EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 2, Article Number 6077
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Existence and Uniqueness of Fixed Points in $MR-Metric$ Spaces and Their Applications

Abed Al-Rahman M. Malkawi

Department of Mathematics, Faculty of Arts and Science, Amman Arab University, Amman 11953, Jordan

Abstract. This paper investigates fixed-point theorems within MR-metric spaces, an extension of standard metric spaces, emphasizing the existence and uniqueness of fixed points for continuous mappings $S : \mathbb{X} \to \mathbb{X}$, where $\mathbb{X}$ is a closed, bounded, and convex subset of a Banach space $(E, \|\cdot\|)$. The study establishes that if S satisfies a contractive condition involving the MR-metric with a constant $k \in [0, 1)$, and a measure of noncompactness condition governed by a function ϕ where $\phi(t) < t$ for $t > 0$, then S possesses a unique fixed point υ^*. The findings have significant applications in solving nonlinear integral equations, ensuring stability of iterative processes, optimization, game theory, economic equilibria, and boundary value problems, showcasing the versatility of MR-metric spaces in addressing noncompact settings and fixed-point problems.

2020 Mathematics Subject Classifications: 47H10, 54H25, 46T99, 47H09

Key Words and Phrases: $MR-metric$ Fixed-point theorems, MR-metric spaces, measure of noncompactness, iterative processes, optimization, game theory, nonlinear integral equations,boundary value problems, Banach spaces

1. Introduction

Fixed-point theory represents a fundamental aspect of functional analysis, playing a pivotal role in mathematics and numerous scientific fields. The idea of a fixed point, where a function maps a specific point to itself, is essential for solving equations, studying dynamic systems, and addressing optimization problems. In this framework, the introduction of generalized metric spaces, including MR-metric spaces, offers a versatile structure for tackling fixed-point challenges, particularly in noncompact contexts.

This study explores MR-metric spaces, which generalize the classical metric space framework to encompass a wider array of applications. By defining a generalized metric M and utilizing the measure of noncompactness, we derive fixed-point theorems for mappings $S : \mathbb{X} \to \mathbb{X}$, where $\mathbb{X}$ represents a closed, bounded, and convex subset of a Banach space. These results ensure the existence and uniqueness of fixed points under certain contractive conditions and noncompactness criteria.

DOI: https://doi.org/10.29020/nybg.ejpam.v18i2.6077

Email address: a.malkawi@aau.edu.jo and math.malkawi@gmail.com (A. Malkawi)

The importance of these findings stems from their wide-ranging applications in both mathematical theory and practical problems, such as nonlinear integral equations, iterative methods, optimization, game theory, and boundary value challenges. This work seeks to establish a solid theoretical framework for employing MR-metric spaces within fixed-point theory, emphasizing their capability to handle intricate and noncompact situations effectively.

For additional information, we direct readers to [1–27].

Definition 1. *[28] Let $\mathbb{X} \neq \emptyset$ denote a non-empty set, and let $R > 1$ be a given real number. A function $M : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to [0, \infty)$ is called an MR-metric if it fulfills the following conditions for all $\upsilon, \xi, \Im \in \mathbb{X}$:*

- $(M1) : M(\upsilon, \xi, \Im) \geq 0$.
- $(M2) : M(\upsilon, \xi, \Im) = 0$ *if and only if* $\upsilon = \xi = \Im$.
- $(M3) : M(\upsilon, \xi, \Im) = M(p(\upsilon, \xi, \Im))$, *for any permutation* $p(\upsilon, \xi, \Im)$ *of* $\upsilon, \xi, \Im$.
- $(M4) : M(\upsilon, \xi, \Im) \leq R\left[M(\upsilon, \xi, \ell_1) + M(\upsilon, \ell_1, \Im) + M(\ell_1, \xi, \Im)\right]$.

A pair $(\mathbb{X}, M)$ that satisfies these properties is called an MR-metric space.

Definition 2. *Let $\{\upsilon_{i_n}\}$ be a sequence in an MR-metric space $(\mathbb{Y}, M)$. The sequence is termed MR-convergent if there exists an element $\upsilon_{i_1} \in \mathbb{Y}$ such that for every $\epsilon > 0$, there exists a positive integer N satisfying $M(\upsilon_{i_n}, \upsilon_{i_m}, \upsilon_{i_1}) < \epsilon$ for all $m, n \geq N$. In this case, the sequence $\{\upsilon_{i_n}\}$ is said to MR-converge to υ_{i_1}, where υ_{i_1} is considered the limit of the sequence.*

Definition 3. *A sequence $\{\upsilon_{i_n}\}$ in an MR-metric space $(\mathbb{Y}, M)$ is called MR-Cauchy if for every $\epsilon > 0$, there exists a positive integer N such that $M(\upsilon i_n, \upsilon i_m, \upsilon i_p) < \epsilon$ for all $m, n, p \geq N$.*

Definition 4. *An MR-metric space (X, M) is considered bounded if there exists a real number $L > 0$ such that $M(\upsilon, \xi, \Im) \leq L$ for all $\upsilon, \xi, \Im \in \mathbb{X}$. In this case, M is referred to as an MR-bound for the metric.*

Definition 5. *Let E be a set subset of an MR-metric space $(\mathbb{X}, M)$ is said to be M-bounded if there exists $L > 0$ such that $M(\upsilon, \xi, \Im) \leq L$ for all $\upsilon, \xi, \Im \in E$.*

Definition 6. *[29] A **Banach space** is a vector space $\mathbb{X}$ over the field $\mathbb{R}$ or $\mathbb{C}$ equipped with a norm $\|\cdot\| : X \to \mathbb{R}$ that satisfies the following conditions:*

(i) ***Positivity:*** $\|\upsilon\| \geq 0$ *for all* $\upsilon \in X$, *and* $\|\upsilon\| = 0$ *if and only if* $\upsilon = 0$.

(ii) ***Scalar Multiplication:*** $\|\alpha\upsilon\| = |\alpha|\|\upsilon\|$ *for all* $\alpha \in \mathbb{R}$ *or* $\mathbb{C}$, *and* $\upsilon \in X$.

(iii) ***Triangle Inequality:*** $\|\upsilon + \xi\| \leq \|\upsilon\| + \|\xi\|$ *for all* $\upsilon, \xi \in X$.

*Moreover, $\mathbb{X}$ is **complete** with respect to the norm, meaning that every Cauchy sequence in $\mathbb{X}$ converges to a limit in $\mathbb{X}$.*

Definition 7. *[30] In measure theory, a **measurable set** is a subset of a set $\mathbb{X}$ that belongs to a σ-algebra $\mathcal{A}$ over $\mathbb{X}$. This means:*

- *The set $\mathbb{X}$ is associated with a measure μ, which is a function defined on a collection $\mathcal{A}$.*
- *$\mathcal{A}$ is a family of subsets of $\mathbb{X}$ satisfying the following conditions:*
  - *(i) $\mathbb{X} \in \mathcal{A}$.*
  - *(ii) If $A \in \mathcal{A}$, then $\mathbb{X} \setminus A \in \mathcal{A}$ (closure under complements).*
  - *(iii) If $\{A_n\}_{n=1}^{\infty} \subset \mathcal{A}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ (closure under countable unions).*

Thus, a measurable set is any element of the σ-algebra $\mathcal{A}$.

2. Fixed Point Theorem in MR-Metric Space within a Banach Space

Fixed-point theory within MR-metric spaces broadens traditional fixed-point results by addressing mappings in more generalized and intricate settings, including noncompact spaces. Here, we focus on MR-metric spaces where $\mathbb{X}$ is a closed, bounded, and convex subset of a Banach space $(E, \|\cdot\|)$. Conditions are derived to ensure that a continuous mapping $S : \mathbb{X} \to \mathbb{X}$ possesses a unique fixed point. This result forms a crucial theoretical foundation for applications in both pure and applied mathematical contexts.

Theorem 1. *Consider an MR-metric space $(\mathbb{X}, M)$, where $\mathbb{X}$ is a closed, bounded, and convex subset of a Banach space $(E, \|\cdot\|)$. Let $S : \mathbb{X} \to \mathbb{X}$ satisfy the following conditions:*

(i) S is continuous.

(ii) For all $\upsilon, \xi, \Im \in \mathbb{X}$,

$$M(S(\upsilon), S(\xi), S(\Im)) \leq k \cdot M(\upsilon, \xi, \Im),$$

where $k \in [0, 1)$ is a constant.

Under these conditions, S has a unique fixed point $\upsilon^* \in \mathbb{X}$, such that $S(\upsilon^*) = \upsilon^*$.

Proof. Let $\upsilon_0 \in \mathbb{X}$ be an arbitrary element, and define a sequence $\{\upsilon_n\}$ by $\upsilon_{n+1} = S(\upsilon_n)$ for all $n \geq 0$.

Using the contraction condition satisfied by S, we have for all $n \geq 0$:

$$M(\upsilon_{n+1}, \upsilon_{n+2}, \upsilon_{n+3}) = M(S(\upsilon_n), S(\upsilon_{n+1}), S(\upsilon_{n+2})) \leq k \cdot M(\upsilon_n, \upsilon_{n+1}, \upsilon_{n+2}).$$

Iterating this inequality yields:

$$M(\upsilon_{n+1}, \upsilon_{n+2}, \upsilon_{n+3}) \leq k^n \cdot M(\upsilon_0, \upsilon_1, \upsilon_2),$$

where $M(\upsilon_0, \upsilon_1, \upsilon_2)$ is a finite constant since M is non-negative.

As $n \to \infty$, the term $k^n \to 0$ because $k \in [0, 1)$. Thus:

$$\lim_{n\to\infty} M(\upsilon_{n+1}, \upsilon_{n+2}, \upsilon_{n+3}) = 0.$$

This implies that the sequence $\{\upsilon_n\}$ is a Cauchy sequence with respect to the MR-metric M.

Since $(\mathbb{X}, M)$ is an MR-metric space and $\mathbb{X}$ is closed in the Banach space E, the completeness of E ensures that the sequence $\{\upsilon_n\}$ converges to some $\upsilon^* \in \mathbb{X}$. That is,

$$\lim_{n\to\infty} \upsilon_n = \upsilon^*.$$

To show that υ^* is a fixed point of S, we use the continuity of S. By definition of S and the convergence of $\{\upsilon_n\}$, we have:

$$\lim_{n\to\infty} S(\upsilon_n) = S\left(\lim_{n\to\infty} \upsilon_n\right).$$

Substituting $\upsilon_{n+1} = S(\upsilon_n)$, we obtain:

$$\upsilon^* = S(\upsilon^*).$$

To prove the uniqueness of the fixed point, suppose there exists another fixed point $\xi^* \neq \upsilon^*$ such that $S(\xi^*) = \xi^*$. Using the contraction condition for S, we have:

$$M(\upsilon^*, \upsilon^*, \upsilon^*) = M(S(\xi^*), S(\xi^*), S(\xi^*)) \leq k \cdot M(\xi^*, \xi^*, \xi^*).$$

Since $M(\upsilon^*, \upsilon^*, \upsilon^*) = 0$ by the properties of the MR-metric, it follows that $M(\xi^*, \xi^*, \xi^*) = 0$. The second axiom of the MR-metric implies $\upsilon^* = \xi^*$, which contradicts the assumption that $\upsilon^* \neq \xi^*$. Hence, υ^* is the unique fixed point of S.

Theorem 2. *Let $(\mathbb{X}, M)$ be an MR-metric space, where $\mathbb{X}$ is a closed, bounded, and convex subset of a Banach space $(E, \|\cdot\|)$. Assume $S : \mathbb{X} \to \mathbb{X}$ is a continuous operator satisfying the following conditions:*

(i) For all $\upsilon, \xi, \Im \in \mathbb{X}$,

$$M(S(\upsilon), S(\xi), S(\Im)) \leq k \cdot M(\upsilon, \xi, \Im),$$

where $k \in [0, 1)$ is a constant.

(ii) The measure of noncompactness μ satisfies:

$$\mu(S(A)) \leq \phi(\mu(A)),$$

for every bounded subset $A \subset \mathbb{X}$, where $\phi : [0, \infty) \to [0, \infty)$ is a non-decreasing function such that $\phi(t) < t$ for all $t > 0$ and $\phi(0) = 0$.

Then S has a unique fixed point $\upsilon^* \in \mathbb{X}$, such that $S(\upsilon^*) = \upsilon^*$.

Proof. Let $\upsilon_0 \in \mathbb{X}$ be arbitrary, and define a sequence $\{\upsilon_n\}$ by $\upsilon_{n+1} = S(\upsilon_n)$ for all $n \geq 0$.

Using the contraction condition of S with respect to the MR-metric M, we have:

$$M(\upsilon_{n+1}, \upsilon_{n+2}, \upsilon_{n+3}) = M(S(\upsilon_n), S(\upsilon_{n+1}), S(\upsilon_{n+2})) \leq k \cdot M(\upsilon_n, \upsilon_{n+1}, \upsilon_{n+2}).$$

By induction, it follows that:

$$M(\upsilon_{n+1}, \upsilon_{n+2}, \upsilon_{n+3}) \leq k^n \cdot M(\upsilon_0, \upsilon_1, \upsilon_2),$$

where $M(\upsilon_0, \upsilon_1, \upsilon_2)$ is a finite constant since M is non-negative. As $n \to \infty$, $k^n \to 0$, and hence:

$$\lim_{n\to\infty} M(\upsilon_{n+1}, \upsilon_{n+2}, \upsilon_{n+3}) = 0.$$

Thus, the sequence $\{\upsilon_n\}$ is a Cauchy sequence under the MR-metric.

Since $\mathbb{X}$ is closed and $(\mathbb{X}, M)$ is complete, the sequence $\{\upsilon_n\}$ converges to some $\upsilon^* \in \mathbb{X}$.

By the continuity of S, we have:

$$\lim_{n\to\infty} S(\upsilon_n) = S\left(\lim_{n\to\infty} \upsilon_n\right),$$

which implies $\upsilon^* = S(\upsilon^*)$. Thus, υ^* is a fixed point of S.

Now, for uniqueness, assume there exists another fixed point $\xi^* \neq \upsilon^*$. Using the contraction condition for S with respect to the MR-metric, we have:

$$M(\upsilon^*, \upsilon^*, \upsilon^*) = M(S(\xi^*), S(\xi^*), S(\xi^*)) \leq k \cdot M(\xi^*, \xi^*, \xi^*).$$

Since $M(\upsilon^*, \upsilon^*, \upsilon^*) = 0$ and $k \in [0, 1)$, it follows that $M(\xi^*, \xi^*, \xi^*) = 0$. By the properties of M, this implies $\upsilon^* = \xi^*$. Hence, the fixed point is unique.

Finally, consider the measure of noncompactness μ. Since $\mathbb{X}$ is closed and bounded, we have:

$$\mu(S(A)) \leq \phi(\mu(A)),$$

for any bounded subset $A \subset \mathbb{X}$. Since $\phi(t) < t$ for all $t > 0$, repeated application of S reduces the measure of noncompactness, ensuring the existence of a fixed point. The uniqueness follows from the argument above.

Example 1. *Fixed Point in MR-Metric Space*

Let $\mathbb{X} = \mathbb{R}$ and define the MR-metric $M : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to [0, \infty)$ by:

$$M(\upsilon, \xi, \Im) = |\upsilon - \xi| + |\xi - \Im| + |\Im - \upsilon|.$$

Let $S : \mathbb{R} \to \mathbb{R}$ be the mapping defined by:

$$S(\upsilon) = \frac{\upsilon}{2}.$$

We verify that S satisfies the contraction condition:

$$M(S(\upsilon), S(\xi), S(\Im)) = \frac{1}{2} M(\upsilon, \xi, \Im),$$

where $k = \frac{1}{2} < 1$. Hence, S satisfies the conditions of the MR-metric space fixed point theorem.

Let $\upsilon_0 \in \mathbb{R}$. Construct the sequence $\upsilon_{n+1} = S(\upsilon_n)$. By iteration:

$$\upsilon_n = \frac{\upsilon_0}{2^n}.$$

As $n \to \infty$, $\upsilon_n \to 0$. Thus, the fixed point of S is $\upsilon^* = 0$.

Example 2. *Fixed Point in Banach Space with Measure of Noncompactness*

Let $\mathbb{X} = C[0,1]$, the Banach space of continuous functions on $[0,1]$ with the norm:

$$\|f\| = \max_{\upsilon \in [0,1]} |f(\upsilon)|.$$

Define the measure of noncompactness μ for a bounded set $A \subset \mathbb{X}$ by:

$$\mu(A) = \inf\{\delta > 0 : A \text{ can be covered by a finite number of sets of diameter } \delta\}.$$

Let $S : C[0,1] \to C[0,1]$ be defined by:

$$(Sf)(\upsilon) = \frac{1}{2} f(\upsilon).$$

We verify that S satisfies:

$$\mu(S(A)) \leq \phi(\mu(A)),$$

where $\phi(t) = \frac{t}{2}$ is non-decreasing with $\phi(t) < t$ for $t > 0$.

For $f \in \mathbb{X}$, consider the sequence $f_{n+1}(\upsilon) = S(f_n)(\upsilon)$. Iterating S gives:

$$f_n(\upsilon) = \frac{1}{2^n} f_0(\upsilon).$$

As $n \to \infty$, $f_n(\upsilon) \to 0$, showing that the fixed point of S is $f^*(\upsilon) = 0$.

Example 3. *Fixed Point in MR-Metric Space with Measure of Noncompactness*

Let $\mathbb{X} = C[0,1]$, the Banach space of continuous functions on $[0,1]$ with the norm:

$$\|f\| = \max_{\upsilon \in [0,1]} |f(\upsilon)|.$$

Define the MR-metric $M : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to [0,\infty)$ by:

$$M(f,g,h) = \|f - g\| + \|g - h\| + \|h - f\|.$$

Consider the operator $S : C[0,1] \to C[0,1]$ defined by:

$$(Sf)(\upsilon) = \frac{1}{2} f(\upsilon).$$

We verify the conditions of the theorem:

1. Contraction Condition in MR-Metric: For all $f, g, h \in C[0,1]$,

$$M(S(f), S(g), S(h)) = \|S(f) - S(g)\| + \|S(g) - S(h)\| + \|S(h) - S(f)\|.$$

Since $S(f)(\upsilon) = \frac{1}{2} f(\upsilon)$, *we have:*

$$\|S(f) - S(g)\| = \frac{1}{2}\|f - g\|.$$

Similarly for other terms, giving:

$$M(S(f), S(g), S(h)) = \frac{1}{2} M(f, g, h).$$

Thus, the contraction constant is $k = \frac{1}{2} < 1$.

2. Measure of Noncompactness Condition: Define the measure of noncompactness μ *on a bounded set* $A \subset C[0,1]$ *as:*

$$\mu(A) = \inf\{\delta > 0 : A \textit{ can be covered by a finite number of sets of diameter } \delta\}.$$

For $S(A) \subset C[0,1]$, *we compute:*

$$\mu(S(A)) = \mu\left(\left\{\frac{1}{2} f : f \in A\right\}\right).$$

By the linearity of S *and the definition of* μ, *we get:*

$$\mu(S(A)) = \frac{1}{2}\mu(A).$$

Define $\phi(t) = \frac{1}{2} t$, *which satisfies* $\phi(t) < t$ *for all* $t > 0$ *and* $\phi(0) = 0$. *Thus:*

$$\mu(S(A)) \leq \phi(\mu(A)).$$

Since both conditions are satisfied, S *has a unique fixed point* $f^* \in C[0,1]$. *Iteratively applying* S, *starting from any* $f_0 \in C[0,1]$, *we find that:*

$$f_n(\upsilon) = \frac{1}{2^n} f_0(\upsilon).$$

As $n \to \infty$, $f_n(\upsilon) \to 0$. *Therefore, the fixed point of* S *is* $f^*(\upsilon) = 0$.

3. Applications of the Theorem

The concept of an MR-metric space plays a critical role in the following applications, as it provides a generalization of the standard metric and enables the handling of noncompact settings, essential for various fixed-point problems.

1. Solving Nonlinear Integral Equations

Example 4. *Consider the nonlinear integral equation:*

$$\upsilon(t) = \int_a^b K(t, s, \upsilon(s))\, ds,$$

where $K : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ *is a continuous function that satisfies a Lipschitz-type condition with respect to the third variable, i.e., there exists a constant* $L > 0$ *such that for all* $t, s \in [a, b]$ *and* $\upsilon_1, \upsilon_2 \in \mathbb{R}$,

$$|K(t, s, \upsilon_1) - K(t, s, \upsilon_2)| \leq L|\upsilon_1 - \upsilon_2|.$$

Step 1: Define the Space Let $\mathbb{X} = C([a, b])$, *the space of continuous functions on the interval* $[a, b]$, *which is a Banach space when equipped with the supremum norm:*

$$\|\upsilon\|_\infty = \sup_{t\in[a,b]} |\upsilon(t)|.$$

Now, equip $\mathbb{X}$ *with the* MR*-metric:*

$$M(\upsilon, \xi, \Im) = \sup_{t\in[a,b]} |\upsilon(t) - \xi(t)| + \sup_{t\in[a,b]} |\xi(t) - \Im(t)|.$$

The MR*-metric generalizes the usual metric by considering the distance between three functions simultaneously, which provides a richer structure to analyze the operator.*

Step 2: Define the Operator Define the operator $S : \mathbb{X} \to \mathbb{X}$ *by:*

$$(S\upsilon)(t) = \int_a^b K(t, s, \upsilon(s))\, ds.$$

The operator S *maps a function* $\upsilon \in \mathbb{X}$ *to another function* $S\upsilon$, *obtained by evaluating the integral.*

Step 3: Verify Continuity of S *Since* $K(t, s, \upsilon)$ *is continuous and satisfies the Lipschitz condition in* $\mathbb{X}$, *the operator* S *is well-defined and continuous on* $\mathbb{X}$. *Specifically, for any* $\upsilon, \xi \in \mathbb{X}$,

$$|(T\upsilon)(t) - (T\xi)(t)| = \left| \int_a^b \big(K(t, s, \upsilon(s)) - K(t, s, \xi(s))\big)\, ds \right|.$$

Using the Lipschitz condition:

$$|(S\upsilon)(t) - (S\xi)(t)| \leq \int_a^b L|\upsilon(s) - \xi(s)|\, ds \leq L\|\upsilon - \xi\|_\infty (b - a).$$

Thus, S *is a contraction in the supremum norm.*

Step 4: Verify Contraction in MR*-Metric For* $\upsilon, \xi, \Im \in \mathbb{X}$,

$$M(S\upsilon, S\xi, S\Im) = \sup_{t\in[a,b]} |(S\upsilon)(t) - (S\xi)(t)| + \sup_{t\in[a,b]} |(S\xi)(t) - (S\Im)(t)|.$$

Using the contraction property derived above:

$$\sup_{t\in[a,b]} |(S\upsilon)(t) - (S\xi)(t)| \leq k \sup_{t\in[a,b]} |\upsilon(t) - \xi(t)|,$$

and similarly:

$$\sup_{t\in[a,b]} |(S\xi)(t) - (S\Im)(t)| \leq k \sup_{t\in[a,b]} |\xi(t) - \Im(t)|.$$

Therefore:

$$M(S\upsilon, S\xi, S\Im) \leq k \cdot M(\upsilon, \xi, \Im),$$

where $k = L(b-a) < 1$ ensures the contraction condition in the MR-metric.

Step 5: Existence and Uniqueness of Fixed Point By the theorem, since S satisfies the contraction condition in the MR-metric and $\mathbb{X}$ is closed, bounded, and convex, there exists a unique fixed point $\upsilon^* \in \mathbb{X}$ such that:

$$\upsilon^*(t) = (S\upsilon^*)(t) = \int_a^b K(t, s, \upsilon^*(s))\, ds.$$

This fixed point υ^* is the unique solution to the nonlinear integral equation.

2. Stability of Iterative Processes

Example 5. *Consider an iterative scheme represented by the update rule:*

$$\upsilon_{n+1} = S(\upsilon_n),$$

where $S : \mathbb{X} \to \mathbb{X}$ is a mapping on a Banach space $\mathbb{X}$ equipped with a norm $\|\cdot\|$. The space $\mathbb{X}$ is further structured with an MR-metric defined as:

$$M(\upsilon, \xi, \Im) = \|\upsilon - \xi\| + \|\xi - \Im\|.$$

Assumptions: 1. S satisfies a contraction condition in the MR-metric:

$$M(S(\upsilon), S(\xi), S(\Im)) \leq k \cdot M(\upsilon, \xi, \Im),$$

for all $\upsilon, \xi, \Im \in \mathbb{X}$, where $k \in [0, 1)$ is a constant.

2. The norm $\|\cdot\|$ ensures that $\mathbb{X}$ is a complete metric space.

Step-by-Step Analysis: Step 1: Understanding the MR-Metric The MR-metric generalizes the standard metric by simultaneously measuring the "distances" between three elements. In this case:

$$M(\upsilon, \xi, \Im) = \|\upsilon - \xi\| + \|\xi - \Im\|.$$

This metric allows the analysis to track how the distances between successive iterations evolve in a dynamic system.

Step 2: Contraction Property of S For any $\upsilon, \xi, \Im \in \mathbb{X}$, the contraction condition ensures:

$$M(S(\upsilon), S(\xi), S(\Im)) = \|S(\upsilon) - S(\xi)\| + \|S(\xi) - S(\Im)\| \leq k(\|\upsilon - \xi\| + \|\xi - \Im\|),$$

where $k \in [0, 1)$. This guarantees that the operator S "brings points closer together" under the MR-metric.

Step 3: Iterative Sequence Start with an initial guess $\upsilon_0 \in \mathbb{X}$ and define the sequence:

$$\upsilon_{n+1} = S(\upsilon_n), \quad n \geq 0.$$

For any n, consider three successive iterates $\upsilon_{n-1}, \upsilon_n, \upsilon_{n+1}$:

$$M(\upsilon_{n+1}, \upsilon_n, \upsilon_{n-1}) = \|\upsilon_{n+1} - \upsilon_n\| + \|\upsilon_n - \upsilon_{n-1}\|.$$

Using the contraction property of S:

$$M(\upsilon_{n+1}, \upsilon_n, \upsilon_{n-1}) \leq k \cdot M(\upsilon_n, \upsilon_{n-1}, \upsilon_{n-2}).$$

By induction:

$$M(\upsilon_{n+1}, \upsilon_n, \upsilon_{n-1}) \leq k^n \cdot M(\upsilon_1, \upsilon_0, \upsilon_{-1}),$$

where υ_{-1} can be taken as the initial condition.

Step 4: Convergence to the Fixed Point Since $k \in [0, 1)$, $k^n \to 0$ as $n \to \infty$. This implies:

$$M(\upsilon_{n+1}, \upsilon_n, \upsilon_{n-1}) \to 0.$$

Decomposing $M(\upsilon_{n+1}, \upsilon_n, \upsilon_{n-1})$, we observe that:

$$\|\upsilon_{n+1} - \upsilon_n\| \to 0 \quad \text{and} \quad \|\upsilon_n - \upsilon_{n-1}\| \to 0.$$

Thus, the sequence $\{\upsilon_n\}$ is Cauchy in $\mathbb{X}$. Since $\mathbb{X}$ is a Banach space, $\{\upsilon_n\}$ converges to a unique point $\upsilon^* \in \mathbb{X}$.

Step 5: Verification of the Fixed Point From the continuity of S, the limit υ^* satisfies:

$$\upsilon^* = \lim_{n\to\infty} \upsilon_n = \lim_{n\to\infty} S(\upsilon_n) = S(\upsilon^*).$$

Hence, υ^* is the unique fixed point of S.

3. Optimization Problems

Example 6. *Consider an iterative scheme represented by the update rule:*

$$\upsilon_{n+1} = S(\upsilon_n),$$

where $S : \mathbb{X} \to \mathbb{X}$ is a mapping on a Banach space $\mathbb{X}$ equipped with a norm $\|\cdot\|$. The space $\mathbb{X}$ is further structured with an MR-metric defined as:

$$M(\upsilon, \xi, \Im) = \|\upsilon - \xi\| + \|\xi - \Im\|.$$

Assumptions: 1. S satisfies a contraction condition in the MR-metric:

$$M(S(\upsilon), S(\xi), S(\Im)) \leq k \cdot M(\upsilon, \xi, \Im),$$

for all $\upsilon, \xi, \Im \in \mathbb{X}$*, where* $k \in [0, 1)$ *is a constant.*

2. The norm $\|\cdot\|$ *ensures that* $\mathbb{X}$ *is a complete metric space.*

Step-by-Step Analysis: Step 1: Understanding the MR*-Metric The* MR*-metric generalizes the standard metric by simultaneously measuring the "distances" between three elements. In this case:*

$$M(\upsilon, \xi, \Im) = \|\upsilon - \xi\| + \|\xi - \Im\|.$$

This metric allows the analysis to track how the distances between successive iterations evolve in a dynamic system.

Step 2: Contraction Property of S *For any* $\upsilon, \xi, \Im \in \mathbb{X}$*, the contraction condition ensures:*

$$M(S(\upsilon), S(\xi), S(\Im)) = \|S(\upsilon) - S(\xi)\| + \|S(\xi) - S(\Im)\| \leq k(\|\upsilon - \xi\| + \|\xi - \Im\|),$$

where $k \in [0, 1)$*. This guarantees that the operator* S *"brings points closer together" under the* MR*-metric.*

Step 3: Iterative Sequence Start with an initial guess $\upsilon_0 \in \mathbb{X}$ *and define the sequence:*

$$\upsilon_{n+1} = S(\upsilon_n), \quad n \geq 0.$$

For any n*, consider three successive iterates* $\upsilon_{n-1}, \upsilon_n, \upsilon_{n+1}$*:*

$$M(\upsilon_{n+1}, \upsilon_n, \upsilon_{n-1}) = \|\upsilon_{n+1} - \upsilon_n\| + \|\upsilon_n - \upsilon_{n-1}\|.$$

Using the contraction property of S*:*

$$M(\upsilon_{n+1}, \upsilon_n, \upsilon_{n-1}) \leq k \cdot M(\upsilon_n, \upsilon_{n-1}, \upsilon_{n-2}).$$

By induction:

$$M(\upsilon_{n+1}, \upsilon_n, \upsilon_{n-1}) \leq k^n \cdot M(\upsilon_1, \upsilon_0, \upsilon_{-1}),$$

where υ_{-1} *can be taken as the initial condition.*

Step 4: Convergence to the Fixed Point Since $k \in [0, 1)$*,* $k^n \to 0$ *as* $n \to \infty$*. This implies:*

$$M(\upsilon_{n+1}, \upsilon_n, \upsilon_{n-1}) \to 0.$$

Decomposing $M(\upsilon_{n+1}, \upsilon_n, \upsilon_{n-1})$*, we observe that:*

$$\|\upsilon_{n+1} - \upsilon_n\| \to 0 \quad \textit{and} \quad \|\upsilon_n - \upsilon_{n-1}\| \to 0.$$

Thus, the sequence $\{\upsilon_n\}$ *is Cauchy in* $\mathbb{X}$*. Since* $\mathbb{X}$ *is a Banach space,* $\{\upsilon_n\}$ *converges to a unique point* $\upsilon^* \in \mathbb{X}$*.*

Step 5: Verification of the Fixed Point From the continuity of S*, the limit* υ^* *satisfies:*

$$\upsilon^* = \lim_{n\to\infty} \upsilon_n = \lim_{n\to\infty} S(\upsilon_n) = S(\upsilon^*).$$

Hence, υ^* *is the unique fixed point of* S*.*

4. Game Theory and Economic Equilibria

Example 7. *In game theory, consider a strategic game where N players aim to optimize their individual payoffs. Let $\mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2 \times \cdots \times \mathbb{X}_N$ represent the strategy space, where $\mathbb{X}_i$ is the strategy set of player i, assumed to be a closed, bounded, and convex subset of a Banach space. The goal is to find a Nash equilibrium $\upsilon^* = (\upsilon_1^*, \upsilon_2^*, \ldots, \upsilon_N^*)$, where no player has an incentive to unilaterally deviate.*

Best-Response Mapping Define the best-response operator T as:

$$S(\upsilon) = \text{BestResponse}(\upsilon) = (S_1(\upsilon), S_2(\upsilon), \ldots, S_N(\upsilon)),$$

where $S_i(\upsilon)$ is the best-response of player i given the strategies of all other players υ_{-i}. Mathematically:

$$S_i(\upsilon) = \arg\max_{\xi \in \mathbb{X}_i} u_i(\xi, \upsilon_{-i}),$$

where u_i is the utility function of player i, and υ_{-i} denotes the strategy profile of all players except i.

MR-Metric Equip the strategy space $\mathbb{X}$ with the MR-metric:

$$M(\upsilon, \xi, \Im) = \|\upsilon - \xi\| + \|\xi - \Im\|,$$

where $\|\cdot\|$ is a norm defined on $\mathbb{X}$. This metric is particularly useful for capturing the distances between successive strategy profiles $\upsilon, \xi, \Im$ during iterative updates.

Assumptions and Contraction Property 1. Convexity of Strategy Sets: The strategy sets $\mathbb{X}_i$ are convex, ensuring the existence of well-defined best responses for each player.

2. Lipschitz Continuity of S: Assume that S satisfies a Lipschitz-type condition in the MR-metric:

$$M(S(\upsilon), S(\xi), S(\Im)) \leq k \cdot M(\upsilon, \xi, \Im),$$

for some constant $k \in [0, 1)$.

3. Existence and Uniqueness: By the conditions of the MR-metric theorem, S has a unique fixed point $\upsilon^* \in \mathbb{X}$, ensuring a unique Nash equilibrium.

Iterative Process Starting from an initial strategy profile $\upsilon^0 \in \mathbb{X}$, the sequence $\{\upsilon^n\}$ is generated iteratively using:

$$\upsilon^{n+1} = S(\upsilon^n).$$

The contraction property of S ensures that $\{\upsilon^n\}$ converges to the fixed point υ^*, which is the Nash equilibrium:

$$S(\upsilon^*) = \upsilon^*.$$

Interpretation of the Fixed Point At the Nash equilibrium $\upsilon^* = (\upsilon_1^*, \upsilon_2^*, \ldots, \upsilon_N^*)$, each player's strategy υ_i^* satisfies:

$$u_i(\upsilon_i^*, \upsilon_{-i}^*) \geq u_i(\xi, \upsilon_{-i}^*), \quad \forall \xi \in \mathbb{X}_i.$$

This implies that no player can improve their utility by unilaterally changing their strategy.

5. Boundary Value Problems

Example 8. *Consider the boundary value problem (BVP):*

$$-u''(\upsilon) = f(\upsilon, u(\upsilon)), \quad \upsilon \in (0,1),$$
$$u(0) = u(1) = 0.$$

This type of problem often arises in physics and engineering, such as in heat conduction, elastic deformation, and electrostatics.

Function Space Define the function space $\mathbb{X} = C([0,1])$*, the space of continuous functions on the interval* $[0,1]$*, equipped with the MR-metric:*

$$M(u, v, w) = \|u - v\|_\infty + \|v - w\|_\infty,$$

where $\|u - v\|_\infty = \sup_{\upsilon \in [0,1]} |u(\upsilon) - v(\upsilon)|$ *is the supremum norm.*

Integral Operator Representation The solution of the boundary value problem can be represented using an integral operator S*, defined as:*

$$(Su)(\upsilon) = \int_0^1 G(\upsilon, s) f(s, u(s))\, ds,$$

where $G(\upsilon, s)$ *is the Green's function for the BVP:*

$$G(\upsilon, s) = \begin{cases} s(1-\upsilon), & \text{if } s \leq \upsilon, \\ \upsilon(1-s), & \text{if } s > \upsilon. \end{cases}$$

The Green's function satisfies the boundary conditions $u(0) = u(1) = 0$ *and accounts for the second-order differential operator.*

Assumptions for $f(\upsilon, u)$ *1. Continuity: The function* $f(\upsilon, u)$ *is continuous in both* υ *and* u*, ensuring the well-posedness of the integral operator* S*.*

2. Lipschitz Condition: There exists a constant $L > 0$ *such that:*

$$|f(\upsilon, u_1) - f(\upsilon, u_2)| \leq L|u_1 - u_2|, \quad \forall \upsilon \in [0,1],\, u_1, u_2 \in \mathbb{R}.$$

This condition ensures that S *is a contraction mapping in the MR-metric.*

Contraction Property in the MR-Metric Let $u, v, w \in \mathbb{X}$*. For the operator* S*, we have:*

$$\|S(u) - S(v)\|_\infty = \sup_{\upsilon \in [0,1]} \left| \int_0^1 G(\upsilon, s)\big(f(s, u(s)) - f(s, v(s))\big)\, ds \right|.$$

Using the Lipschitz condition for f*:*

$$\|S(u) - S(v)\|_\infty \leq L \sup_{\upsilon \in [0,1]} \int_0^1 |G(\upsilon, s)|\, ds \cdot \|u - v\|_\infty.$$

The boundedness of $G(\upsilon, s)$ *ensures that the contraction property is satisfied. Similarly, for the MR-metric:*

$$M(S(u), S(v), S(w)) = \|S(u) - S(v)\|_\infty + \|S(v) - S(w)\|_\infty.$$

Existence and Uniqueness By the MR-metric fixed-point theorem, the contraction property of S guarantees that S has a unique fixed point $u^* \in \mathbb{X}$. This fixed point satisfies:

$$u^*(\upsilon) = \int_0^1 G(\upsilon, s) f(s, u^*(s))\, ds,$$

which is the unique solution to the original boundary value problem.

Iterative Approximation Starting with an initial guess $u_0(\upsilon) \in \mathbb{X}$, the solution can be approximated iteratively using:

$$u_{n+1}(\upsilon) = (Tu_n)(\upsilon) = \int_0^1 G(\upsilon, s) f(s, u_n(s))\, ds.$$

The sequence $\{u_n\}$ converges to the fixed point $u^*(\upsilon)$ under the MR-metric.

Applications 1. Heat Equation: Modeling steady-state heat distribution in a one-dimensional rod. 2. Elasticity: Determining the deflection of a beam under a distributed load. 3. Electrostatics: Solving for potential distributions in one-dimensional domains.

The use of the MR-metric provides a robust framework for analyzing the convergence of solutions and stability of the problem.

4. Conclusion

This paper introduces new fixed-point theorems in the context of **MR-metric spaces**, which extend classical metric space theory. The analysis focuses on continuous self-mappings $S : X \to X$, where X is a closed, bounded, and convex subset of a Banach space. The main results establish the existence and uniqueness of fixed points under two key conditions:

(i) A **contraction condition** with a constant $k \in [0, 1)$.

(ii) A **noncompactness condition** controlled by a function φ satisfying $\varphi(t) < t$ for all $t > 0$.

These theorems provide a significant generalization of classical fixed-point theory, with applications spanning multiple fields, including:

- Solutions to nonlinear integral equations
- Stability analysis of iterative methods
- Optimization problems
- Game theory and economic equilibria
- Boundary value problems

The framework of MR-metric spaces offers a versatile and powerful structure for analyzing noncompact and complex systems, enhancing both theoretical insights and practical applications.

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