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Anti-Diagonals-Parameter Symmetry Model for Square Contingency Tables with Ordinal Classifications

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Abstract. For analyzing square contingency tables having same row and column classifications, analysts are often interested in whether or not the relationship between the row and column variables is symmetric or asymmetric regarding the main-diagonal of the table. However, for the dataset of grip strength test, analysts may be interested in whether or not the relationship between them is symmetric or asymmetric regarding the anti-diagonal, instead of the main-diagonal. This study proposes the anti-diagonals-parameter symmetry model that represents the asymmetric structure regarding the anti-diagonal. Furthermore, this study provides a separation of the anti-symmetry model using the anti-diagonals-parameter symmetry model and an orthogonal separation of the test statistic for the anti-symmetry model. This study shows the advantages of the anti-diagonals-parameter symmetry model by applying it to the real-dataset of grip strength test.

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Key Words and Phrases: Anti-diagonal cell, Separation, Symmetry, Test statistic.

1. Introduction

The main objective of analyzing two-way contingency tables is to evaluate the association between row and column variables. In this study, we address square contingency tables obtained as two-way contingency tables which consist of the same row and column classification. In the analysis of square contingency tables, it is customary to ascertain whether the relationship between the row and column variables is symmetric or asymmetric with respect to the main diagonal. This is due to the fact that the majority of observed frequencies are concentrated in the main diagonal cells, and there is a strong association between row and column variables. Notable models with the symmetric structure are the symmetry (S) model [1], the quasi-symmetry model [2], and the marginal homogeneity model [3]. Additionally, well-known models with the asymmetric structure are the conditional symmetry (CS) model [4], the diagonals-parameter symmetry (DPS) model [5], and the linear DPS (LDPS) model [6].

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We will now examine the dataset presented in Table 1, taken directly from Yamamoto, Aizawa and Tomizawa [7]. This dataset represents the results of grip strength examinations conducted on a cohort of 2356 men between the ages of 15 and 69, as part of the National Health and Nutrition Examination Survey (NHANES) conducted in 2011-2012. Right and left hand grip strength are categorized by the NHANES manual.

Table 1: The grip strength test dataset for men between the ages of 15 and 69

	Left hand					
Right hand	(1)	(2)	(3)	(4)	(5)	Total
Highest (1)	215	124	46	14	2	401
(2)	37	143	165	74	16	435
(3)	7	45	156	166	51	425
(4)	2	20	62	226	210	520
Lowest (5)	1	2	16	61	495	575
Total	262	334	445	541	774	2356

Source: NHANES 2011–2012

In the case of grip strength data, such as that presented in Table 1, analysts may be less concerned with the symmetry or asymmetry between the right and left hand grip strength. However, they may be interested in the symmetry or asymmetry between the dominant and non-dominant hand grip strength. This is because the grip strength of the dominant hand is typically higher than that of the non-dominant hand, with a majority of individuals being right-handed by Intage group self-proposed survey (`https://gallery.intage.co.jp/smartphone-operation/`). Furthermore, Iki [8] and Ando [9, 10] analyzed the grip strength data using models where the relationship between the row and column variables is symmetric or asymmetric with respect to the anti-diagonal of the table, rather than the main diagonal.

The anti-S (AS) model and the anti-CS (ACS) model represent the symmetric and asymmetric structures of cell probabilities regarding the anti-diagonal of the table, respectively. These models are referenced in the following citations: [8, 11]. In this paper, we propose a unified nomenclature for the symmetric and asymmetric models regarding the anti-diagonal, beginning with the term "anti." However, it is noteworthy that Tomizawa [11] referred to the "anti conditional symmetry model" as the "reverse conditional symmetry model."

If the AS model fits the dataset in Table 1 well, we can conclude that the criteria for grip strength classification, as outlined in the NHANES manual, are functioning effectively. Conversely, if the ACS model fits the dataset in Table 1 well, we can interpret the criteria as working poorly. This is due to the following reasons: (i) if the asymmetry parameter of the ACS model is greater than one, people with high grip strength levels are greater than people with low grip strength levels, (ii) if it is less than one, people with high grip strength levels are fewer than people with low grip strength levels, and (iii) if it is equal to one (i.e., the AS model holds), there are equal numbers of people with high and low grip strength levels. It is notable that both the AS and ACS models do not fit to the dataset presented in Table 1 well. For a comprehensive examination of these results, please refer to Section 5.

This study proposes an anti-DPS (ADPS) model, which represents an asymmetric structure with respect to the anti-diagonal of the table. This structure corresponds to the DPS model, which represents an asymmetric structure with respect to the main diagonal. Furthermore, this study provides a separation of the AS model using the ADPS model and a separation of the test statistic for the AS model.

The remainder of this paper is organized as follows. In Section 2, we introduce the ADPS model and other related models, including those described in Section 1. Section 3 presents the separation of the AS model using the ADPS model, while Section 4 illustrates the separation of the test statistic for the AS model. Section 5 illustrates the advantages of the ADPS model through its application to the real dataset presented in Table 1. Section 6 presents concluding remarks and suggests avenues for further research.

2. Models

2.1. Models regarding main-diagonal cells

For $R \times R$ ordinal square tables, we denote π_{ij} for $(i, j) \in A$ as the probability that an observed frequency will fall in the (i, j)th cell of the table, where $A = \{(i, j) \mid i, j = 1, 2, \ldots, R\}$.

In order to ascertain whether the cell probabilities are symmetric or asymmetric with respect to the main diagonal, we consider a class of model which is defined by the following equation:

$$\pi_{ij} = \delta_{ij}\pi_{ji} \quad \text{for } (i, j) \in D,$$

where $D = \{(i, j) \mid i, j = 1, 2, \ldots, R, i < j\}$. When no restrictions are imposed on δ_{ij} for $(i, j) \in D$, the above model is identical to a saturated model. Conversely, when restrictions are imposed on δ_{ij} for $(i, j) \in D$ (i.e., $\delta_{ij} = 1$ for $(i, j) \in D$, $\delta_{ij} = \delta$ for $(i, j) \in D$ and $\delta_{ij} = \delta_{j-i}$ for $(i, j) \in D$), the above model is identical to the S, CS, DPS models, respectively.

The CS model is a special case of the S model because the CS model with $\delta = 1$ is identical to the S model. In other words, if the S model holds then the CS model always holds, although the converse is not necessarily true. As special cases of the DPS model, the DPS model with $\delta_{j-i} = \delta^{j-i}$ for $(i, j) \in D$ is identical to the LDPS model, the DPS model with $\delta_{j-i} = \delta$ for $(i, j) \in D$ is identical to the CS model, and the DPS model with $\delta_{j-i} = 1$ for $(i, j) \in D$ is identical to the S model. Consequently, when the S model holds, the DPS model always holds, although the inverse is not necessarily true. Similarly, when the CS model holds, the DPS model also holds, although the converse is not necessarily true.

Next, we will introduce the separations of the S model via the CS model or the DPS model. The global symmetry (GS) model [12] represents the counterpart model of the CS model in the separation of the S model. The GS model is defined by the following equation:

$$\sum\sum_{(i,j)\in D}\pi_{ij} = \sum\sum_{(i,j)\in D}\pi_{ji}.$$

The GS model represents a structure of symmetry in which the probability of an observation occurring in the upper right non-diagonal cells of the table is equal to the probability of an observation occurring in the lower left non-diagonal cells.

In this context, the notation $G^2(\mathrm{C})$ is used to represent the likelihood ratio chi-squared statistic for testing the goodness of fit of the concerned model C. In their study, Read [12] demonstrated the following separation of the test statistic for the S model:

$$G^2(\mathrm{S}) = G^2(\mathrm{CS}) + G^2(\mathrm{GS}).$$

The marginal diagonal sub-symmetry (MDSS) model [13] represents the counterpart model of the DPS model in the separation of the S model. The MDSS model is defined by the following equation:

$$\sum\sum_{(i,j)\in D_d}\pi_{ij} = \sum\sum_{(i,j)\in D_d}\pi_{ji} \quad (d = 1, 2, \ldots, R-1),$$

where $D_d = \{(i,j) \mid i,j = 1,2,\ldots,R, i < j, j - i = d\}$. Tomizawa and Kato [13] demonstrated the following separation of the test statistic for the S model:

$$G^2(\mathrm{S}) = G^2(\mathrm{DPS}) + G^2(\mathrm{MDSS}).$$

2.2. Models regarding anti-diagonal cells

In order to ascertain whether the cell probabilities are symmetric with respect to the anti-diagonal, rather than the main diagonal, Iki [8] proposed the AS model, which is defined by the following equation:

$$\pi_{ij} = \pi_{j^*i^*} \quad \text{for } (i,j) \in E,$$

where $i^* = R + 1 - i$, $j^* = R + 1 - j$ and $E = \{(i,j) \mid i,j = 1,2,\ldots,R, i + j < R + 1\}$.

Readers may think that the structure of the AS model corresponds to the structure of the S model, in which the categories of one of the row and column variables in a square contingency table are re-labelled, but the structures of the AS and S models are completely different. Let the row and column variables be designated as X and Y, respectively. The cells on the main diagonal of the table imply that the difference between X and Y is equal to zero, which is the average of $X - Y$. Similarly, the cells on the anti-diagonal of the table imply that the sum of X and Y is equal to $R+1$, which is the average of $X+Y$. In particular, the S model implies symmetry with respect to the difference between X and Y, while the AS model implies symmetry with respect to the sum of X and Y.

In order to ascertain whether the cell probabilities are asymmetric with respect to the anti-diagonal, Tomizawa [11] proposed the ACS model, which is defined by the following equation:

$$\pi_{ij} = \delta\pi_{j^*i^*} \quad \text{for } (i,j) \in E.$$

The ACS model is a special case of the AS model because the ACS model with $\delta = 1$ is identical to the AS model. In other words, if the AS model holds then the ACS model always holds, although the converse is not necessarily true.

Let denote f_{ij} for $(i, j) \in A$ as the observed frequency in (i, j)th cell of the table. Table 2 illustrates the ratio of f_{ij} to $f_{j^*i^*}$, which is calculated for each pair $(i, j) \in E$, for the dataset in Table 1. In the case of a dataset for which the ratios of f_{ij} to $f_{j^*i^*}$ are constant, the ACS model fits well. Since the ratios of f_{ij} to $f_{j^*i^*}$ are not constant in this dataset, the ACS model may not be fit the dataset in Table 1 well. Therefore, a model that can express more complex asymmetric structures than the ACS model is required.

Table 2: The ratio of f_{ij} to $f_{j^*i^*}$ is calculated for each pair $(i, j) \in E$

	Left hand				
Right hand	(1)	(2)	(3)	(4)	(5)
Highest (1)	0.434	0.590	0.902	0.875	–
(2)	0.607	0.633	0.994	–	–
(3)	0.438	0.726	–	–	–
(4)	1.000	–	–	–	–
Lowest (5)	–	–	–	–	–

In line with the relationship between the CS and DPS model, in order to express an asymmetric structure that is more complex than that of the ACS model (i.e., an asymmetric structure where the ratios of π_{ij} to $\pi_{j^*i^*}$ for $(i, j) \in E$ are not constant), we propose the ADPS model that is defined by

$$\pi_{ij} = \delta_{R+1-(i+j)}\pi_{j^*i^*} \quad \text{for } (i, j) \in E.$$

The ADPS model represents that the degree of asymmetry varies depending on the distance (i.e., $R + 1 - (i + j)$) from the anti-diagonal. The ADPS model can exhibit greater versatility than the ACS model with regard to the representation of express complex asymmetric structures. As special cases of the ADPS model, the ADPS model with $\delta_{R+1-(i+j)} = \delta$ for $(i, j) \in E$ is identical to the ACS model, and the ADPS model with $\delta_{R+1-(i+j)} = 1$ for $(i, j) \in E$ is identical to the AS model. Consequently, when the AS model holds, the ADPS model always holds, although the inverse is not necessarily true. Similarly, when the ACS model holds, the ADPS model also holds, although the converse is not necessarily true. In Section 3, we will find out a counterpart model of the ACS model for the separation of the AS model, as well as a counterpart model of the ADPS model for the separation of the AS model.

3. Separation of the AS model

This section will examine the separation of the AS model via the ACS model (or the ADPS model). The anti-GS (AGS) model represents the counterpart model of the ACS model for the separation of the AS model. For further information regarding the AGS model, please refer to the following sources: [8, 14–16]. The AGS model is defined by the following equation:

$$\sum\sum_{(i,j)\in E}\pi_{ij} = \sum\sum_{(i,j)\in E}\pi_{j^*i^*}.$$

The AGS model represents a structure of symmetry in which the probability of an observation occurring in the upper left off-anti-diagonal cells of the table is equal to the probability of an observation occurring in the lower right off-anti-diagonal cells.

The separation of the AS model is obtained via the ACS and AGS models in accordance with the following theorem.

Theorem 1. *The AS model holds if and only if both the ACS and AGS models concurrently hold.*

Proof. The initial step is to address the necessary condition. If the AS model holds (i.e., $\pi_{ij} = \pi_{j^*i^*}$ for $(i,j) \in E$), it can be demonstrated that both the AGS model and the ACS model with $\delta = 1$ concurrently hold. Therefore, the necessary condition holds.

We now turn to an examination of the sufficient condition. Assuming that both the ACS and AGS models concurrently hold. Since the AGS model holds, the following equation is derived:

$$\sum\sum_{(i,j)\in E}\pi_{ij} = \sum\sum_{(i,j)\in E}\pi_{j^*i^*}.$$

Moreover, since the AGS model holds, the following equation is derived:

$$(\delta - 1)\sum\sum_{(i,j)\in E}\pi_{j^*i^*} = 0.$$

Given that π_{ij} for $(i,j) \in A$ are positive, it can be concluded $\delta = 1$. Consequently, the sufficient condition is satisfied. The proof is thus complete, as per the aforementioned results.

The counterpart to the ADPS model for separating the AS model is the anti-sum-symmetry (ASS) model [9, 10]. The ASS model is defined by the following equation:

$$\sum\sum_{(i,j)\in E_d}\pi_{ij} = \sum\sum_{(i,j)\in E_d}\pi_{j^*i^*} \quad (d = 2, 3, \ldots, R),$$

where $E_d = \{(i,j) \mid i,j = 1,2,\ldots,R, i+j < R+1, i+j = d\}$.

The separation of the AS model via the ADPS and ASS models is given by the following theorem.

Theorem 2. *The AS model holds if and only if both the ADPS and ASS models hold.*

Proof. The initial step is to address the necessary condition. If the AS model holds (i.e., $\pi_{ij} = \pi_{j^*i^*}$ for $(i,j) \in E$), it can be demonstrated that both the ASS model and ADPS model with $\delta_{R+1-(i+j)} = 1$ for $(i,j) \in E$ concurrently hold. It can thus be concluded that the necessary condition is satisfied.

The sufficient condition will now be considered. Assuming that both the ADPS and ASS concurrently models hold. Since the ASS model holds, the following equation is obtained:

$$\sum\sum_{(i,j)\in E_d}\pi_{ij}=\sum\sum_{(i,j)\in E_d}\pi_{j^*i^*}\quad (d=2,3,\ldots,R).$$

Moreover, since the ADPS model holds, the following equation is derived:

$$\sum\sum_{(i,j)\in E_d}(\delta_{R+1-(i+j)}-1)\pi_{j^*i^*}=0\quad (d=2,3,\ldots,R).$$

Given that π_{ij} for $(i,j)\in A$ are positive, it can be concluded $\delta_{R+1-(i+j)}=1$ for $(i,j)\in E$. Consequently, the sufficient condition is satisfied. The proof is thus complete, as per the aforementioned results.

These separations (i.e.,Theorems 1 and 2) may be useful for seeing a reason for the poor fit of the AS model.

4. Separation of test statistic for the AS model and model selection

Let denote N as sample size (i.e., $N=\sum\sum_{(i,j)\in A}f_{ij}$). We assume that the observed frequencies f_{ij} for $(i,j)\in A$ are distributed as a multinomial distribution with the probability vector $\boldsymbol{\pi}$, where

$$\boldsymbol{\pi}=(\pi_{11},\pi_{12},\ldots,\pi_{1R},\pi_{21},\pi_{22},\ldots,\pi_{2R},\ldots,\pi_{R1},\pi_{R2},\ldots,\pi_{RR})^\top.$$

The symbol $\top$ represents the transpose of vector or matrix.

Let $\hat{e}_{ij}$ for $(i,j)\in A$ denote the maximum likelihood estimator (MLE) of the expected frequencies e_{ij} under the concerned model can be obtained via maximizing the log-likelihood equation under the restrictions of the concerned model. Thus, the $\hat{e}_{ij}$ for $(i,j)\in A$ under the ADPS model can be obtained via maximizing the following Lagrangian regarding $\{\pi_{ij}\},\phi,\{\psi_{ij}\}$, and $\{\delta_{R+1-(i+j)}\}$:

$$L=\sum\sum_{(i,j)\in A}f_{ij}\log\pi_{ij}-\phi\left(\sum\sum_{(i,j)\in A}\pi_{ij}-1\right)-\sum\sum_{(i,j)\in E}\psi_{ij}\left(\pi_{ij}-\delta_{R+1-(i+j)}\pi_{j^*i^*}\right).$$

Although the details are omitted, the $\hat{e}_{ij}$ for $(i,j)\in A$ under the AS, ACS, ADPS, AGS and ASS models are calculated as the equations (1), (2), (3), (4) and (5), respectively. Let denote the sets $E^{c_1}=\{(i,j)\mid i,j=1,2,\ldots,R,i+j=R+1\}$, $E^{c_2}=\{(i,j)\mid i,j=1,2,\ldots,R,i+j>R+1\}$, $S=\{(s,t)\mid s,t=1,2,\ldots,R,s+t<R+1\}$ and $S_{ij}=\{(s,t)\mid s,t=1,2,\ldots,R,s+t<R+1,s+t=i+j\}$.

$$\hat{e}_{ij}=\begin{cases}\dfrac{f_{ij}+f_{j^*i^*}}{2} & ((i,j)\in E)\\ f_{ij} & ((i,j)\in E^{c_1})\\ \dfrac{f_{ij}+f_{j^*i^*}}{2} & ((i,j)\in E^{c_2}).\end{cases}\qquad (1)$$

$$\hat{e}_{ij} = \begin{cases} \dfrac{\sum\sum_{(s,t)\in S} f_{st}}{\sum\sum_{(s,t)\in S} f_{st} + \sum\sum_{(s,t)\in S} f_{t^*s^*}} \cdot (f_{ij} + f_{j^*i^*}) & ((i,j)\in E) \\ f_{ij} & ((i,j)\in E^{c_1}) \\ \dfrac{\sum\sum_{(s,t)\in S} f_{t^*s^*}}{\sum\sum_{(s,t)\in S} f_{st} + \sum\sum_{(s,t)\in S} f_{t^*s^*}} \cdot (f_{ij} + f_{j^*i^*}) & ((i,j)\in E^{c_2}). \end{cases} \tag{2}$$

$$\hat{e}_{ij} = \begin{cases} \dfrac{\sum\sum_{(s,t)\in S_{ij}} f_{st}}{\sum\sum_{(s,t)\in S_{ij}} f_{st} + \sum\sum_{(s,t)\in S_{ij}} f_{t^*s^*}} \cdot (f_{ij} + f_{j^*i^*}) & ((i,j)\in E) \\ f_{ij} & ((i,j)\in E^{c_1}) \\ \dfrac{\sum\sum_{(s,t)\in S_{ij}} f_{t^*s^*}}{\sum\sum_{(s,t)\in S_{ij}} f_{st} + \sum\sum_{(s,t)\in S_{ij}} f_{t^*s^*}} \cdot (f_{ij} + f_{j^*i^*}) & ((i,j)\in E^{c_2}). \end{cases} \tag{3}$$

$$\hat{e}_{ij} = \begin{cases} \dfrac{\sum\sum_{(s,t)\in S} f_{st} + \sum\sum_{(s,t)\in S} f_{t^*s^*}}{2\sum\sum_{(s,t)\in S} f_{st}} \cdot f_{ij} & ((i,j)\in E) \\ f_{ij} & ((i,j)\in E^{c_1}) \\ \dfrac{\sum\sum_{(s,t)\in S} f_{st} + \sum\sum_{(s,t)\in S} f_{t^*s^*}}{2\sum\sum_{(s,t)\in S} f_{t^*s^*}} \cdot f_{ij} & ((i,j)\in E^{c_2}). \end{cases} \tag{4}$$

$$\hat{e}_{ij} = \begin{cases} \dfrac{\sum\sum_{(s,t)\in S_{ij}} f_{st} + \sum\sum_{(s,t)\in S_{ij}} f_{t^*s^*}}{2\sum\sum_{(s,t)\in S_{ij}} f_{st}} \cdot f_{ij} & ((i,j)\in E) \\ f_{ij} & ((i,j)\in E^{c_1}) \\ \dfrac{\sum\sum_{(s,t)\in S_{ij}} f_{st} + \sum\sum_{(s,t)\in S_{ij}} f_{t^*s^*}}{2\sum\sum_{(s,t)\in S_{ij}} f_{t^*s^*}} \cdot f_{ij} & ((i,j)\in E^{c_2}). \end{cases} \tag{5}$$

The test statistic G^2 for the concerned model C is given as

$$G^2(\mathrm{C}) = 2\sum_{i=1}^{R}\sum_{j=1}^{R} f_{ij}\log\left(\frac{f_{ij}}{\hat{e}_{ij}}\right).$$

We consider the situations in which the model C_1 holds if and only if the both models C_2 and C_3 concurrently hold, and the following equation is satisfied:

$$G^2(\mathrm{C}_1) = G^2(\mathrm{C}_2) + G^2(\mathrm{C}_3), \tag{6}$$

where the number of degrees of freedom (DF) for the model C_1 is equal to the sum of those for the models C_2 and C_3. We point out that the likelihood ratio chi-squared statistic for testing goodness-of-fit of the model C_1 assuming that the C_2 model holds true, is $G^2(\mathrm{C}_1) - G^2(\mathrm{C}_2)$ and this is equal to the likelihood ratio chi-squared statistic for testing goodness-of-fit of the C_3 model, i.e., $G^2(\mathrm{C}_3)$.

The present study demonstrates that Theorems 1 and 2 satisfy the equation (6). The following theorem is thus obtained.

Theorem 3.

$$G^2(\mathrm{AS}) = G^2(\mathrm{ACS}) + G^2(\mathrm{AGS}).$$

Proof.

The $\hat{e}_{ij}$ under the AS, ACS, and AGS models are calculated as the equations (1), (2) and (4), respectively. Let notate $\hat{e}_{ij}$ under the model M as $\hat{e}_{ij(\mathrm{M})}$. The product of $f_{ij}/\hat{e}_{ij(\mathrm{ACS})}$ and $f_{ij}/\hat{e}_{ij(\mathrm{AGS})}$ is expressed as follows.

$$\frac{f_{ij}}{\hat{e}_{ij(\mathrm{ACS})}} \cdot \frac{f_{ij}}{\hat{e}_{ij(\mathrm{AGS})}} = \begin{cases} \dfrac{2f_{ij}}{f_{ij}+f_{j^*i^*}} & ((i,j)\in E) \\ 1 & ((i,j)\in E^{c_1}) \\ \dfrac{2f_{ij}}{f_{ij}+f_{j^*i^*}} & ((i,j)\in E^{c_2}). \end{cases}$$

Therefore, the product of $f_{ij}/\hat{e}_{ij(\mathrm{ACS})}$ and $f_{ij}/\hat{e}_{ij(\mathrm{AGS})}$ is equal to the $f_{ij}/\hat{e}_{ij(\mathrm{AS})}$. The proof is completed.

Similarly, as with Theorem 3, the following theorem is derived.

Theorem 4.

$$G^2(\mathrm{AS}) = G^2(\mathrm{ADPS}) + G^2(\mathrm{ASS}).$$

The anti-conditional sum-symmetry (ACSS) model [9, 10] is defined by the following equation:

$$\sum\sum_{(i,j)\in E_d}\pi_{ij} = \delta\sum\sum_{(i,j)\in E_d}\pi_{j^*i^*} \quad (d = 2,\ldots,R).$$

Ando [9] gave the theorems that the ASS model holds if and only if the ACSS and AGS models concurrently hold. Additionally, Ando [9] proposed a novel approach to separating the test statistic for the ASS model:

$$G^2(\mathrm{ASS}) = G^2(\mathrm{ACSS}) + G^2(\mathrm{AGS}).$$

In consequence, the following theorem is derived.

Theorem 5. *The AS model holds if and only if all the ADPS, ACSS and AGS models hold. Additionally, the following separation of the test statistic for the AS model:*

$$G^2(\mathrm{AS}) = G^2(\mathrm{ADPS}) + G^2(\mathrm{ACSS}) + G^2(\mathrm{AGS}).$$

We will present the following theorem that can be derived from Theorems 1 to 5.

Theorem 6. *The ACS model holds if and only if both the ADPS and ACSS models concurrently hold. Additionally, the following separation of the test statistic for the ACS model:*

$$G^2(\mathrm{ACS}) = G^2(\mathrm{ADPS}) + G^2(\mathrm{ACSS}).$$

Table 3 shows the number of DF for testing goodness-of-fit for each model. It must be noted that the number of DF for testing goodness-of-fit of the separation source model is equal to the sum of number of DF for testing goodness-of-fit of the decomposed models.

Table 3: The number of degrees of freedom for testing goodness-of-fit for each model

Models	Degrees of freedom
AS	$R(R-1)/2$
ACS	$(R+1)(R-2)/2$
ADPS	$(R-1)(R-2)/2$
AGS	1
ASS	$R-1$
ACSS	$R-2$

When the models M_1 and M_2 are nested (i.e., the model M_1 is more parsimonious than the model M_2), for testing that the model M_1 holds assuming that the model M_2 model holds true, the likelihood ratio statistics is given as

$$G^2(\mathrm{M}_1 \mid \mathrm{M}_2) = G^2(\mathrm{M}_1) - G^2(\mathrm{M}_2).$$

Under the null hypothesis, the statistics $G^2(\mathrm{M}_1 \mid \mathrm{M}_2)$ has an asymptotic chi-squared distribution with the number of degrees of freedom that is equal to the difference between the number of degrees of freedom for the models M_1 and M_2, see, for example, Agresti [17, Sec 3.4.4].

When the models M_1 and M_2 are not nested. the test statistics $G^2(\mathrm{M}_1 \mid \mathrm{M}_2)$ cannot be used to compare them. Therefore, we need to use other statistics. In this situation, the Akaike information criterion (AIC) [18] and Bayesian information criterion (BIC) [19] are famous information criteria to select the best fitting model among the applied models, and the model with the minimum AIC (or BIC) is the best fitting model. As only the difference between AICs (or BICs) of two models is required for comparing, it can ignore a common part of AIC (or BIC). Therefore, we use modified AIC and BIC defined as

$$\mathrm{AIC}^{+} = G^2 - 2(\text{number of degrees of freedom}),$$
$$\mathrm{BIC}^{+} = G^2 - \log N(\text{number of degrees of freedom}).$$

The model with the minimum AIC^{+} or minimum BIC^{+} is the best fitting model among the applied models.

5. Real data analysis

We re-examine the dataset presented in Table 1. Table 4 provided the values of G^2 and AIC^{+} for each AS, ACS, ADPS, AGS, ASS, and ACSS model applied to the dataset in Table 1.

Table 4: Results of goodness-of-fit test applied each model to the dataset of Table 1

Applied models	Degrees of freedom	G^2	p-value	AIC^{+}
AS	10	167.37*	< 0.01	147.37
ACS	9	43.92*	< 0.01	25.92
ADPS	6	5.35	0.50	-7.35
AGS	1	123.44*	< 0.01	121.44
ASS	4	162.02*	< 0.01	154.02
ACSS	3	38.57*	< 0.01	32.57

The symbol $*$ represents significance at the 0.05 level.

Table 4 showed that the goodness-of-fit of the ADPS model is well and dramatically improves compared to the ACS model. Additionally, the goodness-of-fit of the other models are poor. The values of test statistics $G^2(\mathrm{AS} \mid \mathrm{ACS})$ and $G^2(\mathrm{ACS} \mid \mathrm{ADPS})$ are 123.45 and 38.57, respectively. Therefore, the ACS model is preferable to the AS model, and the ADPS model is preferable to the ACS model. Moreover, in terms of AIC, the ADPS model is the best-fitting model among the models applied to the dataset in Table 1. Table 5 gave the $\hat{e}_{ij}$ under the ADPS model. We see that there is little difference between MLEs of expected frequencies and the observed frequencies. The MLEs of δ_1, δ_2, δ_3, and δ_4 are 0.92, 0.67, 0.59, and 0.43, respectively. The results demonstrate a clear discrepancy between the ratio of men with high and low grip strength levels. It can therefore be concluded that the aforementioned criteria for determining grip strength levels is ineffective. In the future, it is expected that the classification criteria for grip strength may be revised.

In accordance with Theorem 2, the poor goodness-of-fit of the AS model can be attributed to the ASS model, rather than the ADPS model. Additionally, from Theorem 5, it can be concluded that the cause of the poor goodness-of-fit of the AS model is the ACSS and AGS models, rather than the ADPS model.

Table 5: Maximum likelihood estimates of expected frequencies under the anti-diagonals-parameter symmetry (ADPS) model applied to the dataset in Table 1

	Left hand					
Right hand	(1)	(2)	(3)	(4)	(5)	Total
Highest (1)	215	124	46	14	2	401
	(215)	(124.48)	(38.88)	(14.36)	(2)	
(2)	37	143	165	74	16	435
	(36.52)	(147.90)	(158.49)	(74)	(15.64)	
(3)	7	45	156	166	51	425
	(9.22)	(51.23)	(156)	(172.51)	(58.12)	
(4)	2	20	62	226	210	520
	(1.92)	(20)	(55.77)	(221.10)	(209.52)	
Lowest (5)	1	2	16	61	495	575
	(1)	(2.08)	(13.78)	(61.48)	(495)	
Total	262	334	445	541	774	2356

From Theorem 6, it can be concluded that the cause of the poor goodness-of-fit of the ACS model is the ACSS model, rather than the ADPS model.

6. Conclusion

This study proposed the ADPS model as an extension of the ACS model. The ADPS model represents the asymmetric structure of cell probabilities regarding the anti-diagonal cells of the table, and is corresponding to the DPS model, which represents the asymmetric structure of cell probabilities regarding the main-diagonal cells of the table. Furthermore, this study presented the separation of the AS model using the ADPS model (see Theorem 2), and separation of the test statistic for the AS model (see Theorem 4).

The proposed model was demonstrated to offer significant advantages through its application to a real dataset in Table 1. The hypothesis was that males with high grip strength levels are less prevalent than males with low grip strength levels. Furthermore, it was demonstrated that the established criteria for determining grip strength levels were ineffective.

The ADPS model is saturated in the following cells: the 1st row and 1st column cell, the Rth row and Rth column cell, and the anti-diagonal cells of the table. It can be demonstrated that the observed frequencies are equal to the estimated values for the (1,1)th and (R, R)th cells.

Given that observations in square tables tend to concentrate on the main diagonal cells, analysts may wish to consider a model that is saturated on only the anti-diagonal cells. One potential approach is to apply a model corresponding to the LDPS model, which represents an asymmetric structure regarding the main diagonal cells. The double linear diagonals-parameter symmetry model Tahata and Tomizawa [20] has both the structure of the LDPS model with respect to the main diagonal of the table and the structure of the LDPS model with respect to the anti-diagonal of the table. The model comprising solely the latter structure is saturated exclusively on the anti-diagonal cells. However, it is beyond the scope of this study to investigate the utility of this model for application to real-world data. This is a topic for future research.

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