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Vandermonde Determinant for the Generalized Bounded Turning Functions Associated with Gregory Coefficients

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Abstract. This paper explores the class $G_{\mathrm{G}}(\alpha,\delta)$ of analytic functions, which is associated with generalized bounded turning and the generating functions of Gregory coefficients. By using bounds on certain coefficient functionals for functions with a positive real part, we obtain initial Taylor coefficient bounds and logarithmic coefficient bounds of functions and inverse functions within this class. Consequently, we establish upper bounds of the second-order for the Vandermonde determinant, where the entries are Taylor coefficients and logarithmic coefficients of functions and inverse functions. Additionally, we highlight several interesting implications of these results, contributing new insights to this generalized class.

2020 Mathematics Subject Classifications: 30C45, 30C50

Key Words and Phrases: Univalent functions, inverse functions, bounded turning functions, Gregory coefficients, Taylor coefficients, logarithmic coefficients, Vandermonde determinant

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions $f(z)$ that can be expressed as a Taylor series expansion in the open unit disk $E=\{z\in\mathbb{C}:|z|<1\}$, given by

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n\ ,\ z\in E. \tag{1}$$

We denote by S the subclass of $\mathcal{A}$ consisting of univalent functions in E.
The inverse of a function $f(z)\in S$ of the form (1) has a series expansion given by

$$f^{-1}(w)=w+\sum_{n=2}^{\infty}A_nw^n,\ |w|<r_0(f)\,,r_0(f)\geq\frac{1}{4}, \tag{2}$$

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where, in particular, the coefficients A_n for $n = 2, 3, 4$ are expressed in terms of the Taylor coefficients of $f(z) \in S$ as follows:

$$A_2 = -a_2, \tag{3}$$

$$A_3 = -a_3 + 2a_2{}^2, \tag{4}$$

and

$$A_4 = -a_4 + 5a_2a_3 - 5a_2{}^3. \tag{5}$$

Let P denote the class of functions with a positive real part in the open unit disk E. A function $p(z)$ in P has the series expansion

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \ z \in E, \tag{6}$$

that is analytic in E and satisfying the condition $\operatorname{Re}(p(z)) > 0$. Functions in P have often been used to describe geometric properties of functions in $\mathcal{A}$, and to define subclasses in S. Let H denote the class of Schwarz functions $\upsilon(z)$ which are analytic in E, given by

$$\upsilon(z) = \sum_{k=1}^{\infty} b_k z^k, \ z \in E$$

and satisfying $\upsilon(0) = 0$ and $|\upsilon(z)| < 1$. If $p(z) \in P$, then a Schwarz function $\upsilon(z) \in H$ exists such that

$$p(z) = \frac{1 + \upsilon(z)}{1 - \upsilon(z)}, \ z \in E. \tag{7}$$

Let $g_1(z)$ and $g_2(z)$ be two analytic functions in E, with the symbol $\prec$ representing a subordination. The function $g_1(z)$ is subordinate to function $g_2(z)$, denoted $g_1(z) \prec g_2(z)$, if there exists a Schwarz function $\upsilon(z) \in H$ such that $g_1(z) = g_2(\upsilon(z))$. Furthermore, if $g_1(z)$ is univalent in E, then we have the following equivalence

$$g_1(z) \prec g_2(z) \Leftrightarrow g_1(0) = g_2(0)$$

and

$$g_1(E) = g_2(E).$$

Milin [1–3] highlighted the importance of logarithmic coefficients in estimating Taylor coefficients of univalent functions. The inequalities conjectured by Milin attracted much attention, which led to de Branges [4] establishing the Bieberbach conjecture. Logarithmic coefficients also play a significant role in conformal mapping, which helped Kayumov [5] solve Brennan's conjecture. Since then, numerous studies on logarithmic coefficients, which play a central role in the theory of univalent functions, have continued, with examples found in [6–9]. Here, the logarithmic coefficients γ_n, $n \geqslant 1$ of $f(z) \in S$ are defined by

$$\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n. \tag{8}$$

Differentiating (8) and equating coefficients of z^n yield expressions for the logarithmic coefficients in terms of the Taylor coefficients for $f(z) \in S$, specifically for $n = 1, 2, 3$:

$$\gamma_1 = \frac{1}{2}a_2, \tag{9}$$

$$\gamma_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}{a_2}^2\right), \tag{10}$$

and

$$\gamma_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}{a_2}^3\right). \tag{11}$$

The logarithmic coefficients of the inverse functions, denoted Γ_n for $f(z) \in S$ were introduced by Ponnusamy et al. [10]. They are expressed in the series form as

$$\log \frac{f^{-1}(w)}{w} = 2\sum_{n=1}^{\infty} \Gamma_n w^n,\ |w| < \frac{1}{4},$$

where, in particular, for $n = 1, 2, 3$:

$$\Gamma_1 = -\frac{1}{2}a_2, \tag{12}$$

$$\Gamma_2 = -\frac{1}{2}\left(a_3 - \frac{3}{2}{a_2}^2\right), \tag{13}$$

and

$$\Gamma_3 = -\frac{1}{2}\left(a_4 - 4a_2a_3 + \frac{10}{3}{a_2}^3\right). \tag{14}$$

A typical subject in geometric function theory is the study of coefficient functionals, which are equations derived from various combinations of Taylor coefficients for subclasses in S. This comprises the Taylor coefficients of inverse functions, the logarithmic coefficients of functions and inverse functions, as well as the Vandermonde determinant.

The Vandermonde determinant, also known as a discriminant, has many applications in a range of domains. It is used in digital signal processing to compute the discrete Fourier transform (DFT) and the inverse discrete Fourier transform (IDFT), as well as in approximation problems [11]. It is also an important tool in linear algebra, for example, in determining the number of roots of polynomials (see [12]). Vijayalakshmi et al. [11] studied the Vandermonde determinant $V_{q,n}(f)$, where $n, q \geq 1$ and $a_n, n \geq 2$ are the Taylor series coefficients in (1):

$$V_{q,n}(f) = \begin{vmatrix} 1 & a_n & \cdots & {a_n}^{q-1} \\ 1 & a_{n+1} & \cdots & {a_{n+1}}^{q-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & a_{n+q-1} & \cdots & {a_{n+q-1}}^{q-1} \end{vmatrix}, a_1 = 1. \tag{15}$$

It is noted that with one as the first element, this determinant displays a geometric sequence in each row or column. Abdul Wahid et al. [13] introduced the Vandermonde determinant $V_{q,n}(\gamma_f)$, where $n, q \geq 1$ by looking at the logarithmic series coefficients $\gamma_n, n \geq 1$ in (8), because they were inspired by previous studies on Hankel and Toeplitz determinants, which involved both Taylor coefficients and logarithmic coefficients. This determinant is given by [13]

$$V_{q,n}(\gamma_f) = \begin{vmatrix} 1 & \gamma_n & \cdots & {\gamma_n}^{q-1} \\ 1 & \gamma_{n+1} & \cdots & {\gamma_{n+1}}^{q-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \gamma_{n+q-1} & \cdots & {\gamma_{n+q-1}}^{q-1} \end{vmatrix}. \tag{16}$$

Given the importance of the Vandermonde determinant and inspired by the works of Abdul Wahid et al. [14], Shi et al. [15], Obradovi and Tuneski [16], and Hadi et al. [17], which deal with solving determinant and coefficient functional problems for the inverse of analytic functions, it is natural to explore the Vandermonde determinant with A_n and Γ_n replacing a_n and γ_n, respectively. Using this idea, we define the Vandermonde determinant of Taylor coefficients and logarithmic coefficients of inverse functions for $f(z) \in S$, respectively, as follows:

$$V_{q,n}(f^{-1}) = \begin{vmatrix} 1 & A_n & \cdots & {A_n}^{q-1} \\ 1 & A_{n+1} & \cdots & {A_{n+1}}^{q-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & A_{n+q-1} & \cdots & {A_{n+q-1}}^{q-1} \end{vmatrix} \tag{17}$$

and

$$V_{q,n}(\Gamma_{f^{-1}}) = \begin{vmatrix} 1 & \Gamma_n & \cdots & {\Gamma_n}^{q-1} \\ 1 & \Gamma_{n+1} & \cdots & {\Gamma_{n+1}}^{q-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \Gamma_{n+q-1} & \cdots & {\Gamma_{n+q-1}}^{q-1} \end{vmatrix}. \tag{18}$$

Recently, Kazımoğlu et al. [18], Srivastava et al. [19], Tang et al. [20], Murugusundaramoorthy et al. [21], and Al-Hawarya et al. [22] introduced new classes of univalent functions associated with the generating function of Gregory coefficients. The Gregory coefficients, also known as reciprocal logarithmic numbers, second-kind Bernoulli numbers, or Cauchy numbers, are decreasing rational numbers that serve a function similar to Bernoulli numbers and can be found in a wide range of problems, particularly those involving numerical analysis and number theory. The generating function of the Gregory coefficients Λ_n, for $n \geq 0$, is given by

$$\frac{z}{\ln(1+z)} = \sum_{n=0}^{\infty} \Lambda_n z^n = 1 + \frac{1}{2}z - \frac{1}{12}z^2 + \frac{1}{24}z^3 - \frac{19}{720}z^4 + \frac{3}{160}z^5 - \ldots. \tag{19}$$

In connection with this function, we define the following class:

Definition 1. *An analytic function $f(z)$ of the form (1) is said to be in the class $G_{\mathrm{G}}(\alpha,\delta)$ if the following condition is satisfied:*

$$\frac{e^{i\alpha}f'(z)-i\sin\alpha-\delta}{\tau_{\alpha\delta}}\prec\Psi(z),\ \ z\in E,$$

where $\Psi(z)=\frac{z}{\ln(1+z)}$, $\tau_{\alpha\delta}=\cos\alpha-\delta$, $|\alpha|<\pi$, and $0\leqslant\delta<1$.

Remark 1. *Selecting specific values for the parameters α and δ in the class $G_{\mathrm{G}}(\alpha,\delta)$ yields the following classes, which are new and have not yet been studied by others:*

(i) $G_{\mathrm{G}}(\alpha,0)\equiv G_{\mathrm{G}}(\alpha)=\left\{f\in\mathcal{S}:\frac{e^{i\alpha}f'(z)-i\sin\alpha}{\cos\alpha}\prec\Psi(z),z\in E\right\}$

(ii) $G_{\mathrm{G}}(0,\delta)\equiv G_{\mathrm{G}}(\delta):=\left\{f\in\mathcal{S}:\frac{f'(z)-\delta}{1-\delta}\prec\Psi(z),z\in E\right\}$

(iii) $G_{\mathrm{G}}(0,0)\equiv G_{\mathrm{G}}=\{f\in\mathcal{S}:f'(z)\prec\Psi(z),z\in E\}$

The class $G_{\mathrm{G}}(\alpha,\delta)$ is inspired by the generalized class of bounded turning functions $G(\alpha,\delta)$ introduced by Mohamad [23], which satisfies

$$\left\{f\in\mathcal{S}:\mathrm{Re}\left(e^{i\alpha}f'(z)\right)>\delta,\ z\in E\right\}.$$

Then there exists a function $p(z)\in P$ such that [23]

$$\frac{e^{i\alpha}f'(z)-i\sin\alpha-\delta}{\tau_{\alpha\delta}}=p(z)\in P,$$

where $\tau_{\alpha\delta}=\cos\alpha-\delta$, $|\alpha|<\pi$, $0\leqslant\delta<1$, and $\cos\alpha>\delta$.

Remark 2. *Selecting specific values for the parameters α and δ in the class $G(\alpha,\delta)$ results in the following classes:*

(i) $G(\alpha,0)\equiv R(\alpha)=\left\{f\in\mathcal{S}:\mathrm{Re}\left(e^{i\alpha}f'(z)\right)>0,\ z\in E\right\}$

(ii) $G(0,\delta)\equiv R(\delta)=\{f\in\mathcal{S}:\mathrm{Re}(f'(z))>\delta,\ z\in E\}$. *The class $R(\delta)$ is called the class of bounded turning functions of order δ.*

(iii) $G(0,0)\equiv R=\{f\in\mathcal{S}:\mathrm{Re}(f'(z))>0,\ z\in E\}$. *The class R is called the class of bounded turning functions.*

Pioneering researchers like Goel and Mehrok [24], Macgregor [25], Noshiro [26], Silverman and Silvia [27], and Warschawski [28] explored the classes R, $R(\delta)$, and $R(\alpha)$, and nonetheless, it is intriguing to examine these classes in light of the generating functions of Gregory coefficients, leading to the geometric properties of this class and contributing to ongoing developments in geometry function theory.

Therefore, this paper aims to estimate the upper bounds of the Taylor coefficients and logarithmic coefficients of functions and inverse functions belonging to the class $G_{\mathrm{G}}(\alpha,\delta)$ of analytic functions, which is associated with generalized bounded turning and the generating functions of Gregory coefficients. For example, $|a_n|$ $(n=2,3,4,5)$, $|A_n|$ $(n=2,3,4,5)$, $|\gamma_n|$ $(n=1,2,3)$, and $|\Gamma_n|$ $(n=1,2,3)$. As a result, we focus on estimating the upper bounds of the second-order Vandermonde determinant, whose entries are Taylor coefficients and logarithmic coefficients of functions and inverse functions in $G_{\mathrm{G}}(\alpha,\delta)$.

2. Preliminary results

This section gives a few sharp bounds on coefficient functionals for functions with a positive real part, in the form of the following lemmas, to verify our main findings:

Lemma 1. ([29]) *For a function $p(z) \in P$ of the form (6), the sharp inequality $|p_n| \leqslant 2$ holds for each $n \geqslant 1$. The equality holds for the function $p(z) = \frac{1+z}{1-z}$.*

Lemma 2. ([30]) *Let $p(z) \in P$ be a function of the form (6) and $\mu^* \in \mathbb{C}$. Then*

$$|p_n - \mu^* p_k p_{n-k}| \leqslant 2max\{1, |2\mu^* - 1|\}, \ 1 \leqslant k \leqslant n-1.$$

If $|2\mu^* - 1| \geqslant 1$, then the inequality is sharp for the function $p(z) = \frac{1+z}{1-z}$ or its rotations. If $|2\mu^* - 1| < 1$, then the inequality is sharp for the function $p(z) = \frac{1+z^n}{1-z^n}$ or its rotations.

Lemma 3. ([31]) *Let $p(z) \in P$ be a function of the form (6) and $\alpha^*, \beta^*, \gamma^* \in \Re$. Then*

$$\left|\alpha^* {p_1}^3 - \beta^* p_1 p_2 + \gamma^* p_3\right| \leqslant 2|\alpha^*| + 2|\beta^* - 2\alpha^*| + 2|\alpha^* - \beta^* + \gamma^*|.$$

Lemma 4. ([32]) *Let $p(z) \in P$ be a function of the form (6) and $0 < \beta < 1$, $0 < \mu < 1$, and*
$8\beta(1-\beta)\left[(\mu\eta - 2\alpha)^2 + (\mu(\beta+\mu) - \eta)^2\right] + \mu(1-\mu)(\eta - 2\beta\mu)^2 \leq 4\mu^2\beta(1-\mu)^2(1-\beta)$.
Then

$$\left|\alpha {p_1}^4 + \beta {p_2}^2 + 2\mu p_1 p_3 - \frac{3}{2}\eta {p_1}^2 p_2 - p_4\right| \leq 2.$$

3. Main results

This section presents the proof of our main findings, focusing primarily on the upper bounds of Taylor coefficients, logarithmic coefficients, and Vandermonde determinant of second-order of functions and inverse functions belonging to the class $G_{\mathrm{G}}(\alpha, \delta)$.

3.1. Taylor coefficients

We now estimate the upper bounds of the Taylor coefficients of functions and inverse functions in $G_{\mathrm{G}}(\alpha, \delta)$.

Theorem 1. *Let $f(z) \in G_{\mathrm{G}}(\alpha, \delta)$. Then*

$$|a_n| \leq \frac{\tau_{\alpha\delta}}{2n}, \ n = 2, 3, 4, 5,$$

where $\tau_{\alpha\delta} = \cos\alpha - \delta$.

Proof. Let a function $f(z) \in G_{\mathrm{G}}(\alpha, \delta)$ given by (1). Then there exists a Schwarz function $\upsilon(z)$ with $\upsilon(0) = 0$ and $|\upsilon(z)| < 1$ in E such that

$$\frac{e^{i\alpha} f'(z) - i\sin\alpha - \delta}{\tau_{\alpha\delta}} = \Psi(\upsilon(z)), \ z \in E, \tag{20}$$

where $\tau_{\alpha\delta} = \cos\alpha - \delta$.
Define the function $p(z)$ by

$$p(z) = \frac{1 + \upsilon(z)}{1 - \upsilon(z)}, \ z \in E,$$

or equivalently

$$\begin{aligned} \upsilon(z) &= \tfrac{-1+p(z)}{1+p(z)} \\ &= \tfrac{1}{2}p_1 z + \tfrac{1}{2}\left(p_2 - \tfrac{1}{2}p_1{}^2\right) z^2 + \tfrac{1}{2}\left(\tfrac{1}{4}p_1{}^3 - p_1 p_2 + p_3\right) z^3 \\ &+ \tfrac{1}{2}\left(-\tfrac{1}{8}p_1{}^4 + \tfrac{3}{4}p_1{}^2 p_2 - p_1 p_3 - \tfrac{1}{2}p_2{}^2 + p_4\right) z^4 + \cdots . \end{aligned} \tag{21}$$

Using (21) along with the expression $\Psi(\upsilon(z)) = \frac{\upsilon(z)}{\ln(1+\upsilon(z))}$, we obtain

$$\begin{aligned} \Psi(\upsilon(z)) = 1 &+ \tfrac{1}{4}p_1 z + \tfrac{1}{48}\left(12p_2 - 7p_1{}^2\right) z^2 + \tfrac{1}{192}\left(17p_1{}^3 - 56p_1 p_2 + 48p_3\right) z^3 \\ &+ \tfrac{1}{11520}\left(-649p_1{}^4 + 3060p_1{}^2 p_2 - 3360p_1 p_3 - 1680p_2{}^2 + 2880p_4\right) z^4 + \cdots . \end{aligned} \tag{22}$$

Thus, by applying (22) to (20), we have

$$\begin{aligned} & e^{i\alpha}\left[\left(1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + 5a_5 z^4 + \cdots\right) - 1\right] \\ &= \tau_{\alpha\delta}\left[\tfrac{1}{4}p_1 z + \tfrac{1}{48}\left(12p_2 - 7p_1{}^2\right) z^2 + \tfrac{1}{192}\left(17p_1{}^3 - 56p_1 p_2 + 48p_3\right) z^3 \right. \\ &\left. + \tfrac{1}{11520}\left(-649p_1{}^4 + 3060p_1{}^2 p_2 - 3360p_1 p_3 - 1680p_2{}^2 + 2880p_4\right) z^4 + \cdots\right] . \end{aligned} \tag{23}$$

Comparing the coefficients of z^n for $n = 1, 2, 3, 4$ on both sides of (23) gives

$$\left.\begin{aligned} a_2 &= \tfrac{\tau_{\alpha\delta} e^{-i\alpha}}{8} p_1, \\ a_3 &= \tfrac{\tau_{\alpha\delta} e^{-i\alpha}}{144}\left(12p_2 - 7p_1{}^2\right), \\ a_4 &= \tfrac{\tau_{\alpha\delta} e^{-i\alpha}}{768}\left(17p_1{}^3 - 56p_1 p_2 + 48p_3\right), \\ a_5 &= \tfrac{\tau_{\alpha\delta} e^{-i\alpha}}{57600}\left(-649p_1{}^4 + 3060p_1{}^2 p_2 - 3360p_1 p_3 - 1680p_2{}^2 + 2880p_4\right). \end{aligned}\right\} \tag{24}$$

Based on Lemmas 1-4, the equations in (24) can be expressed as follows:

$$|a_2| = \left|\frac{\tau_{\alpha\delta} e^{-i\alpha}}{8} p_1\right|, \tag{25}$$

$$|a_3| = \left|\frac{\tau_{\alpha\delta} e^{-i\alpha}}{144}\left[12\left(p_2 - \frac{7}{12}p_1{}^2\right)\right]\right|, \tag{26}$$

$$|a_4| = \left|\frac{\tau_{\alpha\delta} e^{-i\alpha}}{768}\left[17p_1{}^3 - 56p_1 p_2 + 48p_3\right]\right|, \tag{27}$$

$$|a_5| = \left|\frac{\tau_{\alpha\delta} e^{-i\alpha}}{20}\left[\frac{649}{2880}p_1{}^4 + \frac{1680}{2880}p_2{}^2 + 2\left(\frac{1680}{2880}\right)p_1 p_3 - \frac{3}{2}\left(\frac{2040}{2880}\right)p_1{}^2 p_2 - p_4\right]\right|. \tag{28}$$

It is observed that

$$\begin{aligned}
&|p_1| \leq 2,\\
&\left|p_2 - \tfrac{7}{12}{p_1}^2\right| \leq 2\max\left\{1, \left|2\left(\tfrac{7}{12}\right)-1\right|\right\} = 2,\\
&\left|17{p_1}^3 - 56p_1p_2 + 48p_3\right| \leq 2|17| + 2|56-2(17)| + 2|17-56+48| = 96,\\
&\left|\tfrac{649}{2880}{p_1}^4 + \tfrac{1680}{2880}{p_2}^2 + 2\left(\tfrac{1680}{2880}\right)p_1p_3 - \tfrac{3}{2}\left(\tfrac{2040}{2880}\right){p_1}^2p_2 - p_4\right| \leq 2.
\end{aligned}$$

The upper bounds for $|a_2|$, $|a_3|$, $|a_4|$, and $|a_5|$ result from applying Lemmas 1-4, respectively:

$$\left.\begin{aligned}
|a_2| &\leq \tfrac{\tau_{\alpha\delta}}{8}(2) = \tfrac{\tau_{\alpha\delta}}{4},\\
|a_3| &\leq \tfrac{\tau_{\alpha\delta}}{12}(2) = \tfrac{\tau_{\alpha\delta}}{6},\\
|a_4| &\leq \tfrac{\tau_{\alpha\delta}}{768}(96) = \tfrac{\tau_{\alpha\delta}}{8},\\
|a_5| &\leq \tfrac{\tau_{\alpha\delta}}{20}(2) = \tfrac{\tau_{\alpha\delta}}{10}.
\end{aligned}\right\} \tag{29}$$

Thus, we get the desired bound. This completes the proof of Theorem 1.

Theorem 2. *Let* $f(z) \in G_{\mathrm{G}}(\alpha, \delta)$. *Then*

$$|A_2| \leq \frac{\tau_{\alpha\delta}}{4},$$

$$|A_3| \leq \frac{\tau_{\alpha\delta}}{6},$$

$$|A_4| \leq \frac{\tau_{\alpha\delta}}{24}\left[\left|5\tau_{\alpha\delta}e^{-i\alpha} + 7\right| + 3\right],$$

where $\tau_{\alpha\delta} = \cos\alpha - \delta$.

Proof. By substituting (24) into (3)-(5), we get

$$\left.\begin{aligned}
A_2 &= -\tfrac{\tau_{\alpha\delta}e^{-i\alpha}}{8}p_1,\\
A_3 &= -\tfrac{\tau_{\alpha\delta}e^{-i\alpha}}{288}\left[24p_2 - \left(14 + 9\tau_{\alpha\delta}e^{-i\alpha}\right){p_1}^2\right],\\
A_4 &= -\tfrac{\tau_{\alpha\delta}e^{-i\alpha}}{16}\left[\left(360{\tau_{\alpha\delta}}^2e^{-2i\alpha} + 1120\tau_{\alpha\delta}e^{-i\alpha} + 816\right)\tfrac{{p_1}^3}{2304} + p_3 - \left(\tfrac{5\tau_{\alpha\delta}e^{-i\alpha}+7}{6}\right)p_1p_2\right].
\end{aligned}\right\} \tag{30}$$

Taking the modulus on both sides of the equations in (30) and applying Lemmas 1-3, we can express the equations in (30) as follows:

$$|A_2| = \left|-\frac{\tau_{\alpha\delta}e^{-i\alpha}}{8}p_1\right|, \tag{31}$$

$$|A_3| = \left|-\frac{\tau_{\alpha\delta}e^{-i\alpha}}{288}\left[24\left(p_2 - \left(\frac{14 + 9\tau_{\alpha\delta}e^{-i\alpha}}{24}\right){p_1}^2\right)\right]\right|, \tag{32}$$

$$|A_4| = \left|\frac{\tau_{\alpha\delta}e^{-i\alpha}}{16}\left\{p_1\left(\frac{5\tau_{\alpha\delta}e^{-i\alpha}+7}{6}\right)\left[p_2 - \left(\frac{45{\tau_{\alpha\delta}}^2e^{-2i\alpha} + 140\tau_{\alpha\delta}e^{-i\alpha} + 102}{240\tau_{\alpha\delta}e^{-i\alpha} + 336}\right){p_1}^2\right] - p_3\right\}\right|. \tag{33}$$

It is observed that

$$\begin{aligned}
&|p_1| \le 2,\\
&|p_3| \le 2,\\
&\left|p_2 - \left(\tfrac{14+9\tau_{\alpha\delta}e^{-i\alpha}}{24}\right){p_1}^2\right| \le 2\max\left\{1, \left|2\left(\tfrac{14+9\tau_{\alpha\delta}e^{-i\alpha}}{24}\right)-1\right|\right\} = 2,\\
&\left|p_2 - \left(\tfrac{45{\tau_{\alpha\delta}}^2e^{-2i\alpha}+140\tau_{\alpha\delta}e^{-i\alpha}+102}{240\tau_{\alpha\delta}e^{-i\alpha}+336}\right){p_1}^2\right| \le 2\max\left\{1, \left|2\left(\tfrac{45{\tau_{\alpha\delta}}^2e^{-2i\alpha}+140\tau_{\alpha\delta}e^{-i\alpha}+102}{240\tau_{\alpha\delta}e^{-i\alpha}+336}\right)-1\right|\right\} = 2.
\end{aligned}$$

Thus, the upper bounds for $|A_2|$ and $|A_3|$ are obtained by applying Lemma 1 and Lemma 2, respectively. Meanwhile, the bound for $|A_4|$ follows from the combined application of Lemmas 1 and 2, along with the triangle inequality. This completes the proof of Theorem 2.

3.2. Logarithmic coefficients

Next, we estimate the upper bounds of the logarithmic coefficients for functions and their inverse functions in the class $G_{\mathrm{G}}(\alpha, \delta)$.

Theorem 3. *Let* $f(z) \in G_{\mathrm{G}}(\alpha, \delta)$. *Then*

$$|\gamma_1| \le \frac{\tau_{\alpha\delta}}{8},$$

$$|\gamma_2| \le \frac{\tau_{\alpha\delta}}{12},$$

and

$$|\gamma_3| \le \frac{\tau_{\alpha\delta}}{48}\left[3 + \left|\tau_{\alpha\delta}e^{-i\alpha} + 7\right|\right],$$

where $\tau_{\alpha\delta} = \cos\alpha - \delta$.

Proof. Substituting (24) into (9)-(11) and simplifying, we obtain

$$\gamma_1 = \frac{\tau_{\alpha\delta}e^{-i\alpha}}{16}p_1, \tag{34}$$

$$\gamma_2 = \frac{\tau_{\alpha\delta}e^{-i\alpha}}{2304}\left[96p_2 - \left(57 + 9\tau_{\alpha\delta}e^{-i\alpha}\right){p_1}^2\right], \tag{35}$$

$$\gamma_3 = \frac{\tau_{\alpha\delta}e^{-i\alpha}}{9216}\left[288p_3 - \left(48\tau_{\alpha\delta}e^{-i\alpha} + 336\right)p_1p_2 + \left(3{\tau_{\alpha\delta}}^2e^{-2i\alpha} + 28\tau_{\alpha\delta}e^{-i\alpha} + 102\right){p_1}^3\right]. \tag{36}$$

Using Lemma 1 on (34) yields

$$|\gamma_1| \le \frac{\tau_{\alpha\delta}}{16}(2) = \frac{\tau_{\alpha\delta}}{8}.$$

Applying Lemma 2 to (35) implies that

$$|\gamma_2| = \frac{\tau_{\alpha\delta}}{24}\left|p_2 - \left(\frac{57 + 9\tau_{\alpha\delta}e^{-i\alpha}}{96}\right){p_1}^2\right| \le \frac{\tau_{\alpha\delta}}{24}(2) = \frac{\tau_{\alpha\delta}}{12}.$$

Now, by rearranging the terms in (36), we can rewrite it as

$$|\gamma_3| = \left|\frac{\tau_{\alpha\delta}e^{-i\alpha}}{9216}\left\{288p_3 - \left(48\tau_{\alpha\delta}e^{-i\alpha}+336\right)p_1\left[p_2 - \left(\frac{3{\tau_{\alpha\delta}}^2e^{-2i\alpha}+28\tau_{\alpha\delta}e^{-i\alpha}+102}{48\tau_{\alpha\delta}e^{-i\alpha}+336}\right){p_1}^2\right]\right\}\right|.$$

Consequently, by applying Lemmas 1 and 2, along with the triangle inequality, and simplifying, we obtain

$$|\gamma_3| \le \frac{\tau_{\alpha\delta}}{48}\left[3+\left|\tau_{\alpha\delta}e^{-i\alpha}+7\right|\right].$$

This completes the proof of Theorem 3.

Theorem 4. *Let* $f(z) \in G_{\mathrm{G}}(\alpha,\delta)$. *Then*

$$|\Gamma_1| \le \frac{\tau_{\alpha\delta}}{8},$$

$$|\Gamma_2| \le \frac{\tau_{\alpha\delta}}{12},$$

and

$$|\Gamma_3| \le \frac{\tau_{\alpha\delta}}{48}\left[3+\left|4\tau_{\alpha\delta}e^{-i\alpha}+7\right|\right],$$

where $\tau_{\alpha\delta} = \cos\alpha - \delta$.

Proof. Substituting (24) into (12)-(14), we have

$$\Gamma_1 = -\frac{\tau_{\alpha\delta}e^{-i\alpha}}{16}p_1, \tag{37}$$

$$\begin{aligned}\Gamma_2 &= -\tfrac{\tau_{\alpha\delta}e^{-i\alpha}}{2304}\left[96p_2 - \left(27\tau_{\alpha\delta}e^{-i\alpha}+56\right){p_1}^2\right]\\ &= -\tfrac{\tau_{\alpha\delta}e^{-i\alpha}}{24}\left[p_2 - \left(\tfrac{27\tau_{\alpha\delta}e^{-i\alpha}+56}{96}\right){p_1}^2\right],\end{aligned} \tag{38}$$

$$\begin{aligned}\Gamma_3 &= -\tfrac{\tau_{\alpha\delta}e^{-i\alpha}}{4608}\left[\left(51+56\tau_{\alpha\delta}e^{-i\alpha}+15{\tau_{\alpha\delta}}^2e^{-2i\alpha}\right){p_1}^3 - \left(168+96\tau_{\alpha\delta}e^{-i\alpha}\right)p_1p_2+144p_3\right]\\ &= -\tfrac{\tau_{\alpha\delta}e^{-i\alpha}}{4608}\left\{-p_1\left(168+96\tau_{\alpha\delta}e^{-i\alpha}\right)\left[p_2 - \left(\tfrac{51+56\tau_{\alpha\delta}e^{-i\alpha}+15{\tau_{\alpha\delta}}^2e^{-2i\alpha}}{168+96\tau_{\alpha\delta}e^{-i\alpha}}\right){p_1}^2\right]+144p_3\right\}.\end{aligned} \tag{39}$$

The bounds for $|\Gamma_1|$ and $|\Gamma_2|$ follow from Lemma 1 and Lemma 2, respectively. Meanwhile, the bound for $|\Gamma_3|$ results from applying both Lemmas 1 and 2, along with the triangle inequality. This completes the proof of Theorem 4.

3.3. Vandermonde determinant of Taylor coefficients

In this subsection, we use the results of Theorems 1 and 2 to estimate the upper bounds of the Vandermonde determinant of second-order, where the entries are Taylor coefficients of functions and inverse functions in $G_{\mathrm{G}}(\alpha,\delta)$, that is, $|V_{2,2}(f)|$ and $\left|V_{2,2}\left(f^{-1}\right)\right|$.

Theorem 5. *Let* $f(z) \in G_{\mathrm{G}}(\alpha, \delta)$. *Then*

$$|V_{2,2}(f)| \leq \frac{5\tau_{\alpha\delta}}{12},$$

where $\tau_{\alpha\delta} = \cos\alpha - \delta$.

Proof. In view of (15), we can establish

$$|V_{2,2}(f)| = |a_3 - a_2| \leq |a_3| + |a_2|. \tag{40}$$

Using $|a_2| \leq \frac{\tau_{\alpha\delta}}{4}$ and $|a_3| \leq \frac{\tau_{\alpha\delta}}{6}$ from Theorem 1, we obtain

$$|V_{2,2}(f)| \leq \frac{\tau_{\alpha\delta}}{6} + \frac{\tau_{\alpha\delta}}{4} = \frac{5\tau_{\alpha\delta}}{12}. \tag{41}$$

Thus, we get the desired inequality, thereby completing the proof of Theorem 5.

Theorem 6. *Let* $f(z) \in G_{\mathrm{G}}(\alpha, \delta)$. *Then*

$$\left|V_{2,2}\left(f^{-1}\right)\right| \leq \frac{5\tau_{\alpha\delta}}{12},$$

where $\tau_{\alpha\delta} = \cos\alpha - \delta$.

Proof. From (17), we can establish

$$\left|V_{2,2}\left(f^{-1}\right)\right| = |A_3 - A_2| \leq |A_3| + |A_2|. \tag{42}$$

Making use of $|A_2| \leq \frac{\tau_{\alpha\delta}}{4}$ and $|A_3| \leq \frac{\tau_{\alpha\delta}}{6}$ from Theorem 2, we get

$$\left|V_{2,2}\left(f^{-1}\right)\right| \leq \frac{\tau_{\alpha\delta}}{6} + \frac{\tau_{\alpha\delta}}{4} = \frac{5\tau_{\alpha\delta}}{12}, \tag{43}$$

thereby concluding the proof of Theorem 6.

3.4. Vandermonde determinant of logarithmic coefficients

In this subsection, we use the results of Theorems 3 and 4 to estimate the upper bounds of the Vandermonde determinant of second-order, where the entries are logarithmic coefficients of functions and inverse functions in $G_{\mathrm{G}}(\alpha, \delta)$, that is, $|V_{2,1}(\gamma_f)|$, $|V_{2,2}(\gamma_f)|$, $\left|V_{2,1}\left(\Gamma_{f^{-1}}\right)\right|$, and $\left|V_{2,1}\left(\Gamma_{f^{-1}}\right)\right|$.

Theorem 7. *Let* $f(z) \in G_{\mathrm{G}}(\alpha, \delta)$. *Then*

$$|V_{2,1}(\gamma_f)| \leq \frac{5\tau_{\alpha\delta}}{24}$$

and

$$|V_{2,2}(\gamma_f)| \leq \frac{\tau_{\alpha\delta}}{48}\left[7 + \left|7 + \tau_{\alpha\delta}e^{-i\alpha}\right|\right],$$

where $\tau_{\alpha\delta} = \cos\alpha - \delta$.

Proof. From (16), we have

$$|V_{2,1}(\gamma_f)| = |\gamma_2 - \gamma_1| \leq |\gamma_2| + |\gamma_1| \tag{44}$$

and

$$|V_{2,2}(\gamma_f)| = |\gamma_3 - \gamma_2| \leq |\gamma_3| + |\gamma_2| \,. \tag{45}$$

Substituting $|\gamma_1| \leq \frac{\tau_{\alpha\delta}}{8}$ and $|\gamma_2| \leq \frac{\tau_{\alpha\delta}}{12}$ into (44), as well as $|\gamma_2| \leq \frac{\tau_{\alpha\delta}}{12}$ and $|\gamma_3| \leq \frac{\tau_{\alpha\delta}}{48}\left[3 + \left|\tau_{\alpha\delta}e^{-i\alpha} + 7\right|\right]$ into (45), respectively, yields

$$|V_{2,1}(\gamma_f)| \leq \frac{\tau_{\alpha\delta}}{12} + \frac{\tau_{\alpha\delta}}{8} = \frac{5\tau_{\alpha\delta}}{24}$$

and

$$|V_{2,2}(\gamma_f)| \leq \frac{\tau_{\alpha\delta}}{48}\left[3 + \left|\tau_{\alpha\delta}e^{-i\alpha} + 7\right|\right] + \frac{\tau_{\alpha\delta}}{12} = \frac{\tau_{\alpha\delta}}{48}\left[7 + \left|\tau_{\alpha\delta}e^{-i\alpha} + 7\right|\right].$$

This concludes the proof of Theorem 7.

Theorem 8. *Let* $f(z) \in G_{\mathrm{G}}(\alpha, \delta)$. *Then*

$$\left|V_{2,1}\left(\Gamma_{f^{-1}}\right)\right| \leq \frac{5\tau_{\alpha\delta}}{24}$$

and

$$\left|V_{2,2}\left(\Gamma_{f^{-1}}\right)\right| \leq \frac{\tau_{\alpha\delta}}{48}\left[7 + \left|7 + 4\tau_{\alpha\delta}e^{-i\alpha}\right|\right]$$

where $t_{\alpha\delta} = \cos\alpha - \delta$.

Proof. Using (18), we can establish

$$\left|V_{2,1}\left(\Gamma_{f^{-1}}\right)\right| = |\Gamma_2 - \Gamma_1| \leq |\Gamma_2| + |\Gamma_1| \tag{46}$$

and

$$\left|V_{2,2}\left(\Gamma_{f^{-1}}\right)\right| = |\Gamma_3 - \Gamma_2| \leq |\Gamma_3| + |\Gamma_2| \,. \tag{47}$$

Using the result of Theorem 4, we obtain

$$\left|V_{2,1}\left(\Gamma_{f^{-1}}\right)\right| \leq \frac{\tau_{\alpha\delta}}{12} + \frac{\tau_{\alpha\delta}}{8} = \frac{5\tau_{\alpha\delta}}{24}$$

and

$$\left|V_{2,2}\left(\Gamma_{f^{-1}}\right)\right| \leq \frac{\tau_{\alpha\delta}}{48}\left[3 + \left|4\tau_{\alpha\delta}e^{-i\alpha} + 7\right|\right] + \frac{\tau_{\alpha\delta}}{12} = \frac{\tau_{\alpha\delta}}{48}\left[7 + \left|7 + 4\tau_{\alpha\delta}e^{-i\alpha}\right|\right]$$

This concludes the proof of Theorem 8.

4. Consequences and corollaries

This section explores several new implications of Theorems 1-8, as $G_{\mathrm{G}}(\alpha,\delta)$ generalizes the classes $G_{\mathrm{G}}(\alpha)$, $G_{\mathrm{G}}(\delta)$, and G_{G}.

Selecting $\delta = 0$ from Theorems 1-8, we get the following estimates bounds for the class $G_{\mathrm{G}}(\alpha)$.

Corollary 1. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *and* $f^{-1}(w) = w + \sum_{n=2}^{\infty} A_n w^n$ *be in the class* $G_{\mathrm{G}}(\alpha)$. *Then*

(i) $|a_n| \leq \frac{\cos\alpha}{2n},\ n = 2,3,4,5$

(ii) $|A_2| \leq \frac{\cos\alpha}{4}, |A_3| \leq \frac{\cos\alpha}{6}, |A_4| \leq \frac{\cos\alpha}{24}\left[\left|5e^{-i\alpha}\cos\alpha + 7\right| + 3\right]$

(iii) $|\gamma_1| \leq \frac{\cos\alpha}{8}, |\gamma_2| \leq \frac{\cos\alpha}{12}, |\gamma_3| \leq \frac{\cos\alpha}{48}\left[3 + \left|e^{-i\alpha}\cos\alpha + 7\right|\right]$

(iv) $|\Gamma_1| \leq \frac{\cos\alpha}{8}, |\Gamma_2| \leq \frac{\cos\alpha}{12}, |\Gamma_3| \leq \frac{\cos\alpha}{48}\left[3 + \left|4e^{-i\alpha}\cos\alpha + 7\right|\right]$

(v) $|V_{2,2}(f)| \leq \frac{5\cos\alpha}{12}$

(vi) $\left|V_{2,2}\left(f^{-1}\right)\right| \leq \frac{5\cos\alpha}{12}$

(vii) $|V_{2,1}(\gamma_f)| \leq \frac{5\cos\alpha}{24},\ \ |V_{2,2}(\gamma_f)| \leq \frac{\cos\alpha}{48}\left[7 + \left|7 + e^{-i\alpha}\cos\alpha\right|\right]$

(viii) $\left|V_{2,1}\left(\Gamma_{f^{-1}}\right)\right| \leq \frac{5\cos\alpha}{24},\ \ \left|V_{2,2}\left(\Gamma_{f^{-1}}\right)\right| \leq \frac{\cos\alpha}{48}\left[7 + \left|7 + 4e^{-i\alpha}\cos\alpha\right|\right]$

Taking into account $\alpha = 0$ in Theorems 1-8, we obtain the following estimates bounds for the class $G_{\mathrm{G}}(\delta)$.

Corollary 2. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *and* $f^{-1}(w) = w + \sum_{n=2}^{\infty} A_n w^n$ *be in the class* $G_{\mathrm{G}}(\delta)$. *Then*

(i) $|a_n| \leq \frac{1-\delta}{2n},\ n = 2,3,4,5$

(ii) $|A_2| \leq \frac{1-\delta}{4}, |A_3| \leq \frac{1-\delta}{6}, |A_4| \leq \frac{5(1-\delta)(3-\delta)}{24}$

(iii) $|\gamma_1| \leq \frac{1-\delta}{8}, |\gamma_2| \leq \frac{1-\delta}{12}, |\gamma_3| \leq \frac{(1-\delta)(11-\delta)}{48}$

(iv) $|\Gamma_1| \leq \frac{1-\delta}{8}, |\Gamma_2| \leq \frac{1-\delta}{12}, |\Gamma_3| \leq \frac{(1-\delta)(7-2\delta)}{24}$

(v) $|V_{2,2}(f)| \leq \frac{5(1-\delta)}{12}$

(vi) $\left|V_{2,2}\left(f^{-1}\right)\right| \leq \frac{5(1-\delta)}{12}$

(vii) $|V_{2,1}(\gamma_f)| \leq \frac{5(1-\delta)}{24},\ \ \ |V_{2,2}(\gamma_f)| \leq \frac{(1-\delta)(15-\delta)}{48}$

(viii) $\left|V_{2,1}\left(\Gamma_{f^{-1}}\right)\right| \leq \frac{5(1-\delta)}{24}, \ \left|V_{2,2}\left(\Gamma_{f^{-1}}\right)\right| \leq \frac{(1-\delta)(9-2\delta)}{24}$

Putting $\alpha = 0$ and $\delta = 0$ in Theorems 1-8, we have the following results for the class G_{G}.

Corollary 3. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *and* $f^{-1}(w) = w + \sum_{n=2}^{\infty} A_n w^n$ *be in the class* G_{G}. *Then*

(i) $|a_n| \leq \frac{1}{2n}, \ n = 2, 3, 4, 5$

(ii) $|A_2| \leq \frac{1}{4}$, $|A_3| \leq \frac{1}{6}$, $|A_4| \leq \frac{5}{8}$

(iii) $|\gamma_1| \leq \frac{1}{8}$, $|\gamma_2| \leq \frac{1}{12}$, $|\gamma_3| \leq \frac{11}{48}$

(iv) $|\Gamma_1| \leq \frac{1}{8}$, $|\Gamma_2| \leq \frac{1}{12}$, $|\Gamma_3| \leq \frac{7}{24}$

(v) $|V_{2,2}(f)| \leq \frac{5}{12}$

(vi) $\left|V_{2,2}\left(f^{-1}\right)\right| \leq \frac{5}{12}$

(vii) $|V_{2,1}(\gamma_f)| \leq \frac{5}{24}, \quad |V_{2,2}(\gamma_f)| \leq \frac{5}{16}$

(viii) $\left|V_{2,1}\left(\Gamma_{f^{-1}}\right)\right| \leq \frac{5}{24}, \ \left|V_{2,2}\left(\Gamma_{f^{-1}}\right)\right| \leq \frac{3}{8}$

5. Conclusion

Recent research has sparked considerable interest in Vandermonde determinants. This has inspired us to study the Vandermonde determinant of functions and inverse functions belonging to the class $G_{\mathrm{G}}(\alpha, \delta)$ of analytic functions, which is associated with generalized bounded turning and the generating functions of Gregory coefficients. Furthermore, we have defined Vandermonde determinants whose entries are logarithmic coefficients of functions and inverse functions in S. Thus, in this paper, we have obtained estimates for Taylor coefficients, logarithmic coefficients, and the second-order Vandermonde determinant whose entries are Taylor coefficients and logarithmic coefficients of functions and inverse functions belonging to the class $G_{\mathrm{G}}(\alpha, \delta)$. This extends not only the properties of the class $G_{\mathrm{G}}(\alpha, \delta)$ but also those of $G_{\mathrm{G}}(\alpha)$, $G_{\mathrm{G}}(\delta)$, and G_{G} as shown in Corollaries 1-3. The lemmas in the preliminary section have proven invaluable in establishing upper bounds for coefficient functionals in Theorems 1-8. The findings of this work could be used to further investigate the upper bounds for the second-order Hankel, Toeplitz, and higher-order Vandermonde determinants, specifically within bounded turning functions connected to Gregory coefficients.

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