S'$ with $|\text{ephn}(v; S')| = 2$. Hence, S' is not a 2-step movable hop dominating set. Therefore, $\gamma^2_{mh}(C_n) = |S| = 2 + \sum_{j=1}^{\frac{t-1}{2}} |S_j| = 2 + 4(\frac{t-1}{2}) = 2t$.

Subcase 2: t is even.

Let $S_j = \{v_{8j-7}, v_{8j-6}, v_{8j-5}, v_{8j-4}\}$ for $j = 1, \ldots, \frac{t}{2}$. Let $S = \cup \left(\bigcup_{j=1}^{\frac{t}{2}} S_j\right)$. Then S is a γ^2_{mh}-set in C_n and $\gamma^2_{mh}(C_n) = |S| = (\sum_{j=1}^{\frac{t}{2}} |S_j|) = 4(\frac{t}{2}) = 2t$.

Case 2. $n = 4t + 1$, where $t \leq 3$.

Let $S_j = \{v_{4j-3}, v_{4j-2}\}$ for $j = 1, \ldots, t$. Let $S = \left(\bigcup_{j=1}^{t} S_j\right)$. Then S is a γ^2_{mh}-set in C_n. Therefore, $\gamma^2_{mh}(C_n) = |S| = \sum_{j=1}^{t} |S_j| = 2t$.

Case 3. $n = 4t + 1$, where $t \geq 4$.

Let $S_j = \{v_{4j+4}, v_{4j+5}\}$ for $j = 1, \ldots, t-3$. Let $S = \{v_1, v_2, v_5, v_{n-5}, v_{n-2}\} \cup \left(\bigcup_{j=1}^{t-3} S_j\right)$. Then S is a γ^2_{mh}-set in C_n and $\gamma^2_{mh}(C_n) = |S| = 4 + \sum_{j=1}^{t-3} |S_j| = 5 + 2(t-3) = 2t - 1$.

Case 4. $n = 4t + 2$.

If $n = 6$ and $n = 10$, then $\{v_1, v_4\}$ and $\{v_1, v_2, v_6, v_7\}$ are γ^2_{mh}-sets in C_4 and C_{10}, respetively. Hence, $\gamma^2_{mh}(C_6) = 2$ and $\gamma^2_{mh}(C_10) = 4$. Suppose $t \geq 3$. Let $S_j = \{v_{4j+4}, v_{4j+5}\}$ for $j = 1, \ldots, t-2$. Let $S = \{v_1, v_2, v_5, v_{n-2}\} \cup \left(\bigcup_{j=1}^{t-2} S_j\right)$. Then S is a γ^2_{mh}-set in C_n and $\gamma^2_{mh}(C_n) = |S| = 4 + \sum_{j=1}^{t-2} |S_j| = 4 + 2(t-2) = 2t$.

Case 5. $n = 4t + 3$.

Let $S_j = \{v_{4j+4}, v_{4j+5}\}$ for $j = 1, \ldots, t-1$. Let $S = \{v_1, v_2, v_5\} \cup \left(\bigcup_{j=1}^{t-1} S_j\right)$. Then S is a γ^2_{mh}-set in C_n and $\gamma^2_{mh}(C_n) = |S| = 4 + \sum_{j=1}^{t-1} |S_j| = 3 + 2(t-1) = 2t + 1$. □

If G_1 and G_2 are the copies of graph G in the definition of the shadow graph $D_2(G)$ and if $S_{G_1} \subseteq V(G_1)$ and $S_{G_2} \subseteq V(G_2)$, then the sets S'_{G_1} and S'_{G_2} are the sets given by

$$S'_{G_1} = \{a' \in V(G_2) : a \in S_{G_1}\} \text{ and } S'_{G_2} = \{a \in V(G_1) : a' \in S_{G_2}\}.$$

The next result is found in [20].

Theorem 7. *Let G be a non-trivial connected graph. Then S is a hop dominating set in $D_2(G)$ if and only if one of the following conditions holds:*

(*i*) *S is a hop dominating set in G_1.*

(*ii*) *S is a hop dominating set in G_2.*

(*iii*) *$S = S_{G_1} \cup S_{G_2}$ such that $S_{G_1} \cup S'_{G_2}$ and $S'_{G_1} \cup S_{G_2}$ are hop dominating sets in G_1 and G_2, respectively.*

Theorem 8. *Let G be a non-trivial connected graph. Then $D_2(G)$ admits a 2-step movable hop dominating set. Moreover, a set $S \subseteq V(D_2(G))$ is 2-step movable hop dominating in $D_2(G)$ if and only if one of the following conditions holds:*

(*i*) *S is a 2-step movable hop dominating set in G_1.*

(*ii*) *S is a 2-step movable hop dominating set in G_2.*

(*iii*) *$S = S_{G_1} \cup S_{G_2}$ such that $S_{G_1} \cup S'_{G_2}$ and $S'_{G_1} \cup S_{G_2}$ are 2-step movable hop dominating sets in G_1 and G_2, respectively.*

Proof. Since G is non-trivial and connected, it follows that $D_2(G)$ is connected and $\gamma(D_2(G)) \neq 1$. Thus, $D_2(G)$ admits a 2-step movable hop dominating set by Theorem 2. Let S be a 2-step movable hop dominating set in $D_2(G)$. Set $S_{G_1} = S \cap V(G_1)$ and $S_{G_2} = S \cap V(G_2)$. If $S_{G_2} = \varnothing$, then $S = S_{G_1}$ is a hop dominating set in G_1 by Theorem 7(i). Let $v \in S_{G_1}$. Suppose $S_{G_1} \setminus \{v\}$ is not hop dominating in $D_2(G)$. Since S is a 2-step movable hop dominating set in $D_2(G)$, there exists $w \in V(D_2(G)) \setminus S) \cap N^2_{D_2(G)}(v)$ such that $(S \setminus \{v\}) \cup \{w\}$ is hop dominating in $D_2(G)$. If $w \in V(G_1)$, then $(S \setminus \{v\}) \cup \{w\} = (S_{G_1} \setminus \{v\}) \cup \{w\}$ is hop dominating in G_1 by Theorem 7(i). Suppose $w = u' \in V(G_2)$. Then $u \in V(G_1)$ and $d_{D_2(G)}(v, u') = d_{D_2(G)}(v, u) = d_{G_1}(v, u) = 2$. Since $S_{G_1} \setminus \{v\}$ is not hop dominating in $D_2(G)$, $u \in (V(G_1) \setminus \{v\}) \cap N^2_{G_1}(v)$. Also, since $(S_{G_1} \setminus \{v\}) \cup \{u'\}$ is hop dominating in $D_2(G)$, $(S_{G_1} \setminus \{v\}) \cup \{u\}$ is hop dominating in G_1 by Theorem 7(iii). Therefore, $S = S_{G_1}$ is 2-step movable hop dominating in G_1. Similarly, $S = S_{G_2}$ is 2-step movable hop dominating in G_2 whenever $S_{G_1} = \varnothing$. Finally, suppose $S_{G_1} \neq \varnothing$ and $S_{G_2} \neq \varnothing$. By Theorem 7(iii), $Q = S_{G_1} \cup S'_{G_2}$ and $R = S'_{G_1} \cup S_{G_2}$ are hop dominating sets in G_1 and G_2, respectively. Let $p \in Q$ such that $Q \setminus \{p\} = (S_{G_1} \setminus \{p\}) \cup S'_{G_2}$ is not hop dominating in G_1. Then $S \setminus \{p\}$ is not hop dominating in $D_2(G)$ by Theorem 7. Since S is 2-step movable hop dominating in $D_2(G)$, there exists $q \in (V(D_2(G) \setminus S) \cap N^2_{D_2(G)}(p)$ such that $(S \setminus \{p\}) \cup \{q\}$ is hop dominating in $D_2(G)$. If $q \in V(G_1)$, then $q \in (V((G_1) \setminus Q) \cap N^2_{G_1}(p)$ and $(S \setminus \{p\}) \cup \{q\} = [S_{G_1} \setminus \{p\}) \cup \{q\}] \cup S_{G_2}$. Hence, $[S_{G_1} \setminus \{p\}) \cup \{q\}] \cup S'_{G_2} = (Q \setminus \{p\}) \cup \{q\}$ is hop dominating in G_1 by Theorem 7(iii). Suppose $q = s' \in V(G_2)$. By assumption, $s \in (V(G_1) \setminus Q) \cap N^2_{G_1}(p)$ and $(S \setminus \{p\}) \cup \{q\} = (S_{G_1} \setminus \{p\}) \cup (S_{G_2} \cup \{s'\}$. It follows from Theorem 7(iii) that $(Q \setminus \{p\}) \cup \{s\} = [S_{G_1} \setminus \{p\}) \cup \{s\}] \cup S'_{G_2}$ is hop dominating in

G_1. Thus, Q is 2-step movable hop dominating in G_1. Similarly, R is 2-step movable hop dominating in G_2. Therefore, (i) or (ii) or (iii) holds.

For the converse, suppose (i) holds. Then S is hop dominating in $D_2(G)$ by Theorem 7. Let $x \in S$ such that $S \setminus \{x\}$ is not hop dominating in $D_2(G)$. By assumption, there exists $y \in (V(G_1) \setminus S) \cap N^2_{G_1}(x)$ such that $(S \setminus \{x\}) \cup \{y\}$ is hop dominating in G_1. By Theorem 7(i), $(S\setminus\{x\})\cup\{y\}$ is hop dominating in $D_2(G)$. Note that since $d_{G_1}(x, y) = d_{D_2(G)}(x, y) = 2$, $y \in (V(D_2(G)) \setminus S) \cap N^2_{D_2(G)}(x)$. Thus, S is 2-step movable hop dominating in $D_2(G)$. The same conclusion holds if (ii) holds. Finally, suppose S satisfies (iii). Let $v \in S$ such that $S \setminus \{v\}$ is not hop dominating in $D_2(G)$. Suppose, without loss of generality, that $v \in S_{G_1}$. Then $v \in (S_{G_1} \cup S'_{G_2})$. Suppose $Q = (S_{G_1} \cup S'_{G_2}) \setminus \{v\} = (S_{G_1} \setminus \{v\}) \cup S'_{G_2}$ is hop dominating in G_1. Let $p \notin [(S_{G_1} \setminus \{v\}) \cup S_{G_2}]$. Suppose $p \in V(G_1)$. If $p' \in S_{G_2}$, then $p' \in [(S_{G_1} \setminus \{v\}) \cup S_{G_2}]$ and $d_{D_2(G)}(p, p') = 2$. If $p' \notin S_{G_2}$, then $p \notin Q$. Since Q is hop dominating in G_1, there exists $q \in Q$ such that $d_{G_1}(p, q) = d_{D_2(G)}(p, q) = 2$. If $q \in S_{G_1} \setminus \{v\}$, then $q \in [(S_{G_1} \setminus \{v\}) \cup S_{G_2}]$. Suppose $q \notin S_{G_1} \setminus \{v\}$. Then $q \in S'_{G_2}$. Hence, $q' \in S_{G_2} \subseteq [(S_{G_1} \setminus \{v\}) \cup S_{G_2}]$ and $d_{D_2(G)}(p, q') = d_{D_2(G)}(p, q) = 2$. This implies that $S \setminus \{v\} = [(S_{G_1} \setminus \{v\}) \cup S_{G_2}]$ is hop dominating, a contradiction. Now, since $S_{G_1} \cup S'_{G_2}$ is 2-step movable hop dominating in G_1, there exists $t \in [V(G_1) \setminus (S_{G_1} \cup S'_{G_2})] \cap N^2_{G_1}(v)$ such that $[(S_{G_1} \cup S'_{G_2}) \setminus \{v\}] \cup \{t\}$ is hop dominating in G_1. By Theorem 7, it follows that $[(S_{G_1} \setminus \{v\}) \cup \{t\}] \cup S'_{G_2}$ is hop dominating in $D_2(G)$. Again, this will imply that $[(S_{G_1} \setminus \{v\}) \cup \{t\}] \cup S_{G_2}$ is hop dominating in $D_2(G)$, showing that S is a 2-step movable hop dominating set in $D_2(G)$. □

The next result is a direct consequence of Theorem 8.

Corollary 3. *Let G be a non-trivial connected graph. Then $\gamma^2_{mh}(D_2(G)) = \gamma^2_{mh}(G)$.*

Proof. Let S be a γ^2_{mh}-set in $D_2(G)$. If $S \subseteq V(G_1)$ or $S \subseteq V(G_2)$, then S is a 2-step movable hop dominating set in G_1 or in G_2, respectively, by (i) and (ii) of Theorem 8. Hence, $\gamma^2_{mh}(D_2(G)) = |S| \geq \gamma^2_{mh}(G)$. If $S = S_{G_1} \cup S_{G_2}$, then $S_{G_1} \cup S'_{G_2}$ and $S'_{G_1} \cup S_{G_2}$ are 2-step movable hop dominating sets in G_1 and G_2, respectively, by Theorem 8(iii). Thus, $\gamma^2_{mh}(D_2(G)) = |S_{G_1} \cup S_{G_2}| = |S_{G_1} \cup S'_{G_2}| \geq \gamma^2_{mh}(G)$.

Next, suppose Q is a γ^2_{mh}-set in $G = G_1$. Then Q is 2-step movable hop dominating set in $D_2(G)$ by Theorem 8. Hence, $\gamma^2_{mh}(D_2(G)) \leq |Q| = \gamma^2_{mh}(G)$. This establishes the desired equality.

Theorem 9. *Let G be a non-trivial connected graph. Then $G\overline{G}$ admits a 2-step movable hop dominating set and $2 \leq \gamma^2_{mh}(G\overline{G}) \leq 4$. Moreover, each of the following statements hold:*

(i) *$\gamma^2_{mh}(G\overline{G}) = 2$ if and only if $\gamma_h(G) = 2$ and $\gamma_h(\overline{G}) = 2$.*

(ii) *$\gamma^2_{mh}(G\overline{G}) = 3$ if and only if one of the following conditions holds.*

- *(a)* *$\gamma_h(G) = 2$ and $\gamma_h(\overline{G}) \geq 3$ or $\gamma_h(\overline{G}) = 2$ and $\gamma_h(G) \geq 3$.*
- *(b)* *$\gamma_h(G) = 3$ and $\gamma_h(\overline{G}) \geq 3$ (or $\gamma_h(G) \geq 3$ and $\gamma_h(\overline{G}) = 3$).*

(*c*) *There exist vertices* $x, y, z, w \in V(G)$ *such* $z, w \in N^2_G[x] \cup N^2_G[y[$, $z \in N_G[w]$, *and* $V(G) \setminus [N_G(z) \cup N_G(w)] \neq \varnothing$.

(*d*) *There exist vertices* $p, q, t, s \in V(G)$ *such that* $s \in N^2_G(t)$ *and* $\bar{t}, \overline{s} \in N^2_{\overline{G}}[\{\overline{p}, \overline{q}\}]$.

(*iii*) $\gamma^2_{mh}(G\overline{G}) = 4$ *if and only if* G *does not satisfy any of the properties in* (*i*) *and* (*ii*).

Proof. Since G is non-trivial, $\gamma(G\overline{G}) \neq 1$. Hence, $G\overline{G}$ admits a 2-step movable hop dominating set by Theorem 1. Note that $\{v, \overline{v}\}$ is a hop dominating set in $G\overline{G}$ for each $v \in V(G)$. Thus, $\{v, w, \overline{v}, \overline{w}\}$ is a 2-step movable hop dominating set in $G\overline{G}$ for each pair of distinct vertices v and w of G. Therefore, $2 = \gamma_h(G\overline{G}) \leq \gamma^2_{mh}(G\overline{G}) \leq 4$.

(*i*) Suppose $\gamma^2_{mh}(G\overline{G}) = 2$, say $S = \{p, q\}$ is a γ^2_{mh}-set in $G\overline{G}$. If $p, q \in V(G)$, then S is a hop dominating set of G. If $p, q \in V(\overline{G})$, then S is a hop dominating set of $\overline{G}$. Thus, $\gamma_h(G) = 2$ or $\gamma_h(\overline{G}) = 2$. Suppose $p \in V(G)$ and $q = \bar{t} \in V(\overline{G})$. Suppose $p \neq t$. If $pt \in E(G)$, then $\overline{p}\ \bar{t} \notin E(\overline{G})$. It follows that $t \in N_{G\overline{G}}(p) \cap N_{G\overline{G}}(q)$. If $pt \notin E(G)$, then $\overline{p}\ \bar{t} \in E(\overline{G})$. This implies that $\overline{p} \in N_{G\overline{G}}(p) \cap N_{G\overline{G}}(q)$. Thus, S is not a hop dominating set in $D_2(G)$, a contradiction. Therefore, $p = t$, i.e., $S = \{p, \overline{p}\}$. Now, since S is 2-step movable hop dominating in $D_2(G\overline{G})$, there exists $s \in [V(G\overline{G}) \setminus S] \cap N^2_{G\overline{G}}(\overline{p})$ such that $(S \setminus \{\overline{p}\}) \cup \{s\} = \{p, s\}$ is hop dominating in $G\overline{G}$. If $s \in V(\overline{G})$, then $s = \overline{p}$, a contradiction. Thus, $s \in V(G)$ and $\{p, s\}$ is a hop dominating set in G. This implies that $\gamma_h(G) = 2$. Similarly, $\gamma_h(\overline{G}) = 2$.

For the converse, suppose $\gamma_h(G) = 2$ and $\gamma_h(\overline{G}) = 2$. Let $D = \{x, y\}$ be a γ_h-set in G. Note that whether $xy \in E(G)$ or $xy \notin E(G)$, we find that $(D \setminus \{x\}) \cup \{\overline{y}\} = \{y, \overline{y}\}$ and $(D \setminus \{y\}) \cup \{\overline{x}\} = \{x, \overline{x}\}$ are hop dominating sets. This implies that D is a 2-step movable hop dominating set in $G\overline{G}$. Therefore, $\gamma^2_{mh}(G\overline{G}) = 2$.

(*ii*) Suppose $\gamma^2_{mh}(G\overline{G}) = 3$. If $\gamma_h(G) = 2$ ($\gamma_h(\overline{G}) = 2$), then $\gamma_h(\overline{G}) \geq 3$ (resp. $\gamma_h(G) \geq 3$) by (*i*). Hence, (*a*) holds. Next, suppose $\gamma_h(G) \geq 3$ and $\gamma_h(\overline{G}) \geq 3$. If $\gamma_h(G) = 3$ or $\gamma_h(\overline{G}) = 3$, then (*b*) holds. Suppose $\gamma_h(G) > 3$ and $\gamma_h(\overline{G}) > 3$. Let $S = \{x, y, t\}$ be a γ^2_{mh}-set in $G\overline{G}$. By assumption, $S \cap V(G) \neq \varnothing$ and $S \cap V(\overline{G}) \neq \varnothing$. Suppose $x, y \in V(G)$ and $t = \overline{z} \in V(\overline{G})$. Since $z \in N_{G\overline{G}}(\overline{z})$ and S is hop dominating in $G\overline{G}$, it follows that $z \in N^2_G(x) \cup N^2_G(y)$. Also, since $\gamma_h(G) > 3$ and S is 2-step movable hop dominating in $G\overline{G}$, it follows that $(S \setminus \{\overline{z}\}) \cup \{\overline{w}\} = \{x, y, \overline{w}\}$ is hop dominating in $G\overline{G}$ for some $\overline{w} \in [V(\overline{G}) \setminus \{\overline{z}\}] \cap N^2_{\overline{G}}(\overline{z})$. This implies that $w \in N^2_G(x) \cup N^2_G(y)$, $z \in N_G(w)$, and $V(G) \setminus [N_G(z) \cup N_G(w)] \neq \varnothing$. This shows that (*c*) holds. Suppose $x, y \in V(\overline{G})$ and $t \in V(G)$. Let $x = \overline{p}$ and $x = \overline{q}$. Since $t\bar{t} \in E(G\overline{G})$, $\bar{t} \in N^2_{\overline{G}}[\{\overline{p}, \overline{q}\}]$ because S is hop dominating in $G\overline{G}$. Since $\gamma_h(\overline{G}) > 3$ and S is 2-step movable hop dominating in $G\overline{G}$, there exists $s \in N^2_G(t)$ such that $(S \setminus \{t\}) \cup \{s\} = \{\overline{p}, \overline{q}, s\}$ is hop dominating in $G\overline{G}$. This implies that $\overline{s} \in N^2_{\overline{G}}[\{\overline{p}, \overline{q}\}]$. Thus, (*d*) holds.

For the converse, suppose (*a*) holds. Let $Q = \{c, d\}$ be a γ_h-set of G and let $Q^* = \{c, d, \overline{d}\}$. Since $Q^* \setminus \{\overline{d}\} = \{c, d\}$, $Q^* \setminus \{c\} = \{d, \overline{d}\}$, and $(Q^* \setminus \{\overline{d}\}) \cup \{\overline{c}\} = \{c, \overline{c}\}$ are hop dominating sets in $G\overline{G}$, it follows that $Q^* = \{c, d, \overline{d}\}$ is a γ^2_{mh}-set in $G\overline{G}$. Hence, $\gamma^2_{mh}(G\overline{G}) = 3$. The same conclusion holds if $\gamma_h(G) \geq 3$ and $\gamma_h(\overline{G}) = 2$. Suppose

(b) holds, i.e., $\gamma_h(G) = 3$ and $\gamma_h(\overline{G}) \geq 3$. Let $S = \{x, y, z\}$ be a γ_h-set in G. Since $(S \setminus \{x\}) \cup \{\overline{y}\} = \{y, \overline{y}, z\}$, $(S \setminus \{y\}) \cup \{\overline{x}\} = \{x, \overline{x}, z\}$, and $(S \setminus \{z\}) \cup \{\overline{y}\} = \{x, y, \overline{y}\}$ are hop dominating sets in $G\overline{G}$, it follows that D is a 2-step movable hop dominating set in $G\overline{G}$. Therefore, $\gamma^2_{mh}(G\overline{G}) = 3$. This conclusion also holds if $\gamma_h(G) \geq 3$ and $\gamma_h(\overline{G}) = 3$. Next, suppose (c) holds. Let $D = \{x, y, \overline{z}\}$. Since $V(\overline{G}) \cup \{z\} \subseteq N^2_{G\overline{G}}(\{x, y\})$ and $v \in N^2_{G\overline{G}}(\overline{z})$ for all $v \in V(G) \setminus \{z, x, y\}$ it follows that D is a hop dominating set in $G\overline{G}$. If $z \in \{x, y\}$, say $z = y$, then $D \setminus \{x\} = \{y, \overline{z}\}$ is hop dominating in $G\overline{G}$. Suppose $z \notin \{x, y\}$. Then $(D \setminus \{x\}) \cup \{\overline{y}\} = \{y, \overline{y}, \overline{z}\}$, $(D \setminus \{y\}) \cup \{\overline{x}\} = \{x, \overline{x}, \overline{z}\}$, and $(D \setminus \{\overline{z}\}) \cup \{\overline{w}\} = \{x, y, \overline{w}\}$ are hop dominating in $G\overline{G}$. Therefore, D is a γ^2_{mh}-set in $G\overline{G}$. Hence, $\gamma^2_{mh}(G\overline{G}) = 3$. Finally, suppose (d) holds. Let $Q = \{\overline{p}, \overline{q}, t\}$. Cleary, Q is a hop dominating set in $G\overline{G}$. Suppose $t \in \{p, q\}$, say $t = q$. Then $Q \setminus \{\overline{p}\} = \{t, \overline{q}$ is hop dominating in $G\overline{G}$. Suppose $t \notin \{p, q\}$. Then $(Q \setminus \{\overline{p}\}) \cup \{q\} = \{q, \overline{q}, t\}$, $(Q \setminus \{\overline{q}\}) \cup \{p\} = \{p, \overline{p}, t\}$, and $(Q \setminus \{t\}) \cup \{s\} = \{\overline{p}, \overline{q}, s\}$ are hop dominating sets in $G\overline{G}$. This shows that Q is a γ^2_{mh}-set in $G\overline{G}$. Thus, $\gamma^2_{mh}(G\overline{G}) = 3$.

(iii) This follows from (i) and (ii). □

The next result follows from Theorem 9.

Corollary 4. *Let n be a positive integer and $n \geq 2$. Then*

$$\gamma^2_{mh}(K_n\overline{K}_n) = \begin{cases} 2, & \text{if } n = 2 \\ 3, & \text{if } n = 3 \\ 4, & \text{if } n \geq 4. \end{cases}$$

4. Conclusion

The concepts of 2-step movability of hop dominating sets as well as the parameter 2-step movable hop domination number have been introduced in this paper. Graphs that admit a 2-step movable hop dominating set were characterized. Bounds on the 2-step movable hop domination number were given and graphs that attained these bounds were characterized. It was shown that the difference of the 2-step movable hop domination number and the hop domination can be made arbitrarily large. The 2-step movability of hop dominating sets in the shadow graph and complementary prism were also considered. For interested readers, this newly defined invariant may be studied further for trees and graphs under binary operations and even for its complexity aspects.

Acknowledgements

The authors would like to thank the referees for the invaluable assistance they gave us through their comments and suggestions which led to the improvement of the paper. The authors are also grateful to the Department of Science and Technology - Accelerated Science and Technology Human Resource Development Program (DOST-ASTHRDP)-Philippines and MSU-Iligan Institute of Technology for funding this research.

References

[1] J. Blair, R. Gera, and S. Horton. Movable dominating sensor sets in networks. *Journal of Combinatorial Mathematics and Combinatorial Computing.*, 77:103–123, 2011.

[2] L. Harutyunyan. 1−movable dominating set in wireless sensor networks. *2015 8th IFIP Wireless and Mobile Networking Conference (WMNC), Munich, Germany*, pages 269–276, 2015.

[3] R. Hinampas Jr and S. Canoy Jr. 1−movable domination in graphs. *Applied Mathematical Sciences*, 8(172):8565–8571, 2014.

[4] R. Hinampas Jr and S. Canoy Jr. 1-movable independent domination in graphs. *International Journal of Mathematical Analysis*, 9(2):73–80, 2015.

[5] J. Lomarda and S. Canoy Jr. 1−movable total dominating sets in graphs. *International Journal of Mathematical Analysis*, 8(55):2703–2709, 2014.

[6] J. Lomarda and S. Canoy Jr. 1−movable connected dominating sets in graphs. *Applied Mathematical Sciences*, 9(11):507–514, 2015.

[7] J. Lomarda and S. Canoy Jr. 1−movable total dominating, connected dominating, and double dominating sets in the composition of graphs. *International Journal of Mathematical Analysis*, 9(41):2037–2044, 2015.

[8] C. Natarajan and S. Ayyaswamy. Hop domination in graphs ii. *Versita*, 23(2):187–199, 2015.

[9] S. Ayyaswamy, B. Krishnakumari, B. Natarjan, and Y. Venkatakrishnan. Bounds on the hop domination number of a tree. *Proceedings-Mathematical Sciences.*, 125(4):449–455, 2015.

[10] S. Ayyaswamy, C. Natarajan, and G. Sathiamoorphy. A note on hop domination number of some special families of graphs. *International Journal of Pure and Applied Mathematics.*, 119(12):11465–14171, 2018.

[11] J. Hassan, S. Canoy Jr., and A. Aradais. Hop independent sets in graphs. *Eur. J. Pure Appl. Math.*, 15(2):467–477, 2022.

[12] J. Hassan and S. Canoy Jr. Hop independent hop domination in graphs. *Eur. J. Pure Appl. Math.*, 15(4):1783–1796, 2022.

[13] M. Henning and N. Rad. On 2-step and hop dominating sets in graphs. *Graphs and Combinatorics.*, 33(4):913–927, 2017.

[14] S. Canoy Jr., R. Mollejon, and J. G. Canoy. Hop dominating sets in graphs under binary operations. *Eur. J. Pure Appl. Math.*, 12(4):1455–1463, 2019.

[15] S. Canoy Jr. and G. Salasalan. Revisiting domination, hop domination, and global hop domination in graphs. *Eur. J. Pure Appl. Math.*, 14:1415–1428, 2021.

[16] S. Canoy Jr. and G. Salasalan. Locating-hop domination in graphs. *Kyungpook Mathematical Journal.*, 62:193–204, 2022.

[17] Y. Pabilona and H. Rara. Connected hop domination in graphs under some binary operations. *Asian-Eur. J. Math.*, 11(5):1850075–1–1850075–11, 2018.

[18] R. Rakim, H. Rara, and C.J. Saromines. Perfect hop domination in graphs. *Applied Mathematical Sciences*, 12(13):635–649, 2018.

[19] G. Salasalan and S. Canoy Jr. Global hop domination numbers of graphs. *Eur. J.*

Pure Appl. Math., 14(1):112–125, 2021.

[20] S. Canoy Jr. J. Hassan and C.J. Saromines. Convex hop domination in graphs. *Eur. J. Pure Appl. Math.*, 16(1):319–335, 2023.