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Secure Hop Dominating Sets in Graphs

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Abstract. Let G be an undirected (simple) graph with vertex and edge sets $V(G)$ and $E(G)$, respectively. A hop dominating set S in G is secure hop dominating if for each $v \in V(G) \setminus S$, there exists $w \in S \cap N_G^2(v)$ such that $(S \setminus \{w\}) \cup \{v\}$ is hop dominating in G. The minimum cardinality of a secure hop dominating in G, denoted by $\gamma_{sh}(G)$, is called the secure hop domination number of G. In this paper, we show that the difference $\gamma_{sh}(G) - \gamma_h(G)$ can be made arbitrarily large, where $\gamma_h(G)$ is the hop domination number of G. We give bounds on the secure hop domination number and characterize those graphs which attain these bounds. The value of the newly defined parameter is determined for some classes of graphs. Moreover, we characterize the secure hop dominating sets in the shadow graph and complementary prism and determine the value of the parameter for each of these graphs.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: Hop domination, secure hop domination number, shadow graph, complementary prism

1. Introduction

In 2003, Cockayne et al. [1] introduced and studied secure domination, a variant of the standard domination concept. As used to model a protection strategy in a given network, a secure dominating set may be viewed as one consisting of guards that protect the network from possible attacks. It is ensured that a guard can respond to a certain attack in some nearby vertex and as the guard moves to this location to defend the attack, the protection or security of the whole network is not compromised. The concept and some of its variants have been considered and studied in [2], [3], [4], [5], [6], [7], [8], [9], and [10].

Another domination-related concept was introduced by Natarajan et al. in [11]. This parameter, called hop domination parameter, and some of its variants have been the

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subject of interest in a number of recent studies (see [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], and [23]). In this paper, we introduce and study initially the new variant secure hop domination. The motivation stems from the fact that domination and hop domination have many similar applications in networks. It is easily observed from its definition that every graph admits a secure hop dominating set; in fact, the vertex set of a graph is such a set. We give bounds on the parameter and give necessary and sufficient conditions for a hop dominating set to be secure hop dominating. We also study the newly defined parameter in the shadow graph and complementary prism.

2. Terminology and Notation

Let $G = V(G), E(G))$ be an undirected graph. For any two vertices u and v of G, the distance $d_G(u, v)$ is the length of a shortest path joining u and v. Any u-v path of length $d_G(u, v)$ is called a *u-v geodesic*. The interval $I_G[u, v]$ consists of u, v, and all vertices lying on a u-v geodesic. The interval $I_G(u, v) = I_G[u, v] \setminus \{u, v\}$. Vertices u and v are adjacent (or neighbors) if $uv \in E(G)$. The set of neighbors of a vertex u in G, denoted by $N_G(u)$, is called the *open neighborhood* of u. The *closed neighborhood* of u is the set $N_G[u] = N_G(u) \cup \{u\}$. If $X \subseteq V(G)$, the *open neighborhood* of X is the set $N_G(X) = \bigcup_{u\in X} N_G(u)$. The *closed neighborhood* of X is the set $N_G[X] = N_G(X) \cup X$.

A set $D \subseteq V(G)$ is a *dominating set* in G if for every $v \in V(G) \setminus D$, there exists $u \in D$ such that $uv \in E(G)$, that is, $N_G[D] = V(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in G. Any dominating set in G with cardinality $\gamma(G)$, is called a γ-set in G. If $\gamma(G) = 1$ and $\{v\}$ is a dominating set in G, then we call v a *dominating vertex* in G. A dominating set $D \subseteq V(G)$ is *secure dominating* in G if for every $v \in V(G) \setminus D$, there exists $w \in D \cap N_G(v)$ such that $(D \setminus \{w\}) \cup \{v\}$ is a dominating set in G.

A vertex v in G is a *hop neighbor* of vertex u in G if $d_G(u, v) = 2$. The set $N^2_G(u) = \{v \in V(G) : d_G(v, u) = 2\}$ is called the *open hop neighborhood* of u. The *closed hop neighborhood* of u is given by $N^2_G[u] = N^2_G(u) \cup \{u\}$. The *open hop neighborhood* of $X \subseteq V(G)$ is the set $N^2_G(X) = \bigcup_{u\in X} N^2_G(u)$. The *closed hop neighborhood* of X is the set $N^2_G[X] = N^2_G(X) \cup X$. If $S \subseteq V(G)$ and $v \in S$, then a vertex $w \in V(G) \setminus S$ is an *external private hop neighbor* of v if $N^2_G(w) \cap S = \{v\}$. The set containing all the external private hop neighbors of v with respect to S is denoted by $ephn(v; S)$.

A set $S \subseteq V(G)$ is a *hop dominating set* in G if $N^2_G[S] = V(G)$, that is, for every $v \in V(G)\setminus S$, there exists $u \in S$ such that $d_G(u, v) = 2$. The minimum cardinality among all hop dominating sets in G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. Any hop dominating set with cardinality equal to $\gamma_h(G)$ is called a γ_h-set.

A hop dominating set S is *secure hop dominating* if for each $v \in V(G) \setminus S$, there exists $w \in S \cap N^2_G(v)$ such that $(S \setminus \{w\}) \cup \{v\}$ is a hop dominating set in G. The minimum cardinality among all secure hop dominating sets of G, denoted by $\gamma_{sh}(G)$, is called the *secure hop domination number* of G. Any secure hop dominating set with cardinality equal to $\gamma_{sh}(G)$ is called a γ_{sh}-set.

The complement of graph G, denoted by $\overline{G}$, is the graph with $V(\overline{G}) = V(G)$ such that $vw \in E(\overline{G})$ if and only if $vw \notin E(G)$. The *shadow graph* $D_2(G)$ of graph G is constructed by taking two copies of G, say G_1 and G_2, and then joining each vertex $u \in V(G_1)$ to the neighbors of its corresponding vertex $u' \in V(G_2)$. For a graph G, the complementary prism $G\overline{G}$ is the graph formed from the disjoint union of G and its complement $\overline{G}$ by adding a perfect matching between corresponding vertices of G and $\overline{G}$. In simple terms, the graph $G\overline{G}$ is formed from $G \cup \overline{G}$ by adding the edge $v\overline{v}$ for every vertex $v \in V(G)$, where $\overline{v}$ is the vertex of $\overline{G}$ corresponding to vertex v of G.

For other graph theoretic terms not mentioned here, readers may refer to [24] and [25].

3. Results

Given a graph G, the vertex set $V(G)$ is a secure hop dominating set of G. Thus, every graph admits a secure hop dominating set.

Remark 1. *Let $G_1, G_2, \ldots, G_k$ be the components of a graph G. Then S is a hop dominating set in G if and only if $S_j = S \cap V(G_j)$ is a hop dominating set in G_j for each $j \in [k] = \{1, 2, \cdots, k\}$. Moreover, $\gamma_h(G) = \sum_{j=1}^{k} \gamma_h(G_j)$.*

Theorem 1. *Let $G_1, G_2, \ldots, G_k$ be the components of G. Then $\gamma_{sh}(G) = \sum_{j=1}^{k} \gamma_{sh}(G_j)$.*

Proof. Suppose S is a secure hop dominating set in G. Then, by Remark 1, $S = \cup_{j\in[k]} S_j$ and $S_j = S \cap V(G_j)$ is a hop dominating set in G_j for each $j \in [k]$. For $j \in [k]$, let $x \in V(G_j) \setminus S_j$. Then $x \in V(G) \setminus S$. Since S is a secure hop dominating set in G, there exists $y \in S \cap N_G^2(x)$ such that

$$(S \setminus \{y\}) \cup \{x\} = [(S_j \setminus \{y\}) \cup \{x\}] \cup [\cup_{i\in[k]\setminus\{j\}} S_i]$$

is a hop dominating dominating set in G. Thus, $(S_j \setminus \{y\}) \cup \{x\}$ is a hop dominating dominating set in G_j. Since j was arbitrarily chosen, it follows that S_j is a secure hop dominating set in G_j for each $j \in [k]$. Therefore,

$$\gamma_{sh}(G) = |S| = \sum_{j=1}^{k} |S_j| \geq \sum_{j=1}^{k} \gamma_{sh}(G_j).$$

Next, suppose that D_j is a γ_{sh}-set in G_j for each $j \in [k]$. Since each D_j is a hop dominating set of G_j, $D = \cup_{j\in[k]} D_j$ is a hop dominating set in G by Remark 1. Let $v \in V(G) \setminus D$. Then $v \in V(G_t) \setminus D_t$ for a unique $t \in [k]$. Since D_t is a secure hop dominating set in G_t, there exists $w \in D_t \cap N_{G_t}^2(v)$ such that $(D_t \setminus \{w\}) \cup \{v\}$ is a hop dominating set in G_t. By Remark 1,

$$(D \setminus \{w\}) \cup \{v\} = [(D_t \setminus \{w\}) \cup \{v\}] \cup [\cup_{i\in[k]\setminus\{t\}} D_i]$$

is a hop dominating dominating set in G. Hence, D is a secure hop dominating set in G and

$$\gamma_{sh}(G) \leq |D| = \sum_{j=1}^{k} |D_j| = \sum_{j=1}^{k} \gamma_{sh}(G_j).$$

Therefore, the assertion holds. □

Theorem 2. *Let G be any graph. Then $\gamma_h(G) \leq \gamma_{sh}(G)$. Moreover, for each positive integer n, there exists a connected graph G such that $\gamma_{sh}(G) - \gamma_h(G) = n$. In particular, the difference $\gamma_{sh}(G) - \gamma_h(G)$ can be made arbitrarily large.*

Since every secure hop dominating set is hop dominating, it follows that $\gamma_h(G) \leq \gamma_{sh}(G)$.

Next, let n be a positive integer. Consider the graph G in Figure 1 obtained from the complete graph K_{n+2}, where $V(K_{n+2}) = \{z_1, z_2, \cdots, z_{n+2}\}$, by adding the edges vw and wz_1. The set $\{v, w\}$ is a γ_h-set in G. Thus, $\gamma_h(G) = 2$. Let D be a γ_{sh}-set in G. If $w \notin D$, then $D = \{v, z_2, \cdots, z_{n+2}\}$ or $D = \{z_1, z_2, \cdots, z_{n+2}\}$ because D is a hop dominating set. Hence, $|D| = n + 2$. Suppose $w \in D$ and let $z_j \in V(G) \setminus D$ for some $j \in \{2, 3, \cdots, n + 2\}$. Since D is secure hop dominating, $(D \setminus \{w\}) \cup \{z_j\}$ is hop dominating. Note that $N_G^2(z_j) \cap [\{z_2, z_3, \cdots, z_{n+2}\} \setminus \{z_j\}] = \varnothing$. This implies that $\{z_2, z_3, \cdots, z_{n+2}\} \setminus \{z_j\} \subseteq D$. Hence, $D = \{v, w\} \cup [\{z_2, \cdots, z_{n+2}\} \setminus \{z_j\}]$ or $D = \{w\} \cup [\{z_1, z_2, \cdots, z_{n+2}\} \setminus \{z_j\}]$. It follows that $|D| = n + 2$. Therefore, $\gamma_{sh}(G) = n + 2$ and $\gamma_{sh}(G) - \gamma_h(G) = n$.

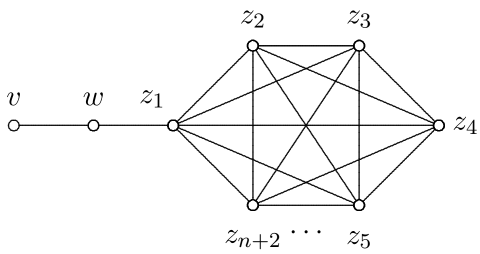

Figure 1: Graph G with $\gamma_{sh}(G) - \gamma_h(G) = n$

Theorem 3. *Let G be any graph and let S be a hop dominating set in G. Then S is a secure hop dominating set in G if and only if for each $v \in V(G) \setminus S$ there exists $w \in S \cap N_G^2(v)$ such that $ephn(w; S) \subseteq N_G^2[v]$.*

Proof. Suppose S is a secure hop dominating set in G. Let $v \in V(G) \setminus S$. Since S is secure hop dominating, there exists $w \in S \cap N_G^2(v)$ such that $S_v = (S \setminus \{w\}) \cup \{v\}$ is hop dominating. Let $z \in ephn(w; S)$. Then $N_G^2(z) \cap S = \{w\}$. Since S_v is a hop dominating set, it follows that $z \in N_G^2[v]$. Thus, $ephn(w; S) \subseteq N_G^2[v]$.

For the converse, suppose that the given property holds. Let $p \in V(G) \setminus S$. Then by assumption, there exists $q \in S \cap N_G^2(p)$ such that $ephn(q; S) \subseteq N_G^2[p]$. Let $S_p =$

$(S \setminus \{q\}) \cup \{p\}$ and let $x \in V(G) \setminus S$. If $x = q$, then $q \in N_G^2(p) \subseteq N_G^2[S_p]$. Suppose $x \neq q$. If $x \notin ephn(q; S)$, then there exists $y \in (S \setminus \{q\}) \cap N_G^2(x)$ since S is a hop dominating set in G. Hence, $x \in N_G^2(y) \subseteq N_G^2[S_p]$. Next, suppose that $x \in ephn(q; S)$. Then by assumption, $x \in N_G^2[p] \subseteq N_G^2[S_p]$. Therefore, S_p is a hop dominating set in G. Since p was arbitrarily chosen, it follows that S is a secure hop dominating in G. □

Corollary 1. *Let G be a non-trivial graph and let S be a hop dominating set in G. If for each $v \in V(G) \setminus S$ there exists $w \in S \cap N_G^2(v)$ with $|ephn(w; S)| = 0$ or $|ephn(w; S)| \geq 1$ such that $d_G(v, p) = 2$ for all $p \in ephn(w; S) \setminus \{v\}$, then S is a secure hop dominating set in G.*

Proof. Suppose S satisfies the given property. Let $v \in V(G) \setminus S$. By assumption, there exists $w \in S \cap N_G^2(v)$ satisfying the condition. If $|ephn(w; S)| = 0$, then $ephn(w; S) = \varnothing \subseteq N_G^2[v]$. Suppose $|ephn(w; S)| \geq 1$. Then $d_G(v, p) = 2$ for all $p \in ephn(w; S) \setminus \{v\}$ by assumption. Thus, $ephn(w; S) \subseteq N_G^2[v]$. Therefore, S is a secure hop dominating set by Theorem 3. □

Theorem 4. *$\gamma_{sh}(K_n) = \gamma_{sh}(\overline{K}_n) = n$ for every positive integer n.*

Proof. Let $G \in \{K_n, \overline{K}_n\}$. Since the only hop dominating set in G is $V(G)$, it follows that $V(G)$ is the only secure hop dominating. Therefore, $\gamma_{sh}(G) = n$.

Lemma 1. *Let G be a non-trivial graph and let $S = \{p, q\}$ be a hop dominating set in G. Then $ephn(p; S) = V(G) \setminus (N_G^2[q] \cup \{p\})$ and $ephn(q; S) = V(G) \setminus (N_G^2[p] \cup \{q\})$.*

Proof. Note that since S is hop dominating, $d_G(p, q) \neq 2$. Let $x \in ephn(p; S)$. Then $x \in V(G) \setminus S$ and $N_G^2(x) \cap S = \{p\}$. It follows that $x \in V(G) \setminus (N_G^2[q] \cup \{p\})$. Hence, $ephn(p; S) \subseteq V(G) \setminus (N_G^2[q] \cup \{p\})$. Now, let $z \in V(G) \setminus (N_G^2[q] \cup \{p\})$. Then $z \neq p$ and $z \notin N_G^2[q]$. Since S is hop dominating, it follows that $z \in N_G^2(p)$. This implies that $z \in ephn(p; S)$. Thus, $V(G) \setminus (N_G^2[q] \cup \{p\}) \subseteq ephn(p; S)$, showing the desired equality. Similarly, the second equality also holds. □

Theorem 5. *Let G be any graph of order n. Then $1 \leq \gamma_{sh}(G) \leq n$. Moreover, each of the following statements holds:*

(i) *$\gamma_{sh}(G) = 1$ if and only if $G = K_1$.*

(ii) *$\gamma_{sh}(G) = 2$ if and only if there exist two distinct vertices $v, w \in V(G)$ satisfying the following conditions:*

- *(p_1) $N_G^2[\{v, w\}] = V(G)$ and $N_G^2(v) \cap N_G^2(w) = \varnothing$.*
- *(p_2) For each $x \notin \{v, w\}$ such that $x \in N_G^2(v)$ (or $x \in N_G^2(w)$), it holds that $V(G) \setminus (N_G^2[w] \cup \{v\}) \subseteq N_G^2[x]$ (resp. $V(G) \setminus (N_G^2[v] \cup \{w\}) \subseteq N_G^2[x]$).*

(iii) *$\gamma_{sh}(G) = n$ if and only if every component of G is complete.*

Proof. Clearly, $1 \leq \gamma_{sh}(G) \leq n$.

(i) Suppose $\gamma_{sh}(G) = 1$, say $S = \{v\}$ is a γ_{sh}-set in G. Since S cannot be a hop dominating set if G is non-trivial, it follows that $G = K_1$. Conversely, if $G = K_1$, then $\gamma_{sh}(G) = 1$.

(ii) Suppose $\gamma_{sh}(G) = 2$. Let $D = \{v, w\}$ be a γ_{sh}-set of G. Since D is hop dominating, $V(G) = N_G^2[\{v, w\}]$. Suppose $p \in N_G^2(v) \cap N_G^2(w)$. Since D is a secure hop dominating set, we may assume that $S_p = (D \setminus \{v\}) \cup \{p\} = \{p, w\}$ is hop dominating (otherwise, $\{p, v\}$ is hop dominating). Let $q \in N_G(p) \cap N_G(w)$. Then $q \notin N_G^2[S_p]$. This implies that D is not hop dominating, a contradiction. Thus, (p_1) holds. Next, let $x \in V(G) \setminus D$. Assume without loss of generality that $x \in N_G^2(v)$ (hence, $x \notin N_G^2(w)$). Since D is a secure hop dominating set in G, it follows that $D_x = (D \setminus \{v\}) \cup \{x\} = \{w, x\}$ is a hop dominating set in G. Let $z \in ephn(v; D)$. Then $N_G^2(z) \cap D = \{v\}$. Since D_x is hop dominating, we must have $z \in N_G^2[x]$. Hence, $ephn(v; D) \subseteq N_G^2[x]$. By Lemma 1, (p_2) holds.

For the converse, suppose that there exist distinct vertices $v, w \in V(G)$ satisfying properties (p_1) and (p_2). Set $S = \{v, w\}$. Then S is a hop dominating set by (p_1). Let $x \in V(G) \setminus S$. By (p_1), $x \in N_G^2(v) \setminus N_G^2(w)$ or $x \in N_G^2(w) \setminus N_G^2(v)$. If $x \in N_G^2(v) \setminus N_G^2(w)$ ($x \in N_G^2(w) \setminus N_G^2(v)$), then $V(G) \setminus (N_G^2[w] \cup \{v\}) \subseteq N_G^2[x]$ (resp. $V(G) \setminus (N_G^2[v] \cup \{w\}) \subseteq N_G^2[x]$) by (p_2). By Lemma 1 and Theorem 3, S is a secure hop dominating set in G. Since G is non-trivial, $\gamma_{sh}(G) = |S| = 2$.

(iii) Suppose $\gamma_{sh}(G) = n$. Suppose there exists a component of H of G that is not complete. Then there exists $v \in V(H) \subseteq V(G)$ such that $N_H^2(v) = N_G^2(v) \neq \varnothing$, say $w \in N_H^2(v)$. Set $S = V(G) \setminus \{w\}$. Then clearly, S is a hop dominating set in G. Since $S_w = (S \setminus \{v\}) \cup \{w\} = V(G) \setminus \{v\}$ is also hop dominating, it follows that S is a secure hop dominating set. This implies that $\gamma_{sh}(G) \leq |S| = n - 1$, a contradiction. Therefore, every component of G is complete.

For the converse, suppose that every component of G is complete. Let $G_1, G_2, \ldots, G_k$ be the components of G. By assumption and Theorem 4, $\gamma_{sh}(G_j) = |V(G_j)|$ for each $j \in [k] = \{1, 2, \cdots, k\}$. Thus, $\gamma_{sh}(G) = \sum_{j=1}^{k} \gamma_{sh}(G_j) = n$ by Theorem 1. □

Lemma 2. *Let G be a graph of order $n \geq 3$ such that $\gamma(G) = 1$. If $|D(G)| \geq 2$, where $D(G) = \{v \in V(G) : N_G[v] = V(G)\}$, then $\gamma_{sh}(G) \geq 3$.*

Proof. If $G = K_n$, then $\gamma_{sh}(G) = n \geq 3$. So suppose $G \neq K_n$ and let S be a γ_{sh}-set in G. Since S is a hop dominating set, $D(G) \subseteq S$. Hence, if $|D(G)| \geq 3$, then $\gamma_{sh}(G) = |S| \geq 3$. Suppose $|D(G)| = 2$. Since $|N_G^2(v)| = 0$ for every $v \in D(G)$ and S is a hop dominating set, it follows that $S \neq D(G)$. This implies that $2 = |D(G)| < |S| = \gamma_{sh}(G)$. This proves the assertion. □

Theorem 6. *Let G be a non-trivial graph of order n such that $\gamma(G) = 1$. Then $\gamma_{sh}(G) = 2$ if and only if $G = K_{1,n-1}$, that is, G has a unique dominating vertex v and $|N_G(x)| = 1$ for $x \in V(G) \setminus \{v\}$.*

Proof. Suppose $\gamma_{sh}(G) = 2$. If $n = 2$, then $G = K_2 = K_{1,1}$. Suppose $n \geq 3$ and let S be a γ_{sh}-set in G. By the contrapositive of Lemma 2, $|D(G)| = 1$, where $D(G) = \{u \in V(G) : N_G[u] = V(G)\}$ that is, G has a unique dominating vertex, say v. This implies that $S = \{v, w\}$ for some $w \in V(G) \setminus \{v\}$. Note that since S is a hop dominating set, $|N_G(w)| = 1$. Suppose there exists $x \in V(G) \setminus S$ with $|N_G(x)| \geq 2$. Since S is a secure hop dominating set and v is a dominating vertex, it follows that $x \in N_G^2(w)$ and $S_x = (S \setminus \{w\}) \cup \{x\} = \{v, x\}$ is a hop dominating set in G. This, however, is not possible because a vertex $y \in N_G(x) \setminus \{v\}$ is not in $N_G^2[S_x]$. Therefore, $|N_G(x)| = 1$ for every $x \in V(G) \setminus \{v\}$. Accordingly, $G = K_{1,n-1}$.

For the converse, suppose that $G = K_{1,n-1}$. Then $\gamma_{sh}(G) \geq 2$ by Theorem 5(i). Let $v_0 \in V(G)$ be such that $|N_G(v_0)| = n - 1$ and let $q \in V(G) \setminus \{v_0\}$. Then v_0 and q satisfy the properties (p_1) and (p_2) of Theorem 5(ii). Therefore, $\gamma_{sh}(G) = 2$. □

Remark 2. *There are graphs G with $\gamma_{sh}(G) = 2$ such that $\gamma(G) \neq 1$.*

To see this, consider $G \in \{P_4, C_4, H\}$ in Figure 2. Clearly, $\gamma(G) = 2 \neq 1$. It can be verified easily that the blackened vertices form a γ_{sh}-set in G.

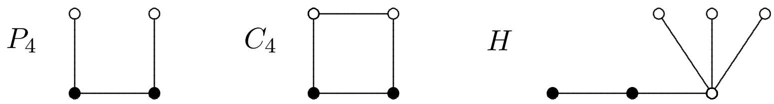

Figure 2: Graph G with $\gamma_{sh}(G) = 2$ and $\gamma(G) = 2$

The next result is a consequence of Theorem 1, Theorem 6, and Theorem 5(ii).

Corollary 2. *Let G be a graph of order n where $3 \leq n \leq 4$. If $\gamma_{sh}(G) = 2$, then $G \in \{P_3, P_4, C_4, K_{1,3}\}$.*

Proof. By Theorem 1 and Theorem 5(ii), none of the disconnected graphs G satisfies $\gamma_{sh}(G) = 2$. From Theorem 6, it follows that $\gamma_{sh}(P_3) = \gamma_{sh}(K_{1,3}) = 2$. Suppose $n = 4$. As seen in Remark 2, $\gamma_{sh}(P_4) = \gamma_{sh}(C_4) = 2$. If G is connected and $G \notin \{P_4, C_4, K_{1,3}\}$, then $\gamma(G) = 1$ and either G has more that two dominating vertices or contains a single dominating vertex and another vertex with two neighbors. Thus, $\gamma_{sh}(G) \geq 3$ by Theorem 6. Therefore, $G \in \{P_3, P_4, C_4, K_{1,3}\}$. □

Proposition 1. *Let n be any positive integer. Then*

$$\gamma_{sh}(P_n) = \begin{cases} n & \text{if } n \in \{1, 2\} \\ 2 & \text{if } n = 3 \\ 2t & \text{if } n = 4t, t \geq 1 \\ 2t+1 & \text{if } n = 4t+1, t \geq 1 \\ 2t+2 & \text{if } n = 4t+2, t \geq 1 \\ & \text{or } n = 4t+3, t \geq 1. \end{cases}$$

Proof. Let $P_n = [v_1, v_2, \dots, v_n]$. Then $\gamma_{sh}(P_n) = n$ for $n \in \{1, 2\}$ by Corollary 4, and $\gamma_{sh}(P_3) = 2$ by Theorem 6. Suppose $n \geq 4$ and consider the following cases:

Case 1. $n = 4t$.

Let $S_j = \{v_{4j-3}, v_{4j-2}\}$ for each $j \in \{1, 2, \dots, t\}$ and set $S = \cup_{j=1}^{t} S_j$. Then S is a secure hop dominating set and $|S| = \sum_{j=1}^{t} |S_j| = 2t$. Let S' be a hop dominating set such that $|S'| < |S|$. Then one can find a vertex $z \in V(P_n) \setminus S'$ such that for each $v \in S' \cap N^2_{P_n}(z)$, either $|ephn(v; S')| = 1$ and $v \notin ephn(v; s)$ or $|ephn(v; S')| = 2$. Hence, S' is not a secure hop dominating set. Thus, S is a γ_{sh}-set in P_n and $\gamma_{sh}(P_n) = |S| = 2t$.

Case 2. $n = 4t + 1 (t \geq 1)$.

Let $S_j = \{v_{4j}, v_{4j+1}\}$ for each $j \in \{1, 2, \dots, t\}$. Then $S = \{v_1\} \cup [\cup_{j=1}^{t} S_j]$ is a hop dominating set in P_n. Since $|ephn(v; S)| \leq 1$ for every $v \in S$, S is a secure hop dominating set by Corollary 1. Following an argument in the preceding case, any hop dominating set S' with $|S'| < |S|$ is not secure hop dominating. Therefore, $\gamma_{sh}(P_n) = |S| = 1 + \sum_{j=1}^{t} |S_j| = 2t + 1$.

Case 3. $n = 4t + 2$ or $4t + 3$.

Consider the set $S_j = \{v_{4j-3}, v_{4j-2}\}$ for each $j \in \{1, 2, \dots, t + 1\}$. Then $S = \cup_{j=1}^{t+1} S_j$ is a γ_{sh}-set in P_n. Therefore, $\gamma_{sh}(P_n) = |S| = \sum_{j=1}^{t+1} |S_j| = 2(t + 1) = 2t + 2$.

This proves the assertion. □

Proposition 2. *Let n be any positive integer such that $n \geq 3$. Then*

$$\gamma_{sh}(C_n) = \begin{cases} 3 & \text{if } n \in \{3, 5\} \\ 2t & \text{if } n = 4t, t \geq 1 \\ & \text{or } n = 4t + 1, t \geq 2 \\ & \text{or } n = 4t + 2, t \geq 1 \\ 2t + 1 & n = 4t + 3, t \geq 1. \end{cases}$$

Proof. Let $C_n = [v_1, v_2, \dots, v_n, v_1]$. If $n = 3$, then $\gamma_{sh}(C_3) = \gamma_{sh}(K_3) = 3$ by Corollary 4. If $n = 5$, then $D = \{v_1, v_2, v_4\}$ is a γ_{sh}- set in C_5. Hence, $\gamma_{sh}(C_5) = 3$. Now, suppose $n \notin \{3, 5\}$. Consider the following cases:

Case 1. $n = 4t$ or $n = 4t + 1$.

Let $S_j = \{v_{4j-3}, v_{4j-2}\}$ for each $j \in \{1, 2, \dots, t\}$. Then $S = \cup_{j=1}^{t} S_j$ is γ_{sh}-set in C_n. It follows that $\gamma_{sh}(C_n) = |S| = \sum_{j=1}^{t} |S_j| = 2t$.

Case 2. $n = 4t + 3$.

Set $S_j = \{v_{4j+3}, v_{4j+4}\}$, $j = 1, 2, \dots, t-1$. Then $S = \{v_1, v_2, v_n\} \cup [\cup_{j=1}^{t-1} S_j]$ is γ_{sh}-set in C_n. Hence, $\gamma_{sh}(C_n) = |S| = 3 + \sum_{j=1}^{t-1} |S_j| = 3 + 2(t-1) = 2t + 1$.

This proves the theorem. □

If G_1 and G_2 are the copies of graph G in the definition of the shadow graph $D_2(G)$ and if $S_{G_1} \subseteq V(G_1)$ and $S_{G_2} \subseteq V(G_2)$, then the sets S'_{G_1} and S'_{G_2} are the sets given by

$$S'_{G_1} = \{a' \in V(G_2) : a \in S_{G_1}\} \text{ and } S'_{G_2} = \{a \in V(G_1) : a' \in S_{G_2}\}.$$

The next result is obtained by Hassan et al. in [26].

Theorem 7. *Let G be a non-trivial connected graph. Then S is a hop dominating set in $D_2(G)$ if and only if one of the following conditions holds:*

(i) S is a hop dominating set in G_1.

(ii) S is a hop dominating set in G_2.

(iii) $S = S_{G_1} \cup S_{G_2}$ such that $S_{G_1} \cup S'_{G_2}$ and $S'_{G_1} \cup S_{G_2}$ are hop dominating sets in G_1 and G_2, respectively.

Theorem 8. *Let G be a non-trivial connected graph. Then a set $S \subseteq V(D_2(G))$ is secure hop dominating in $D_2(G)$ if and only if one of the following conditions holds:*

(i) S is a secure hop dominating set in G_1.

(ii) S is a secure hop dominating set in G_2.

(iii) $S = S_{G_1} \cup S_{G_2}$ such that $S_{G_1} \cup S'_{G_2}$ and $S'_{G_1} \cup S_{G_2}$ are secure hop dominating sets in G_1 and G_2, respectively.

Proof. Let S be a secure hop dominating set in $D_2(G)$. Set $S_{G_1} = S \cap V(G_1)$ and $S_{G_2} = S \cap V(G_2)$. If $S_{G_2} = \varnothing$, then $S = S_{G_1}$ is a hop dominating set in G_1 by Theorem 7(i). Let $x \in V(G_1) \setminus S_{G_1}$. Since S is a secure hop dominating set in $D_2(G)$, there exists $w \in S \cap N^2_{D_2(G)}(x)$ such that $(S \setminus \{w\}) \cup \{x\}$ is hop dominating in $D_2(G)$. Since $S_{G_2} = \varnothing$, it follows that $w \in S_{G_1}$. Thus, $(S \setminus \{w\}) \cup \{x\} = (S_{G_1} \setminus \{w\}) \cup \{x\}$ is hop dominating in G_1 by Theorem 7(i). Therefore, $S = S_{G_1}$ is secure hop dominating in G_1. Similarly, $S = S_{G_2}$ is secure hop dominating in G_2 whenever $S_{G_1} = \varnothing$. Finally, suppose $S_{G_1} \neq \varnothing$ and $S_{G_2} \neq \varnothing$. By Theorem 7(iii), $L = S_{G_1} \cup S'_{G_2}$ and $M = S'_{G_1} \cup S_{G_2}$ are hop dominating sets in G_1 and G_2, respectively. Let $p \in V(G_1) \setminus L$. Since S secure hop dominating in $D_2(G)$, there exists $q \in S \cap N^2_{D_2(G)}(p)$ such that $S_p = (S \setminus \{q\}) \cup \{p\}$ is hop dominating in $D_2(G)$. Suppose $q \in S_{G_1}$. Then $S_p = (S \setminus \{q\}) \cup \{p\} = [(S_{G_1} \setminus \{q\}) \cup \{p\}] \cup S_{G_2}$. Since S_p is hop dominating in $D_2(G)$, it follows from Theorem 7(iii) that

$$[(S_{G_1} \setminus \{q\}) \cup \{p\}] \cup S'_{G_2} = [(S_{G_1} \cup S'_{G_2}) \setminus \{q\}]) \cup \{p\}$$

is hop dominating in G_1 Theorem 7(iii). Suppose $q = t' \in S_{G_2}$. Then $t \in S'_{G_2} \cap N^2_{G_1}(p)$ and $S_p = (S_{G_1} \cup \{p\}) \cup [(S_{G_2} \setminus \{t'\}]$. Since S_p is hop dominating in $D_2(G)$,

$$(S_{G_1} \cup \{p\}) \cup [(S'_{G_2} \setminus \{t\}] = [(S_{G_1} \setminus \{t\}) \cup \{p\}] \cup S'_{G_2} = [(S_{G_1} \cup S'_{G_2}) \setminus \{t\}] \cup \{p\}$$

is hop dominating in G_1 by Theorem 7(iii). Thus, L is secure hop dominating in G_1. Similarly, M is secure hop dominating in G_2. Therefore, one of (i), (ii), and (iii) holds.

For the converse, suppose (i) holds. Then S is hop dominating in $D_2(G)$ by Theorem 7(i). Let $z \in V(D_2(G)) \setminus S$. Suppose $z \in V(G_1)$. Since S is secure hop dominating in G_1, there exists $q \in S \cap N^2_G(z)$ such that $(S \setminus \{q\}) \cup \{z\}$ is hop dominating in G_1. By Theorem 7, $(S \setminus \{q\}) \cup \{z\}$ is hop dominating in $D_2(G)$. Next suppose $z = t' \in V(G_2)$. If $t \in S$, then $d_{D_2(G)}(t, t') = 2$. If $a \in ephn(t; S)$, then $d_{D_2(G)}(a, t) = d_{D_2(G)}(a, t') = 2$. This implies that $ephn(t; S) \subseteq N^2_{D_2(G)}[t']$. Hence, $(S \setminus \{t\}) \cup \{t'\}$ is a hop dominating set in $D_2(G)$. Suppose $t \notin S$. Since S is a secure hop dominating set in G_1, there exists $s \in S \cap N^2_{G_1}(t)$ such that $S_t = (S \setminus \{s\}) \cup \{t\}$ is hop dominating in G_1. Thus, S_t is hop dominating in $D_2(G)$. Set $S_{t'} = (S \setminus \{s\}) \cup \{t'\}$ and let $p \in V(D_2(G)) \setminus S_{t'}$. Then $p \notin S \setminus \{s\}$ and $p \neq t'$. Suppose first that $p \in V(G_1)$. If $p \in \{s, t\}$, then $p \in N^2_{D_2(G)}(t')$. Suppose $p \notin \{s, t\}$. Since S_t is hop dominating, there exists $r \in (S_t \setminus \{s\}) \cap N^2_{G_1}(p)$. It follows that $r \in S_{t'} \cap N^2_{D_2(G)}(p)$. Suppose $p = b' \in V(G_2)$. By considering b and following the preceding arguments, it can be shown that there exists $d \in S_{t'} \cap N^2_{D_2(G)}(p)$. Hence, $S_{t'}$ is hop dominating in $D_2(G)$. Therefore, S is a secure hop dominating set in $D_2(G)$. The same conclusion holds if (ii) holds. Finally, suppose (iii) holds. Then, by Theorem 7(iii), $S = S_{G_1} \cup S_{G_2}$ is hop dominating. Let $x \in V(D_2(G)) \setminus S$. Then $x \notin S_{G_1} \cup S_{G_2}$. We may assume that $x \in V(G_1) \setminus S_{G_1}$. If $x' \in S_{G_2}$, then $x' \in S \cap N^2_{D_2(G)}(x)$ and $(S \setminus \{x'\}) \cup \{x\}$ is hop dominating in $D_2(G)$. Suppose $x' \notin S_{G_2}$. Then $x \notin S'_{G_2}$. This implies that $x \notin V(G_1) \setminus (S_{G_1} \cup S'_{G_2}$. Since $S_{G_1} \cup S'_{G_2}$ is secure hop dominating in G_1, there exists $y \in (S_{G_1} \cup S'_{G_2}) \cap N^2_{G_1}(x)$ such that

$$[(S_{G_1} \cup S'_{G_2}) \setminus \{y\}] \cup \{x\} = [(S_{G_1} \setminus \{y\}) \cup \{x\}] \cup S'_{G_2}$$

is hop dominating in G_1. By Theorem 7(iii), $[(S_{G_1} \setminus \{y\}) \cup \{x\}] \cup S_{G_2}$ is hop dominating in G_1. Thus, $(S \setminus \{y\}) \cup \{x\} = [(S_{G_1} \setminus \{y\}) \cup \{x\}] \cup S_{G_2}$ is hop dominating in $D_2(G)$. Therefore, S is secure hop dominating in $D_2(G)$. □

The next result is a direct consequence of Theorem 8.

Corollary 3. *Let G be a non-trivial connected graph. Then $\gamma_{sh}(D_2(G)) = \gamma_{sh}(G)$.*

Proof. Let S be a γ_{sh}-set of $G = G_1$. Then S is a secure hop dominating set of $D_2(G)$ by Theorem 8. Hence, $\gamma_{sh}(D_2(G)) \leq |S| = \gamma_{sh}(G)$.

Next, suppose S' is a γ_{sh}-set of $D_2(G)$. If $S' \subseteq V(G_1)$ or $S' \subseteq V(G_2)$, then S' is a secure hop dominating set of G by (i) and (ii) of Theorem 8. It follows that $\gamma_{sh}(G) \leq |S'| = \gamma_{sh}(D_2(G))$. If $S' = S_{G_1} \cup S_{G_2}$, then $S_{G_1} \cup S'_{G_2}$ is secure hop dominating in G by

Theorem 8(iii). Hence,

$$\gamma_{sh}(G) \le |S_{G_1} \cup S'_{G_2}| = |S_{G_1} \cup S_{G_2}| = |S'| = \gamma_{sh}(D_2(G)).$$

This establishes the desired equality. □

Lemma 3. *Let G be a graph. Then $S = \{x, \overline{y}\}$, where $x, y \in V(G)$, is a hop dominating set in $G\overline{G}$ if and only if $x = y$.*

Proof. Suppose S is a hop dominating set in $G\overline{G}$. Suppose $x \neq y$. If $xy \in E(G)$, then $y \notin N^2_{G\overline{G}}(S)$ because $y\overline{y} \in E(G\overline{G})$. If $xy \notin E(G)$, then $\overline{x}\ \overline{y} \in E(\overline{G})$. Thus, $\overline{x} \notin N^2_{G\overline{G}}(S)$. In both cases, we obtain a contradiction. Thus, $x = y$.

For the converse, suppose $x = y$. Then clearly, $S = \{x, \overline{x}\}$ is a hop dominating set in $G\overline{G}$. □

Theorem 9. *Let G be a graph. Then $2 \le \gamma_{sh}(G\overline{G}) \le 4$. Moreover, each of the following statements hold:*

(i) $\gamma_{sh}(G\overline{G}) = 2$ *if and only if* $G \in \{K_1, K_2, \overline{K}_2\}$.

(ii) $\gamma_{sh}(G\overline{G}) = 3$ *if and only if* $G \notin \{K_2, \overline{K}_2\}$ *and one of the following conditions holds:*

- (i_1) $\gamma_h(G) = 2$ *or* $\gamma_h(\overline{G}) = 2$.
- (i_2) *There exists a secure hop dominating set S of G with $|S| = 3$ such that $ephn(v; S) = 0$ for some $v \in S$ or a secure hop dominating set S of $\overline{G}$ with $|S| = 3$ such that $ephn(\overline{v}; S) = 0$ for some $\overline{v} \in S$.*
- (i_3) *There exist vertices $x, y, z, \in V(G)$ such $z \in N^2_G[\{x, y\}]$, and $d_G(v, w) = 2$ for all $v, w \in V(G) \setminus N^2_G(\{x, y\})$, where $v \neq w$.*
- (i_4) *There exist vertices $x, y, z \in V(G)$ such $\overline{z} \in N^2_{\overline{G}}[\{\overline{x}, \overline{y}\}]$, and $d_{\overline{G}}(\overline{v}, \overline{w}) = 2$ for all $\overline{v}, \overline{w} \in V(\overline{G}) \setminus N^2_{\overline{G}}[\{\overline{x}, \overline{y}\}]$, where $v \neq w$*

(iii) $\gamma_{sh}(G\overline{G}) = 4$ *if and only if G does not satisfy any of the properties in (i) and (ii).*

Proof. Since $G\overline{G}$ is non-trivial, it follows that $2 \le \gamma_{sh}(G\overline{G})$. Since $\{v, \overline{v}\}$ is a hop dominating set in $G\overline{G}$ for each $v \in V(G)$, $\{v, w, \overline{v}, \overline{w}\}$ is a secure hop dominating set in $G\overline{G}$ for each pair of distinct vertices v and w of G. Therefore, $2 \le \gamma_{sh}(G\overline{G}) \le 4$.

(i) Suppose $\gamma_{sh}(G\overline{G}) = 2$, say $S = \{p, q\}$ is a γ_{sh}-set in $G\overline{G}$. Suppose $G \notin \{K_1, K_2, \overline{K}_2\}$. Then $|V(G)| \ge 3$. Suppose $p, q \in V(G)$. Choose any $s \in V(G) \setminus S$. Then $\overline{s} \in V(G\overline{G}) \setminus S$. Since S is secure hop dominating in $G\overline{G}$, $\{p, \overline{s}\}$ or $\{q, \overline{s}\}$ is hop dominating in $G\overline{G}$. According to Lemma 3, this is impossible. Similarly, we arrived at a contradiction if $p, q \in V(\overline{G})$. Suppose now that $p \in V(G)$ and $q = \overline{c} \in V(\overline{G})$. By Lemma 3, $p = c$, i.e., $S = \{p, \overline{p}\}$. Let $z \in V(G) \setminus \{p\}$. Then, by Lemma 3, $(S \setminus \{\overline{p}\}) \cup \{z\} = \{p, z\}$ is a hop dominating set in G. Suppose G is disconnected. Pick $w \in V(H)$ where H is a component of G such that $p \notin V(H)$. Then $(S \setminus \{\overline{p}\}) \cup \{w\} = \{p, w\}$ is not a hop dominating set, a contradiction. Hence, G is connected. Since $G \notin \{K_1, K_2\}$, we may choose a vertex d such that $N_G(p) \cap N_G(d) \neq \varnothing$. This implies that $(S \setminus \{\overline{p}\}) \cup \{d\} = \{p, d\}$ is not a hop dominating set in G, a contradiction. Therefore, $G \in \{K_1, K_2, \overline{K}_2\}$.

For the converse, suppose first that $G = K_1$. Then $G\overline{G} = K_2$ and $\gamma_{sh}(G\overline{G}) = 2$. If $G \in \{K_2, \overline{K}_2\}$, then $G\overline{G} = P_4$. By Theorem 1, $\gamma_{sh}(G\overline{G}) = 2$.

(ii) Suppose $\gamma_{sh}(G\overline{G}) = 3$. Then $G \notin \{K_2, \overline{K}_2\}$ by (i). If $\gamma_h(G) = 2$ or $\gamma_h(\overline{G}) = 2$, then (i_1) is satisfied. Suppose $\gamma_h(G) > 2$ and $\gamma_h(\overline{G}) > 2$. Let $D = \{x, y, s\}$ be a γ_{sh}-set in $G\overline{G}$. We may assume that $D \subseteq V(G)$. Then D is a secure hop dominating set in G. Suppose $ephn(v; S) \neq 0$ for all $v \in D$. Let $u' \in V(\overline{G})$ where $u \notin \{x, y, s\}$. Since D is secure hop dominating in $G\overline{G}$, there exists $w \in D$, say $w = x$, such that $(D \setminus \{x\}) \cup \{\overline{u}\}$ is hop dominating in $G\overline{G}$. This, however, is not possible because $ephn(x; S) \neq 0$. Thus, there exists $v \in D$ such that $ephn(v; S) = 0$. This shows that (i_2) holds. Next, suppose that $x, y \in V(G)$ and $s = \overline{z} \in V(\overline{G})$. If $\{x, y\}$ is a hop dominating set in G, then $\gamma_h(G) = 2$ and we find that (i_1) holds. Suppose $\{x, y\}$ is not a hop dominating set in G. Since $z\overline{z} \in E(G\overline{G})$ and D is a hop dominating set in $G\overline{G}$, it follows that $z \in N_G^2(\{x, y\})$. Let $v \in V(G) \setminus N_G^2(\{x, y\})$. Since D is a secure hop dominating set in $G\overline{G}$, $D_v = (D \setminus \{\overline{z}\}) \cup \{v\}) = \{x, y, v\}$ is a hop dominating set in $G\overline{G}$. Let $w \in V(G) \setminus [N_G^2(\{x, y\}) \cup \{v\}]$. Since D_v is hop dominating in $G\overline{G}$, $w \in N_G^2(v)$. Hence, $d_G(v, w) = 2$ for any pair of distinct vertices $v, w \in V(G) \setminus N_G^2(\{x, y\})$. This shows that (i_3) holds. Similarly, (i_4) holds.

For the converse, suppose (i_1) holds. Note that since $G \notin \{K_1, K_2, \overline{K}_2\}$, it follows that $\gamma_{sh}(G\overline{G}) \geq 3$. Let $\{x, y\}$ be a γ_h-set of G and let $Q = \{x, y, \overline{x}\}$. Then clearly, Q is a hop dominating set in $G\overline{G}$. Let $z \in V(G\overline{G}) \setminus Q$. Suppose $z \in V(G)$. Since $\{x, y\}$ is hop dominating in G, $z \in N_G^2(\{x, y\})$. Both sets $(Q \setminus \{y\}) \cup \{z\}$ and $(Q \setminus \{x\}) \cup \{z\}$ are hop dominating in $G\overline{G}$. Suppose $z = \overline{s} \in V(\overline{G})$. If $s = y$, then $\overline{s} \in N_{G\overline{G}}^2(x)$ and $(Q \setminus \{x\}) \cup \{\overline{s}\} = \{y, \overline{s}, \overline{x}\}$ is hop dominating in $G\overline{G}$. Suppose $s \neq y$. Then $\overline{s} \in N_{G\overline{G}}^2(y)$ and $(Q \setminus \{y\}) \cup \{\overline{s}\} = \{x, \overline{s}, \overline{x}\}$ is hop dominating in $G\overline{G}$. Hence, Q is secure hop dominating in $G\overline{G}$ and $\gamma_{sh}(G\overline{G}) = |Q| = 3$. The same conclusion holds if $\{x, y\}$ is a γ_h-set in $\overline{G}$. Suppose now that (i_2) holds. We may assume that there exists a secure hop dominating set $S = \{a, b, c\}$ of G with $|S| = 3$ and $ephn(a; S) = 0$. Clearly, S is hop dominating in $G\overline{G}$. Moreover, because of the conditions that S is secure hop dominating in G and $ephn(a; S) = 0$, S is hop dominating in $G\overline{G}$. Hence, $\gamma_{sh}(G\overline{G}) = |S| = 3$. Suppose (i_3) holds, i.e., there exist vertices $x, y, z, \in V(G)$ such $z \in N_G^2[\{x, y\}]$, and $d_G(v, w) = 2$ for all $v, w \in V(G) \setminus N_G^2(\{x, y\})$, where $v \neq w$. Let $R = \{x, y, \overline{z}\}$. Then R is a hop dominating set in $G\overline{G}$. Let $u \in V(G\overline{G}) \setminus R$. Clearly, if $u = k' \in V(\overline{G})$, then $(R \setminus \{x\}) \cup \{u\}$ is hop dominating if $k \neq x$ and $(R \setminus \{y\}) \cup \{u\}$ is hop dominating if $k \neq y$. Suppose $u \in V(G)$. If $u \in N_G^2(\{x, y\})$, say $u \in N_G^2(x)$, then $(R \setminus \{x\}) \cup \{u\}$ is hop dominating in $G\overline{G}$. If $u \notin N_G^2(\{x, y\})$, then $(R \setminus \{\overline{z}\}) \cup \{u\}$ is hop dominating in $G\overline{G}$ because of the additional assumption that $d_G(v, w) = 2$ for all $v, w \in V(G) \setminus N_G^2(\{x, y\})$, where $v \neq w$. Hence, R is a secure hop dominating set in $G\overline{G}$ and $\gamma_{sh}(G\overline{G}) = |R| = 3$. The same conclusion holds if (i_4) is assumed.

(iii) This follows from (i) and (ii). □

The next result follows from Theorem 9.

Corollary 4. *Let n be a positive integer and $n \geq 2$. Then*

$$\gamma_{sh}(K_n\overline{K}_n) = \begin{cases} 2, & \text{if } n = 2 \\ 3, & \text{if } n = 3 \\ 4, & \text{if } n \geq 4. \end{cases}$$

4. Conclusion

Secure hop domination was introduced and initially investigated in this study. Bounds on the parameter were given and graphs which attain these bounds were characterized. It was shown that the difference of the secure hop domination number and the hop domination number can be made arbitrarily large. A necessary and sufficient condition for a hop dominating set to be secure hop dominating was obtained. Moreover, the secure hop dominating sets in the shadow graph and complementary prism were characterized. These characterizations were used to determine the values of the parameter for these graphs. The newly defined parameter can be studied further for trees and even for graphs resulting from some binary operations. Moreover, it may be interesting to consider and investigate the complexity of the secure hop dominating set problem.

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